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A BRYCE AND COSSEY TYPE THEOREM IN A CLASS OF LOCALLY FINITE GROUPS

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In this paper the subgroup-closed saturated Fitting formations of radical locally finite groups with min-p for all p are fully characterised. Moreover the study of a class of generalised nilpotent groups introduced by Ballester-Bolinches and Pedraza is continued.

1. INTRODUCTION

The concepts of formations and Fitting classes were introduced in finite soluble groups by Gaschütz and by Fischer, Gaschütz and Hartley. Both theories form an important and a well-established part of the theory of finite soluble groups (see [**6**]).

Probably one of the most celebrated results connecting these two theories is a theorem of Bryce and Cossey establishing that a subgroup-closed Fitting formation of finite soluble groups is saturated [**3**].

The above theories have been successfully extended to various classes of infinite groups, specially locally finite-soluble, and there is a vast literature on the subject (see [**5**]). The definition of saturated formation has usually been done locally, because an infinite group may not have (many) maximal subgroups and so not have a satisfactory Frattini subgroup (see [**5**, Chapter 6]). Fitting classes are defined differently in different classes of groups; for example, the usual subnormality closure conditions can be replaced by seriality, ascendancy or descendancy.

In this note we study saturated formations and Fitting classes based on certain infinite groups. These groups are elements of the class $c\overline{\mathcal{L}}$ of radical locally finite groups satisfying min-p for every prime p. Dixon has developed in this class a satisfactory formation theory and interesting results on Fitting classes (see [**5**]). He uses the more general definition of Fitting classes, depending on the concept of serial subgroup, although in this universe serial subgroups are in fact ascendant.

In this paper, by a *Fitting class* of $c\overline{\mathcal{L}}$-groups we mean a subclass $\mathcal{F}$ of $c\overline{\mathcal{L}}$ satisfying:

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(F1) If $G \in \mathcal{F}$ and N is a normal subgroup of G, then $N \in \mathcal{F}$.

(F2) If $G \in c\overline{\mathcal{L}}$ is generated by normal $\mathcal{F}$-subgroups, then $G \in \mathcal{F}$.

This definition is used in this paper since it mirrors more precisely the definition of a finite Fitting class.

Notice that if $G \in c\overline{\mathcal{L}}$, then for each Fitting class $\mathcal{F}$, G has a unique maximal normal $\mathcal{F}$-subgroup of G, generated by all normal $\mathcal{F}$-subgroups of G. We call this subgroup the *$\mathcal{F}$-radical* of G.

It seems interesting to wonder whether it is possible to prove a Bryce and Cossey type theorem in the class $c\overline{\mathcal{L}}$.

Unfortunately, a subgroup-closed Fitting $c\overline{\mathcal{L}}$-formation is not saturated in general. The class $\mathcal{B}$ of all $c\overline{\mathcal{L}}$-groups such that each proper subgroup has a proper normal closure is a counterexample as it is shown in [2]. However, the characterisation of the saturated $c\overline{\mathcal{L}}$-formations given in [1] allows us to give a criterion for a subgroup-closed $c\overline{\mathcal{L}}$-Fitting formation to be saturated (Theorem 2). The paper is completed by proving some interesting results concerning the class $\mathcal{B}$ introduced above.

We use [5] and [6] as our main references for notation, terminology and results on classes of groups.

2. Known results

For the sake of completeness, we collect in this section some known results which are used to prove our theorems.

Definition 1. [9] *Let U be a subgroup of a group G. Consider the properly ascending chains of subgroups*

$$U = U_0 < U_1 < \cdots < U_\alpha = G$$

from U to G. Define $m(U)$ to be the least upper bound of the types α of all such chains.

A proper subgroup U of G is said to be a major subgroup *of G if $m(U) = m(V)$ whenever $U \leqslant V < G$.*

Tomkinson [9] shows that every proper subgroup of a group is contained in a major subgroup. Therefore the intersection $\mu(G)$ of all major subgroups of a group G is always a proper subgroup of G, which coincides with the Frattini subgroup of G, $\Phi(G)$, if G is finitely generated.

Definition 2. *A $c\overline{\mathcal{L}}$-formation $\mathcal{F}$ is said to be* E_μ-closed *if $\mathcal{F}$ enjoys the following properties:*

(E1) A $c\overline{\mathcal{L}}$-group G is in $\mathcal{F}$ if and only if $G/\mu(G) \in \mathcal{F}$.

(E2) A semiprimitive group G is an $\mathcal{F}$-group if and only if G is the union of an ascending chain $\{G_i \mid i \geqslant 0\}$ of finite $\mathcal{F}$-subgroups.

Recall that a group G is semiprimitive if it is the semidirect product $G = [D]M$, where M is a finite soluble group of trivial core and D is a divisible Abelian group such that every proper M-invariant subgroup of D is finite.

The following result will be used throughout the paper without further comment:

THEOREM 1. [1] *Let $\mathcal{F}$ be a $c\overline{\mathcal{L}}$-formation. Then $\mathcal{F}$ is E_μ-closed if and only if $\mathcal{F}$ is a saturated $c\overline{\mathcal{L}}$-formation.*

3. A Bryce and Cossey type Theorem

Recall that a class $\mathcal{F}$ is said to be a *$c\overline{\mathcal{L}}$-Fitting formation* if $\mathcal{F}$ is a $c\overline{\mathcal{L}}$-Fitting class which is also a $c\overline{\mathcal{L}}$-formation.

THEOREM 2. *Let $\mathcal{F}$ be a subgroup-closed $c\overline{\mathcal{L}}$-Fitting formation. Then $\mathcal{F}$ is saturated if and only $\mathcal{F}$ satisfies the following property:*

(α) *If G is a Chernikov group which is the union of an ascending chain $\{G_i \mid i \geqslant 0\}$ of finite subgroups, then G belongs to $\mathcal{F}$ if and only if G_i belongs to $\mathcal{F}$ for all i.*

PROOF: Suppose that $\mathcal{F}$ is a subgroup-closed $c\overline{\mathcal{L}}$-Fitting formation satisfying property (α). We prove that $\mathcal{F}$ is E_μ-closed. The second condition follows from property (α) which is satisfied by hypothesis (notice that a semiprimitive group is a Chernikov group). Therefore we must prove that a $c\overline{\mathcal{L}}$-group G belongs to $\mathcal{F}$ if $G/\mu(G)$ is in $\mathcal{F}$.

Assume that there exists a prime p such that $O_{p'}(G) = 1$. Then, by [**5**, (2.5.13)], G is a Chernikov group. Then, if G^0 is the radicable part of G, it follows that G/G^0 is finite and G^0 is an Abelian p-group which is a direct product of finitely many copies of quasicyclic p-groups. Moreover, $\mu(G/G^0) = \Phi(G/G^0)$, the Frattini subgroup of G/G^0, and $\mu(G)G^0/G^0$ is contained in $\Phi(G/G^0)$. Hence $(G/G^0)/\Phi(G/G^0)$ belongs to $\mathcal{F}^*$, where $\mathcal{F}^*$ denotes the class of all finite $\mathcal{F}$-groups. Since $\mathcal{F}^*$ is a subgroup-closed Fitting formation of finite soluble groups, it follows that $\mathcal{F}^*$ is saturated by the Bryce and Cossey Theorem. Therefore $G/G^0 \in \mathcal{F}$.

On the other hand, since G is a Chernikov group, it follows that $\mu(G)$ is finite by [**10**, (1.2)]. Consequently the $\mathcal{F}$-residual N of G is finite because $N \leqslant \mu(G)$. Moreover N is also contained in G^0. We may assume $N \neq 1$. Therefore there exists $i \geqslant 1$ such that N is contained in $\Omega_i(G^0)$, which is the subgroup of G^0 generated by the elements of G^0 whose order divides p^i. Let $1 = G_0 \leqslant G_1 \leqslant G_2 \leqslant \cdots$ be an ascending chain of finite subgroups of G such that $G = G^0G_j$ for all j and $G = \bigcup_{j \geqslant 1} G_j$. Since $\Omega_{i+1}(G^0)$ is finite, we have that there exists a positive integer n for which $\Omega_{i+1}(G^0)$ is contained in G_n. Let m be an integer such that $m \geqslant n$. Since $\Omega_{i+1}(G^0) \trianglelefteq G$ and $\Omega_{i+1}(G^0) \leqslant G_m$, it follows that $\Phi(\Omega_{i+1}(G^0)) \leqslant \Phi(G_m)$. On the other hand, $N \leqslant \Omega_i(G^0) \leqslant \Phi(\Omega_{i+1}(G^0))$. Hence $G_m/\Phi(G_m) \in \mathcal{F}^*$ because $\mathcal{F}$ is subgroup-closed. Since $\mathcal{F}^*$ is saturated, we have

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that $G_m \in \mathcal{F}$. Applying property (α) to the family $\{G_m \mid m \geqslant n\}$, we conclude that $G \in \mathcal{F}$.

In the general case, we have that $O_{p'}\big(G/O_{p'}(G)\big) = 1$ for all p. Moreover

$$\mu(G)O_{p'}(G)/O_{p'}(G) \leqslant \mu\big(G/O_{p'}(G)\big).$$

Therefore $\big(G/O_{p'}(G)\big)/\mu\big(G/O_{p'}(G)\big) \in \mathcal{F}$ and so $G/O_{p'}(G) \in \mathcal{F}$ for all p by the above argument. Since $\mathcal{F}$ is a formation, it follows that $G/\Big(\bigcap_p O_{p'}(G)\Big) \in \mathcal{F}$. Then $G \in \mathcal{F}$.

Conversely, assume that $\mathcal{F}$ is saturated and let G be a Chernikov group which is the union of an ascending chain $\{G_i \mid i \geqslant 0\}$ of finite subgroups. If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$ for all i because $\mathcal{F}$ is subgroup-closed. Suppose that $G_i \in \mathcal{F}$ for all i. Let M be a major subgroup of G. Then $G/\operatorname{core}_G(M)$ is either a finite and soluble primitive group or $G/\operatorname{core}_G(M)$ is semiprimitive by [**1**]. If $G/\operatorname{core}_G(M)$ is finite, then

$$G/\operatorname{core}_G(M) = G_i \operatorname{core}_G(M)/\operatorname{core}_G(M)$$

for some i and so $G/\operatorname{core}_G(M) \in \mathcal{F}$. If $G/\operatorname{core}_G(M)$ is semiprimitive, then it is the union of the ascending chain

$$\big\{G_i \operatorname{core}_G(M)/\operatorname{core}_G(M) \mid i \geqslant 0\big\}$$

of finite $\mathcal{F}$-subgroups. Since $\mathcal{F}$ is E_μ-closed, it follows that $G/\operatorname{core}_G(M) \in \mathcal{F}$. Now, $\mathcal{F}$ is a formation and

$$\mu(G) = \bigcap_{M \text{ major in } G} \operatorname{core}_G(M).$$

Consequently $G/\mu(G) \in \mathcal{F}$ and so $G \in \mathcal{F}$ because $\mathcal{F}$ is E_μ-closed. □

4. The class $\mathcal{B}$

Recall that $\mathcal{B}$ is the class of all groups $G \in c\overline{\mathcal{L}}$ such that every proper subgroup of G has proper normal closure. $\mathcal{B}$ is a class of generalised nilpotent groups in the sense of [**8**] and it enjoys many properties of nilpotent type. In fact, the following results hold.

Theorem 3. [**2**] *The following statements are pairwise equivalent:*

1. *G is a $\mathcal{B}$-group.*
2. *$G/\mu(G)$ is a $\mathcal{B}$-group.*
3. *$G' \leqslant \mu(G)$.*
4. *Every major subgroup of G is a normal subgroup of G.*
5. *G is a locally nilpotent group and each Sylow subgroup of G is nilpotent.*
6. *G is locally nilpotent and $G^0 \leqslant Z(G)$.*

As a consequence, a Chernikov group is nilpotent if and only if it is a $\mathcal{B}$-group

Theorem 4. [**2**] *$\mathcal{B}$ is a subgroup-closed $c\overline{\mathcal{L}}$-Fitting formation.*

Let G be the semidirect product of a quasicyclic 2-group H by a cyclic group of order 2 acting on H by inverting each of its elements. Then G is the union of an ascending chain of finite $\mathcal{B}$-subgroups, but G is not a $\mathcal{B}$-group. Therefore $\mathcal{B}$ is not saturated.

Following [2], we denote by $\delta(G)$ the $\mathcal{B}$-radical of a group $G \in c\overline{\mathcal{L}}$.

Theorem 5. *Let G be a $c\overline{\mathcal{L}}$-group, and let H be a subgroup of G. Then H is a descendant $\mathcal{B}$-subgroup of G if and only if H is contained in $\delta(G)$.*

Proof: Assume first that H is a descendant $\mathcal{B}$-subgroup of G. Then H is, in particular, a locally nilpotent serial subgroup of G. By [7, Lemma 1.31], H is contained in $h(G)$, the Hirsch-Plotkin radical of G. Since $\delta(h(G))$ is contained in $\delta(G)$ and H is a descendant $\mathcal{B}$-subgroup of $h(G)$, we may suppose without loss of generality that G is locally nilpotent.

Let p be any prime and let G_p be the unique Sylow p-subgroup of G. We prove that the unique Sylow p-subgroup H_p of H is contained in $\delta(G_p)$.

It is clear that $H_p = H \cap G_p$. Hence H_p is descendant in G_p. Since G_p satisfies min-p, we have that it satisfies the minimum condition on subgroups. Therefore H_p is actually subnormal in G_p. But H_p is nilpotent. Hence $H_p \leqslant F(G_p)$, the Fitting subgroup of G_p. By [5, 2.5.13], G_p is a Chernikov group. Hence $F(G_p) = \delta(G_p)$ and so $H_p \leqslant \delta(G_p)$.

Consequently

$$H = \operatorname*{Dr}_p H_p \leqslant \operatorname*{Dr}_p \delta(G_p) \leqslant \delta(G).$$

Conversely, suppose that H is a subgroup of G such that $H \leqslant \delta(G)$. Then we construct a descending series from G to H by transfinite recursion. We define $H_0 = G$ and suppose that, given an ordinal α, for each ordinal $\beta < \alpha$, H_β is defined and it is a descendant subgroup of G containing H.

Assume that $\alpha - 1$ exists. Then $H_{\alpha-1} \cap \delta(G)$ is a normal $\mathcal{B}$-subgroup of $H_{\alpha-1}$ and so $H_{\alpha-1} \cap \delta(G) \leqslant \delta(H_{\alpha-1})$. If $\delta(H_{\alpha-1})$ is a proper subgroup of $H_{\alpha-1}$, we define H_α to be $\delta(H_{\alpha-1})$. On the other hand, if $\delta(H_{\alpha-1}) = H_{\alpha-1}$ then $H_{\alpha-1}$ is a $\mathcal{B}$-group. Then, if H is a proper subgroup of $H_{\alpha-1}$, we have that the normal closure $H^{H_{\alpha-1}}$ of H in $H_{\alpha-1}$ is properly contained in $H_{\alpha-1}$. Then we define $H_\alpha = H^{H_{\alpha-1}}$. If $H = H_{\alpha-1}$, we define $H_\alpha = H$. Consider now the case in which α is a limit ordinal. Then we define H_α to be the intersection of all H_β, where $\beta < \alpha$. It is clear that H_α is a descendant subgroup of G containing H.

If λ is an ordinal of cardinality greater than that of G, then $H = H_\lambda$. Consequently H is a descendant subgroup of G. ∎

It is well-known that a finite group is nilpotent if and only if every subgroup is subnormal. As a consequence of the above theorem we have:

Corollary 1. *A $c\overline{\mathcal{L}}$-group G is a $\mathcal{B}$-group if and only if every subgroup of G is descendant in G.*

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Our next result characterises $\delta(G)$ in terms of the μ-chief factors defined in [4]. We recall the basic definitions and results of that paper.

DEFINITION 3. *Let G be a $c\overline{\mathcal{L}}$-group and I a totally ordered index set. A family $\Gamma = \{V_i, \Lambda_i \mid i \in I\}$ of pairs of normal subgroups of G is said to be a μ-series of type I if for each $i, k \in I$ the following holds:*

1. $V_i \leqslant \Lambda_i$,
2. $\Lambda_i \leqslant V_k$ *when* $i < k$,
3. $G \setminus 1 = \bigcup_{i \in I} (\Lambda_i \setminus V_i)$,
4. $(\Lambda_i \cap G^0)/(V_i \cap G^0)$ *is a divisible group.*

The subgroups V_i and Λ_i are called the terms of Γ, and the groups Λ_i/V_i are the factors of the μ-series.

If Γ and Ω are two μ-chief series of a group G, we shall say that Γ is a μ-refinement of Ω when each term of the series Ω is a term of the series Γ too. A μ-series without proper μ-refinements is a μ-chief series. The factors of such a series are called μ-chief factors.

THEOREM 6. [4] *Let G be a $c\overline{\mathcal{L}}$-group and let H/K be a μ-chief factor of G. Then either G^0 covers or avoids H/K. This is to say:*

$$H \leqslant KG^0 \text{ or } H \cap G^0 \leqslant K.$$

Moreover, if G^0 covers H/K, then we have:

1. $H/K \simeq \overbrace{C_{p^\infty} \times \cdots \times C_{p^\infty}}^{n}$, *for some positive integer n, and p prime.*
2. *H/K is divisibly irreducible as G-module.*
3. *$H \cap G^0/K \cap G^0$ is a μ-chief factor of G isomorphic to H/K.*

If G^0 avoids H/K, then H/K is a minimal normal subgroup of G/K. Thus:

1. *H/K is a finite elementary Abelian p-group, for some p prime.*
2. *H/K is irreducible viewed as G-module.*
3. *HG^0/KG^0 is a μ-chief factor of G isomorphic to H/K.*

THEOREM 7. [4] *Let us consider a group G in $c\overline{\mathcal{L}}$ and let Γ and Ω be two μ-chief series of G. Then we can establish a one-to-one correspondence between the factors of both series such that the corresponding factors are isomorphic.*

THEOREM 8. *Let G be a $c\overline{\mathcal{L}}$-group and Γ a μ-chief series of G. Then we have:*

$$\delta(G) = \bigcap \{C_G(H/K) \mid H/K \text{ is a factor in } \Gamma\}.$$

PROOF: Let H/K be a μ-chief factor of G. If H/K is finite, then H/K is a chief factor of G. Since $\delta(G)$ is contained in $h(G)$, the Hirsch-Plotkin radical of G, and $h(G)$

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centralises H/K, it follows that $\delta(G) \leqslant C_G(H/K)$. On the other hand, if H/K is divisible, then H/K is contained in $(\delta(G/K))^0$, the radicable part of $\delta(G/K)$. Since $\delta(G/K)$ is a $\mathcal{B}$-group, it follows that $(\delta(G/K))^0 \leqslant Z(\delta(G/K))$. Now $\delta(G)K/K \leqslant \delta(G/K)$. Hence $[H, \delta(G)] \leqslant K$.

Let $C = \bigcap\{C_G(H/K) \mid H/K \text{ is a factor in } \Gamma\}$. It is clear that C is a normal subgroup of G. Refining Γ to a chief series of G, we have that C centralises every chief factor of G in such a series. Therefore C is locally nilpotent. Let p be a prime and let C_p be the unique Sylow p-subgroup of C. We consider the series $\Gamma \cap C_p$, which is the intersection of the terms of Γ with C_p. Since Γ has only finitely many p-factors, it follows that the above series has finite length.

Let H/K be a μ-chief factor of G in Γ. Then $[C, H] \leqslant K$. This implies that $[C_p, H \cap C_p] \leqslant K \cap C_p$. Consequently $(H \cap C_p)/(K \cap C_p)$ is central in $C_p/(K \cap C_p)$ and then C_p has a chain of finite length with central factors. Therefore C_p is nilpotent for each prime p and so C is a $\mathcal{B}$-group. This implies that $C \leqslant \delta(G)$ and the theorem is proved. ∎

It is known that a finite group is nilpotent if and only if every chief factor of G is central. In our case, the following is true:

Corollary 2. *A $c\overline{\mathcal{L}}$-group G is a $\mathcal{B}$-group if and only if every μ-chief factor of G is central.*

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