A NEW APPROACH FOR THE CONSTRUCTION OF LONG-PERIODIC PERTURBATIONS

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ABSTRACT. The aim of this work is to study perturbations of planets of a period of some thousands of years. The use of an analytical method allows us to separate all different influences, e.g. near resonances and is combined with the very precise method of the numerical integration. The truncation to low orders can be avoided which is made by analytical methods in using developments with respect to the small parameters inclinations and eccentricities. For this purpose a special form of the Lagrange Equations is used where the terms containing the inverse distancefrom the planet to the perturbing one are separated as it is the most difficult to compute. To develop this a specific formulation has been found where the short periodic terms can precisely be determined. Although the development seems to be of a certain complexity the small numbers of quantities used can be tabulated once and for all in a specific problem. It should be possible to integrate the new form of the Lagrange Equations within a reasonable computer-time to determine the long periodic perturbations.

1. INTRODUCTION

To formulate the problem of constructing the long periodic perturbations of the planets with the aid of analytical methods is a priori restricted to a low order of the inclinations i and the eccentricities e. On the other hand the determination of the motion of the nodes and the perihelia is also dependant on the choice of numbers of the near resonances fixed in advance, which contributes to the solution of the problem.

The first aim of this paper is to extend the realm of validity of the series expansions with respect to the small parameters e and i to higher orders as it has to be done in methods based on trigonometric series. Another truncation is always made in looking only for solutions which are of low order in the masses. Therefore our knowledge of the long periodic terms in the motion of the planets is very uncertain. Although work has been done to understand

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the longer periods of some thousands of years by Hill (1889), Brouwer and van Woerkom (1950) and Anolik et al.(1969) the truncation mentioned above limits our understanding. Recently P. Bretagnon (1974) studied the long periodic variations of the planetary system in introducing long periodic terms of fourth order in eccentricities and inclinations. He was then taking into account the short periodic terms in providing the perturbations of the first order with regard to the masses. Another approach is made by Cohen et al.(1973) by a numerical integration in using rectangular coordinates over the period of one million years. This method mixes up all the different influences on the periodic perturbations, e.g. the role of the near resonances cannot precisely be determined. In the diagrams represented by these authors one can very well separate some main perturbations (e.g. the multiples of the great inequality Jupiter-Saturn) but no analysis of the periods mentioned above is made.

Therefore we try to combine an analytical method - which allows us to introduce any desired near resonance term - with a numerical integration by using the special advantage of the high degree of accuracy of the last mentioned. The advantage of our method will be a triple one:

1st we avoid the truncation due to the low powers in the inclinations and eccentricities
2nd we are not restricted to a low order of masses
3rd the single influence of a group of near resonances can be estimated carefully.

For this purpose we have chosen the system of the variables of Lagrange. In a first part the Lagrange Equations are formed in a special closed manner separating the terms dependant on the inverse distance $1/\Delta$ planet - perturbing planet from the other one, because the most complicated development arises from evaluating the quantity Δ^{-s}.

σ being the vector of the osculating elements the equations can be written as follows

$$\frac{d\sigma}{dt} = \frac{F(\sigma,\sigma')}{\Delta^3} + \frac{G(\sigma,\sigma')}{r'^3} \tag{1}$$

As we can see later the short periodic terms appear through the quantities of the form $(\frac{r}{a})^n \tau^m$, r being the radius vector, a the semimajor axes, $\tau=\exp\sqrt{-1}\,(v+\tilde{\omega})$, $v+\tilde{\omega}$ being the true longitude. The most heavy work is done on finding a development of the inverse distance to be able to evaluate the expressions mentioned above. Starting from the development

$$F(\frac{r}{a},\frac{a'}{r'},\tau,\tau',h,u,\ (\alpha^2)) \tag{2}$$

with α=a'/a, h=h(e,e',r/a,a'/r') and u=u(i,i',τ,τ') we used the convenient solution of Abu el Ata and Chapront (1974) where the $\Psi(\alpha^2)$ have been calculated very precisely even for larger α. One could use the method of the series expansion of Δ^{-s} in Legendre polynomials having the disadvantage for the planetary problem to be not suitable for large values of α .

Instead of the quantities h and u which are of the order of e and i^2 we transformed the development F into the new one G. The establishement of the form

$$G(\Psi(\alpha^2),\Gamma_1(i,i'),\Gamma_2(i,i'),(r/a)^n\tau^m,(a'/r')^{n'}\tau'^{m'}) \tag{3}$$

put in evidence the short periodic terms through the quantities (a/r)τ . These short periodic terms can be computed with the aid of the Hansen coefficients with a great accuracy without truncation in the form introduced by Brumberg (1973). Note that the Γ_1 and Γ_2 are of the order of thesquare of the inclinations. The calculated expression for the inverse distance is now combined with the equation (1) and enables to compute the long periods by a numerical integration of the system of the Lagrange Equations for the osculating elements used in taking the mean values of equation (1).

2. THE LAGRANGE EQUATIONS

We used the variables of Lagrange for the perturbing function in the following form

$$\begin{aligned} &a \text{ the semimajor axes}\\ &\lambda \text{ the mean longitude}\\ &z = e \exp \sqrt{-1}\,\tilde{\omega}\\ &\xi = \sin(\tfrac{i}{2}) \exp \sqrt{-1}\,\Omega \end{aligned} \tag{4}$$

introduced by Chapront et al. (1975). We were able to find an expression where the inverse distance is isolated by starting from the equations givenin the paper mentioned above

$$\begin{aligned} \frac{da}{dt} &= \frac{2a^2}{\varphi}\, \mathrm{Im}\bar{z}(\rho R_1+\pi R_2)+\mathrm{Im}(\bar{\tau}\pi)R_2\\ \frac{d\lambda}{dt} &= n-\mathrm{Re}\left\{(2\bar{\rho}+\alpha\varphi\Psi\bar{z})(\rho R_1+\pi R_2)\right\}+\left[\frac{\psi}{\varphi}\mathrm{Im}(\bar{z}\rho)\mathrm{Im}(\bar{\tau}\pi)+\frac{1}{\varphi}\,\sigma_\lambda\right]R_2\\ \frac{dz}{dt} &= -ja\varphi(\rho R_1+\pi R_2)+\left[\frac{1}{\varphi}(zr+\rho)\mathrm{Im}(\bar{\tau}\pi)+jz\frac{1}{\varphi}\,\sigma_\lambda\right]R_2\\ \frac{d\xi}{dt} &= \left[\rho-\xi\mathrm{Re}(\bar{\xi}\rho)\right]\frac{1}{\varphi}\,\sigma_\xi R_2 \end{aligned} \tag{5}$$

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All the quantities like ς, π, φ, ψ, δ_λ and δ_ξ are functions of the Lagrange variables used; j signifies $\sqrt{-1}$. The R_1 and R_2 are the terms of the disturbing function where one has to distinguish on one hand for an inner planet perturbed by an outer one

$$R_1 = -\frac{\mu}{\Delta^3} \quad ; \quad R_2 = \mu\left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right) \quad ; \quad \mu = \frac{nam'}{1+m} \tag{6}$$

and on the other hand for an outer planet perturbed from an inner one (see the paper by Chapront et al. (1975)).

$$R_2 = \mu'\left(\frac{1}{\Delta^3} - \frac{1}{r^3}\right) \quad ; \quad \mu' = \frac{n'a'm}{1+m'} \quad ; \quad R_1 = -\frac{u'}{\Delta^3} \tag{7}$$

The separation of the terms dependant on the inverse distance leads to

$$\begin{aligned} \frac{da}{dt} &= \mu\left(\frac{A}{\Delta^3} - \frac{A'}{r'^3}\right) \\ \frac{d\lambda}{dt} &= n-\mu\left(\frac{\Lambda}{\Delta^3} - \frac{\Lambda'}{r'^3}\right) \\ \frac{dz}{dt} &= \mu\left(\frac{Z}{\Delta^3} - \frac{Z'}{r'^3}\right) \\ \frac{d\xi}{dt} &= \mu Z''\left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right) \end{aligned} \tag{8}$$

The according functions A, A', Λ, Λ', Z, Z' and Z'' are

$$\begin{aligned} A &= \frac{2a^2}{\varphi^2} \mathrm{Im}(\bar{c}\pi) - \mathrm{Im}\,\bar{z}l \\ A' &= \frac{2a^2}{\varphi} \mathrm{Im}(\bar{z}\pi) - \mathrm{Im}(\bar{c}\pi) \\ \Lambda &= \mathrm{Re}(ml) + p \\ \Lambda' &= \mathrm{Re}(m\pi) - p \\ Z &= ja\varphi l + q \\ Z' &= ja\varphi\pi - q \\ Z'' &= [\varsigma - \xi\,\mathrm{Re}(\bar{\xi}\varsigma)\ \frac{1}{\varphi}\ \delta_\xi \end{aligned} \tag{9}$$

with $l = \pi - \varsigma$; $m = 2\bar{\varsigma} + a\varphi\psi\bar{z}$; $p = \frac{\psi}{\varphi}\mathrm{Im}(\bar{z}\varsigma)\mathrm{Im}(\bar{c}\pi) + \frac{1}{\varphi}\ \delta_\lambda$;

$$q = \frac{1}{\varphi}(zr+\varsigma)\,\mathrm{Im}(\bar{c}\pi) + jz\frac{1}{\varphi}\ \delta_\lambda \quad .$$

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3. THE INVERSE DISTANCE

For the analytical development of the inverse distance we took the original form of Abu el Ata and Chapront (1975)

$$\Delta^{-\sigma} = r'^{-s}(2-\delta_j^o)\,\mathrm{Re} \sum_{j=o}^{j_m} \sum_{n=o}^{E(\omega/2)} \sum_{m=o}^{\omega-2n} \frac{(\sigma/2)_n}{(1)_n} \alpha^n \varphi^{(j)}_{n+\frac{\sigma}{2},m} (\alpha^2) * \tag{10}$$

$$* \left(\frac{r}{a}\tau \frac{a'}{r'} \bar{\tau}'\right)^j h^m \left(\frac{ra'}{ar'}\, u\right)^n$$

The development above should be transformed to a new expression which separates the $(r/a)^n \tau^m$. It could be possible to use the method of development of Δ^{-1} in Legendre polynomials of the form

$$\Delta^{-1} = \frac{a}{a'} \sum \frac{r^k}{r'^{k+1}} P_k(\cos H) \tag{11}$$

where r and r' are the radius vectors (prime signifies the perturbing planet) and H is the angular distance between the two planets. This is very convenient for the satellite problem where α is rather small.

Our purpose was to have the advantage of the Laplace development in which the functions $\varphi(\alpha^2)$ can be computed with a high degree of accuracy. Therefore we took directly

$$\Delta^{-\sigma} = r'^{-\sigma}(2-\delta_j^o)\,\mathrm{Re} \sum_{j=0}^{j_M} \sum_{n=o}^{E(\omega/2)} \sum_{m=o}^{\omega-2n} \psi^{(j)}_{n,m}(\alpha^2) P^{j+n} \tau^j \bar{\tau}'^j h^m u^n \tag{12}$$

with $P = \frac{ra'}{ar'}$ and $\psi^{(j)}_{n,m} = \frac{(\sigma/2)_n}{(1)_n} \alpha^n \varphi^{(j)}_{n-\frac{\sigma}{2},m}$

Taking the binomial development for $h = P^2 - 1$

$$h^n = \sum_{v=o}^{n} (-1)^v \binom{n}{v} P^{2(n-v)}$$

leads to the form

$$\Delta^{-\sigma} = r'^{-\sigma}(2-\delta_j^o)\,\mathrm{Re} \sum_{j=o}^{j_M} \sum_{n=o}^{E(\omega/2)} F_n^{(j)} u^n \tag{13}$$

with $F_n^{(j)} = \sum_{w=o}^{\omega-2n} \sum_{v=w}^{\omega-2n} (-1)^{v-w} \binom{v}{v-w} \psi^{(j)}_{n,v} P^{2w+n}$

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As an illustration one can realize for the seventh order of the eccentricities (ω=7) for the coefficients of u^2 (fourth order of i or i')

$$F_2^{(j)} = P^2(\Psi_{2,0}-\Psi_{2,1}+\Psi_{2,2}-\Psi_{2,3}) + P^4(\Psi_{2,1}-2\Psi_{2,2}+3\Psi_{2,3}) + $$
$$+ P^6(\Psi_{2,2}-3\Psi_{2,3}) + P^8\Psi_{2,3}$$

This leads to the following development of the inverse radius vector which seems to be convenient for the plane case (i=o, i'=o, u=o, n=o):

$$\Delta^{-\sigma} = r'^{-\sigma}(2-\delta_j^o)\,\mathrm{Re}\sum_{j=o}^{j_M}\sum_{w=o}^{\omega-2n}\sum_{v=w}^{\omega-2n}(-1)^{v-w}\binom{v}{v-w}\Psi_{n,v}^{(j)}P^{2w+n+j}\tau^j\bar{\tau}'^{\,j} \tag{14}$$

For the development in inclinations one has to evaluate

$$u=2\mathrm{Re}(\bar{\tau}(\Gamma_1\tau'+\Gamma_2\bar{\tau}')) \text{ with } \Gamma_1=\Gamma_1(i^2,i'^2);\ \Gamma_2=\Gamma_2(i^2,i'^2)$$

Putting $\beta=\Gamma_1\tau'+\Gamma_2\bar{\tau}'$ we can express u by $u=\bar{\tau}\beta+\tau\bar{\beta}$ where $\bar{\beta}$ signifies the conjugate complex of β. We find for β^n

$$\beta^n = \sum_{p=o}^{n}\binom{n}{p}\Gamma_1^{n-p}\tau'^{\,n-p}\Gamma_2^{p}\bar{\tau}'^{\,p} \tag{15}$$

The power of u can now be developed by

$$u^n = \sum_{r=o}^{n}\sum_{p=o}^{n-r}\sum_{q=o}^{r}V_{r,p,q}\,\tau^{2r-n}\tau'^{\,n+2(q-r-p)} \tag{16}$$

with $V_{r,p,q}=\dfrac{n!}{(n-r-p)!\,p!\,q!\,(r-q)!}\,\Gamma_1^{n-r-p}\Gamma_2^{p}\bar{\Gamma}_1^{r-q}\bar{\Gamma}_2^{q}$

giving the final expression for $\Delta^{-\sigma}$

$$\Delta^{-\sigma}=\left(\frac{a'}{r}\right)^{\sigma}(2-\delta_j^o)\,\mathrm{Re}\sum_{j=o}^{j_M}\sum_{n=o}^{E(\omega/2)}\sum_{w=o}^{\omega-2n}\sum_{v=w}^{\omega-2n}\sum_{r=o}^{n}\sum_{p=o}^{n-r}\sum_{q=o}^{r} \tag{17}$$
$$(-1)^{v-w}\binom{v}{v-w}V_{r,p,q}\Psi_{n,v}^{(j)}\left(\frac{ra'}{ar'}\right)^{2w+n+j}\tau^{j+2r-n}\tau'^{\,-j+n+2(q-r-p)}$$

In those expressions the $(r/a)^n\tau^m$ are developped with Hansen coefficients as follows

$$\left(\frac{r}{a}\right)^n\tau^m = \sum_{q=-oo}^{+oo}X_q^{n,m}(e)\exp\sqrt{-1}\,qM \tag{18}$$

This elegant method for computing them has been elaborated by Brumberg (1967). Although it seems to be a very cumbersome calculation because of the great number of series many coefficients disappear in the practical computation due to specific properties of the Hansen coefficients which are the symmetry

$$X_o^{n,m} = X_o^{n,-m}$$

and the disappearance of terms when $n \in [-2, -|m|-1]$. Note that in the equation (17) it remains the coefficients with index 0 when we take the mean values that is to say

$$\left(\frac{r}{a}\right)^n \tau^m = X_o^{n,m}$$

The next work to be done is the numerical integration of the system of the Lagrange Equations to find the long periods in the variables used. This can be done by integrating only the system of long periodic terms

$$\frac{d\sigma^{(n)}}{dt} = \sum_{k \neq n} G_k(a^{(n)}, a^{(k)}, z^{(n)}, z^{(k)}, \zeta^{(n)}, \zeta^{(k)}) \tag{19}$$

for k perturbing planets; the index (n) signifies the regarded planet. The short periodic terms in λ are absent in taking the mean values of the equation (17) as it is explained above.

4. CONCLUSIONS

The new approach in this paper for constructing the long periodic terms is done in combining the classical method of analytical development with the very useful instrument of a numerical integration. As mentioned above in the classical works one does not know the truncation errors made in restrictions

1) of developments with respect to the small parameters e and i
2) in taking into account only low order of masses.

It should be mentioned that to analyze the effects order by order with respect to the perturbing masses it is possible to integrate numerically the system in the same way as it is done by the analytical methods in an iterative process. The method of using only a numerical integration is mixing up all different influences and does not allow to draw conclusions on the specific effects which play an important role in regard to the long periodic perturbations. We established a strict form of the Lagrange Equations and have the whole advantage of the powerful instrument of variation of constants. If we select some specific arguments to add to the system (8), e.g. the great inequality Jupiter-Saturn and integrate it together we should be able to evaluate rigorously the role of the near resonances which is always a fundamental one. On the other hand the numerical integration permits to integrate the system with the desired precision, withous taking into account truncation problems which are neglectable in the periods regarded, when we can take a suitable step length.

The special form of the equations (17) on regarding the small parameters e and i has allowed to work only on certain problems, but with slight changes it should be possible to establish also formulas for larger eccentricities and inclinations, as it has been done by Kozai (1962). Although the complexity of the Lagrange Equations established in this paper seems to lead to a cumbersome work for integrating the system numerically there is only a small number of quantities like $\Psi(\alpha^2)$, $\Gamma(i,i')$ and $(a/r)^n\tau^m$. These specific quantities can be tabulated in a certain regarded problem once and for all and at least should give a very precise determination of the periodic perturbations of some thousands of years in our planetary system.

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