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On descending cohomology geometrically

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Abstract

In this paper, motivated by a problem posed by Barry Mazur, we show that for smooth projective varieties over the rationals, the odd cohomology groups of degree less than or equal to the dimension can be modeled by the cohomology of an abelian variety, provided the geometric coniveau is maximal. This provides an affirmative answer to Mazur's question for all uni-ruled threefolds, for instance. Concerning cohomology in degree three, we show that the image of the Abel–Jacobi map admits a distinguished model over the rationals.

Introduction

We consider the problem of determining when the cohomology of a smooth projective variety X over a field K can be modeled by an abelian variety. For example, for a prime number ℓ different from the characteristic of K, it is well known that the first ℓ-adic cohomology group $H^1(X_{\overline{K}}, \mathbb{Q}_\ell)$ can be modeled by the Albanese variety $\mathrm{Alb}(X)$ in the sense that it is isomorphic as a $\mathrm{Gal}(K)$-representation to $H^1(\mathrm{Alb}(X)_{\overline{K}}, \mathbb{Q}_\ell)$. Our primary motivation is a special case of a problem posed by Mazur [Maz14] at the birthday conference for Joe Harris in 2011. To fix ideas, suppose that X is a smooth projective variety over a number field K. For each prime number ℓ and non-negative integer n, the cohomology group $H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell)$ admits a continuous action by $\mathrm{Gal}(K)$, and $H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ has weight one. Now further suppose that $h^{p,q} = 0$ for $p + q = 2n+1$, $p, q \neq n$. Thanks to Mazur's 'Newton over Hodge' theorem [Maz72], the twisted Galois representation $H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ is still effective; each eigenvalue of Frobenius is actually an algebraic integer. Thus, $H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ could be the cohomology group of some abelian variety in the sense that there is no obvious obstruction to the existence of an abelian variety J/K such that for all prime numbers ℓ there is an isomorphism of $\mathrm{Gal}(K)$-representations

$$H^1(J_{\overline{K}}, \mathbb{Q}_\ell) \cong H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)). \tag{0.1}$$

Mazur's question is to determine when such an abelian variety J/K actually exists.

Note that an isomorphism as in (0.1) only characterizes the abelian variety J/K up to isogeny. This ambiguity persists even if one works with $\mathbb{Z}_\ell$-cohomology groups. Indeed, there exist elliptic curves E_1 and E_2 over a number field such that, for each ℓ, the Tate modules $T_\ell E_1$ and $T_\ell E_2$

are isomorphic as Galois representations; and yet E_1 and E_2 are not isomorphic, even over $\mathbb{C}$ [Zar08, §12]. Following Mazur, the *phantom isogeny class* for X, in degree n, is the isogeny class of abelian varieties satisfying (0.1). Any member of this isogeny class will be called a *phantom abelian variety* for X. (We make the analogous definition for varieties over an arbitrary base field.)

This type of question has of course been considered in many contexts, and results in the literature provide a great deal of motivation. To begin, from the arithmetic perspective, the theorems of Tate and Honda [Tat66, Tat71, Hon68] imply (under certain further hypotheses; see §1.4) that for a prime $\mathfrak{p}$ of good reduction for X, there exists an abelian variety $J_{\mathfrak{p}}/\kappa(\mathfrak{p})$ such that, if a phantom abelian variety J/K were to exist, then the reduction of J modulo $\mathfrak{p}$ would be isogenous to $J_{\mathfrak{p}}$. Further motivation comes from complex geometry. Under our hypotheses on the Hodge numbers, the intermediate Jacobian $J^{2n+1}(X_{\mathbb{C}}) = H^{n,n+1}(X_{\mathbb{C}})/H^{2n+1}(X_{\mathbb{C}}, \mathbb{Z})$ is equal to the algebraic intermediate Jacobian, and is consequently an abelian variety. Since the dual abelian variety $\widehat{J}^{2n+1}(X_{\mathbb{C}})$ satisfies $H^1(\widehat{J}^{2n+1}(X_{\mathbb{C}}), \mathbb{Z}_\ell) = H^{2n+1}(X_{\mathbb{C}}, \mathbb{Z}_\ell(n))$, if the intermediate Jacobian were isogenous to an abelian variety that descended to K, this would provide a candidate for a phantom abelian variety. As justification for this approach, recall that a theorem of Deligne [Del72] implies that for a complete intersection of odd dimension with Hodge level 1, the intermediate Jacobian descends to K and its dual provides a phantom abelian variety.

In this paper, we establish the existence of phantom abelian varieties for more general smooth projective varieties, but under some further hypotheses on the cohomology. Namely, we make the stronger, yet natural, hypothesis that $H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q})$ has geometric coniveau n, meaning that $H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q})$ is supported on a subvariety of $X_{\mathbb{C}}$ of codimension n. In fact, assuming the Tate conjecture for $\widehat{J} \times_K X$, the geometric coniveau-n part is an *a priori* upper bound on what can be modeled directly with an abelian variety.

THEOREM A (Theorem 2.1). *Suppose that X is a smooth projective variety of pure dimension over a field $K \subseteq \mathbb{C}$. If $H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q})$ has geometric coniveau n, for some integer n with $2n + 1 \leqslant \dim X$, then there exist an abelian variety J over K, with $J_{\mathbb{C}}$ isogenous to the intermediate Jacobian $J^{2n+1}(X_{\mathbb{C}})$, and a correspondence on $\widehat{J} \times_K X$ that induces, for each prime ℓ, an isomorphism of $\mathrm{Gal}(K)$-representations*

$$H^1(\widehat{J}_{\overline{K}}, \mathbb{Q}_\ell) \cong H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)). \tag{0.2}$$

Moreover, its inverse is induced by a correspondence on $X \times_K \widehat{J}$.

We note that the theorem provides an affirmative answer to Mazur's question for all threefolds with CH_0 supported on a surface (e.g., for all uni-ruled threefolds; see Corollary 5.2) and, assuming the generalized Hodge conjecture, for all threefolds with $h^{3,0} = 0$. More generally, Theorem 2.1 provides for all n a phantom for the geometric coniveau part $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ as soon as X satisfies the Lefschetz standard conjecture.

In Theorem A, operating only under the hypothesis of geometric coniveau n, we find a suitable phantom J/K and show that its dual is geometrically isogenous to the intermediate Jacobian $J^{2n+1}(X_{\mathbb{C}})$. This does not imply that $J^{2n+1}(X_{\mathbb{C}})$ itself descends to K. In contrast, in the special case $n = 1$, we can single out an isomorphism class (in fact, a model over K of the dual of the intermediate Jacobian) within the phantom isogeny class. More generally, without any

hypothesis on geometric coniveau, we extend Murre's work [Mur85] on algebraic representatives to the setting of non-algebraically closed fields to show (see Corollary 5.2) the following result.

THEOREM B. *Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$. The abelian variety $J_a^3(X_{\mathbb{C}})$ that is the image of the Abel–Jacobi map $AJ : A^2(X_{\mathbb{C}}) \to J^3(X_{\mathbb{C}})$ has a distinguished model J over K making the Abel–Jacobi map $AJ : A^2(X_{\mathbb{C}}) \twoheadrightarrow J_a^3(X_{\mathbb{C}})$ $\mathrm{Aut}(\mathbb{C}/K)$-equivariant. Moreover, $\hat{J}$ provides a phantom for $\mathrm{N}^1 H^3(X_{\overline{K}}, \mathbb{Q}_\ell(1))$.*

We also establish results regarding specialization of algebraic representatives (§ 6.1), as well as algebraically closed base change of algebraic representatives (Theorem 3.7). As suggested by Lenny Taelman, our results extending Murre's work on algebraic representatives can be interpreted in a functorial framework to provide algebraic representatives over any perfect field. More precisely, let X be a smooth projective variety over a perfect field K. Denote by $\mathrm{A}_\natural^2(X)$ the contravariant functor on the category of smooth integral varieties over K to the category of abelian groups given by families of codimension-two cycles on X whose geometric fibers are algebraically trivial. Then there exist an abelian variety $\mathrm{Ab}^2(X)$ over K and a natural transformation of contravariant functors $\mathrm{A}_\natural^2(X) \to \mathrm{Ab}^2(X)$ that is initial among all natural transformations of contravariant functors $\mathrm{A}_\natural^2(X) \to A$ to an abelian variety A over K. This approach will appear in future work by the authors.

In concomitant work in preparation by the authors, following [Bea77], we provide a geometric construction of the phantom abelian variety when X is an odd-dimensional quadric bundle over a surface over a field of characteristic $\neq 2$. Mazur has also asked for the relationship between phantom abelian varieties and Albaneses of Hilbert and Kontsevich spaces associated to X. In future work we will discuss how this question relates to Theorem B.

Outline. In § 1, we review the notions of geometric, Hodge, and Tate coniveau filtrations and relate the final step of these filtrations on odd-degree cohomology groups with the cohomology of curves, the Abel–Jacobi map, and Mazur's question on phantoms. The case of finite fields is discussed in § 1.4. We prove our results on phantom abelian varieties in two parts that complement each other: Theorem A (Theorem 2.1) gives general criteria for the existence of a phantom, and Theorem B (Theorem 5.1 and Corollary 5.2) shows that the image of the second Abel–Jacobi map descends to a field of definition. The proof of Theorem 2.1 relies essentially on polarizations of Hodge structures, carried out in § 2, while the proof of Theorem 5.1 is based on Murre's theory of algebraic representatives [Mur85], and is carried out in §§ 3–5. In § 3, we review the results of Murre on algebraic representatives for smooth projective varieties defined over algebraically closed fields and prove in Theorem 3.7 that algebraic representatives, if they exist, behave well under base change along algebraically closed fields. In § 4, we extend Murre's theory of algebraic representatives to the setting of Galois field extensions and prove in Theorem 4.4 that the algebraic representative, if it exists, inherits a Galois descent datum. In § 5, we prove Theorem 5.1, providing the connection between algebraic representatives and phantoms. In § 6, we discuss specialization of algebraic representatives and phantom abelian varieties, as well as some results concerning algebraic representatives for higher codimension cycles.

Conventions. A *variety* over a field is a geometrically reduced separated scheme of finite type over that field. A *curve* (respectively *surface*) is a variety of pure dimension 1 (respectively 2). Given a variety X over a field, $\mathrm{CH}^i(X)$ denotes the Chow group of codimension-i cycles modulo rational equivalence, and $\mathrm{A}^i(X) \subseteq \mathrm{CH}^i(X)$ denotes the subgroup of cycles algebraically equivalent to 0. For an abelian variety A over a field K, we denote by $\widehat{A}$ the dual abelian variety $\mathrm{Pic}_{A/K}^0$.

1. Phantoms and coniveau filtrations

The aim of this section is to highlight the relevance of the geometric coniveau filtration in the context of Mazur's question. The main question we will tackle in this work is Question 1.4, where we ask whether the deepest part of the geometric coniveau filtration on odd-degree cohomology groups can be modeled by abelian varieties. To this end we discuss Hodge, Tate, and geometric coniveau filtrations, and their link to the Abel–Jacobi map. One of the key points for later arguments is Proposition 1.3 stating that the deepest part of the geometric coniveau filtration can be realized via the cohomology of a curve.

1.1 Hodge theory and geometric Galois representations

While our motivating problem is posed purely in terms of Galois representations, it has a natural Hodge-theoretic analogue.

Question 1.1 [Voi14, Question 2.43]. Let X be a smooth complex projective variety. Given a weight-i Hodge substructure $L \subseteq H^i(X, \mathbb{Q})$ of Hodge coniveau r, so that

$$L_{\mathbb{C}} = L^{i-r,r} \oplus \cdots \oplus L^{r,i-r}$$

(and $L(r)$ is an effective weight-$(i - 2r)$ Hodge structure), does there exist a smooth complex projective variety Y admitting a Hodge substructure of $L' \subseteq H^{i-2r}(Y, \mathbb{Q})$ with $L' \cong L(r)$?

In the case that $i = 2n+1$ is odd, and L has Hodge coniveau n, the question has an affirmative answer. Indeed, recall that the algebraic intermediate Jacobian, $J_{\mathrm{alg}}^{2n+1}(X) \subseteq J^{2n+1}(X)$, is defined to be the largest complex subtorus with tangent space contained in $H^{n,n+1}(X)$. This is an abelian variety (e.g., [Voi07, p. 304]) and clearly $L(n)$ can be obtained as a Hodge substructure of $H^1(\widehat{J}_{\mathrm{alg}}^{2n+1}, \mathbb{Q})$.

For Mazur's question we are interested in an ℓ-adic analogue of Question 1.1. Now let K be a field, such as a number field, which is finitely generated over the prime field. A $\mathrm{Gal}(K)$-representation V_ℓ over $\mathbb{Q}_\ell$ has weight i if, for each unramified prime v of K (and each embedding $\overline{\mathbb{Q}}_\ell \to \mathbb{C}$), each eigenvalue of Frobenius at v has size $\sqrt{q_v^i}$, where q_v is the cardinality of the residue field at v. We say that V_ℓ is entire if all eigenvalues of Frobenius are algebraic integers. We say that V_ℓ is effective of weight w if V_ℓ and $\mathrm{Hom}(V_\ell, \mathbb{Q}_\ell(-w))$ are entire and of weight w. For a smooth projective variety X over K, by Poincaré duality and hard Lefschetz, we have $\mathrm{Hom}(H^i(X_{\overline{K}}, \mathbb{Q}_\ell), \mathbb{Q}_\ell(-i)) \cong H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$, so that a geometric representation $H^i(X_{\overline{K}}, \mathbb{Q}_\ell(\nu))$ is effective (of weight $i - 2\nu$) if and only if it is entire (e.g., [Jan94, p. 269]).

Question 1.2. Let X be a smooth projective variety over a field K that is finitely generated over the prime field. Given a $\mathrm{Gal}(K)$-subrepresentation $L_\ell \subseteq H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ with the property that $L_\ell(r)$ is effective (of weight $i - 2r$), does there exist a smooth projective variety Y over K admitting a $\mathrm{Gal}(K)$-subrepresentation $L'_\ell \subseteq H^{i-2r}(Y_{\overline{K}}, \mathbb{Q}_\ell)$ with $L'_\ell \cong L_\ell(r)$?

Unlike the case of Hodge structures, even when $i = 2n + 1$ is odd, and $L_\ell(n)$ is effective of weight 1, it is not known exactly when Question 1.2 will have an affirmative answer. Theorem A provides an affirmative answer under further hypotheses on the geometric coniveau; we review coniveau in the sections below. (See § 1.4 for a discussion of the connection between Question 1.2 over finite fields and the theory of Honda–Tate; see [ACV16] for a discussion of the extent to which Y may be taken to be geometrically integral.)

1.2 Coniveau filtrations and the generalized Hodge and Tate conjectures

Questions 1.1 and 1.2 are related to the generalized Hodge and Tate conjectures [Gro68]. Our treatment here follows [Jan94, §3]. Let X be a smooth projective variety over a field K. The *geometric coniveau filtration* $\mathrm{N}^\nu H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ is defined by

$$\mathrm{N}^\nu H^i(X_{\overline{K}}, \mathbb{Q}_\ell) = \sum_{\substack{Z \subseteq X \\ \text{closed, codim } \geqslant \nu}} \mathrm{Ker}(H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \to H^i(X_{\overline{K}} \backslash Z_{\overline{K}}, \mathbb{Q}_\ell)). \tag{1.1}$$

By resolution of singularities and by Deligne's theory of mixed Hodge structures when char $K = 0$, or by de Jong's theory of alterations and by the Weil conjectures when K is a finite field, the geometric coniveau filtration can also be characterized by the following formula (see [Ill06, §4.4(d)]):

$$\mathrm{N}^\nu H^i(X_{\overline{K}}, \mathbb{Q}_\ell) = \sum_{\substack{f : Y \to X \\ \dim Y = \dim X - \nu \\ Y \text{ smooth proj}}} \mathrm{Im}(f_* : H^{i-2\nu}(Y_{\overline{K}}, \mathbb{Q}_\ell(-\nu)) \to H^i(X_{\overline{K}}, \mathbb{Q}_\ell)). \tag{1.2}$$

Now suppose that K is a subfield of $\mathbb{C}$. The *geometric coniveau filtration on Betti cohomology* $\mathrm{N}^\nu H^i(X_{\mathbb{C}}, \mathbb{Q})$ is defined in the same way. Note that usually a class $\alpha \in H^i(X_{\mathbb{C}}, \mathbb{Q})$ is said to have geometric coniveau ν if it vanishes when restricted to the complement of a closed *complex* subvariety of $X_{\mathbb{C}}$ of pure codimension ν. However, using a standard argument involving spreading out, specialization, and then Galois averaging, it suffices as indicated above to consider projective varieties Z, and smooth projective varieties Y, of pure dimension $\dim X - \nu$ defined over K.

By virtue of the fact that the canonical comparison isomorphisms between Betti, de Rham, and étale cohomologies are compatible with Gysin maps, and even cycle maps [Del82, §1], there are canonical isomorphisms

$$(\mathrm{N}^\nu(H^i(X_{\mathbb{C}}, \mathbb{Q}))) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \mathrm{N}^\nu H^i(X_{\overline{K}}, \mathbb{Q}_\ell), \tag{1.3}$$

which justify our apparent abuse of notation. The generalized Hodge and Tate conjectures relate these geometric coniveau filtrations to two other filtrations.

Call a $\mathbb{Q}$-Hodge structure V effective if $V^{p,q} = 0$ unless $p, q \geqslant 0$. The *Hodge coniveau filtration* N_H^ν of $H^i(X_{\mathbb{C}}, \mathbb{Q})$ has term $\mathrm{N}_H^\nu H^i(X_{\mathbb{C}}, \mathbb{Q})$ defined to be the span of all Hodge substructures $V \subseteq H^i(X_{\mathbb{C}}, \mathbb{Q})$ such that $V(\nu)$ is effective, i.e., such that $V^{p,q} = 0$ unless $p, q \geqslant \nu$. The Gysin map provides a natural inclusion

$$\mathrm{N}^\nu H^i(X_{\mathbb{C}}, \mathbb{Q}) \subseteq \mathrm{N}_H^\nu H^i(X_{\mathbb{C}}, \mathbb{Q}); \tag{1.4}$$

the *generalized Hodge conjecture* (for X, in degree i) asserts equality in (1.4).

Similarly, if K is finitely generated over the prime field, define a filtration $\mathrm{N}_\ell^\nu H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ as the span of all sub-$\mathrm{Gal}(K)$-representations $V_\ell \subset H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ such that $V_\ell(\nu)$ is effective (of weight $i - 2\nu$). Again, the Gysin map (this time, for étale cohomology) provides a natural inclusion

$$\mathrm{N}^\nu H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \subseteq \mathrm{N}_\ell^\nu H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \tag{1.5}$$

and the *generalized Tate conjecture* (for X, in degree i) asserts equality in (1.5). Sometimes, we will abuse notation slightly and denote the rth twist of step ν in the geometric coniveau filtration by

$$\mathrm{N}^\nu H^i(X_{\overline{K}}, \mathbb{Q}_\ell(r)) := (\mathrm{N}^\nu H^i(X_{\overline{K}}, \mathbb{Q}_\ell)) \otimes \mathbb{Q}_\ell(r);$$

we employ similar notation for the other filtrations (also on Betti cohomology).

Important to us will be the following characterization of the final step of the geometric coniveau filtration for odd-degree cohomology groups.

Proposition 1.3 [Via13, §1.2.3]. *Suppose that X is a smooth projective variety over a field K, which is either of characteristic zero or finite, and let $\ell \neq \operatorname{char}(K)$ be a prime. Let n be a non-negative integer. Then there exist a smooth projective curve C over K and a correspondence $\gamma \in \operatorname{CH}^{n+1}(C \times_K X) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that the induced morphism of $\operatorname{Gal}(K)$-representations*

$$\gamma_* : H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \to H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$$

has image $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$. If $K \subseteq \mathbb{C}$, then the morphism of Hodge structures

$$\gamma_* : H^1(C_{\mathbb{C}}, \mathbb{Q}) \to H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$$

has image $\mathrm{N}^n H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$. Moreover, $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ is a semisimple representation of $\operatorname{Gal}(K)$.

Proof. This is proved in [Via13] in the case of Betti cohomology (see the formula $\widetilde{\mathrm{N}}^{\lfloor i/2 \rfloor} H_i(X) = \mathrm{N}^{\lfloor i/2 \rfloor} H_i(X)$ in §1.2.3 therein). The proof adapts to our more general setting: up to working component-wise, we may and do assume that X is connected, say of dimension d. Since $H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ is a finite-dimensional $\mathbb{Q}_\ell$-vector space, from the characterization of coniveau given in (1.2), there exist a smooth projective variety Y of dimension $d_Y = d - n$ over K and a K-morphism $f : Y \to X$ such that

$$\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)) = \operatorname{Im}(f_* : H^1(Y_{\overline{K}}, \mathbb{Q}_\ell) \to H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))).$$

By Bertini [Poo04], let $\iota : C \hookrightarrow Y$ be a one-dimensional smooth linear section of Y. The hard Lefschetz theorem [Del80, Theorem 4.1.1] states that intersecting with C yields an isomorphism

$$\iota_* \iota^* : H^1(Y_{\overline{K}}, \mathbb{Q}_\ell) \hookrightarrow H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \twoheadrightarrow H^{2d_Y - 1}(Y_{\overline{K}}, \mathbb{Q}_\ell(d_Y - 1)).$$

The Lefschetz standard conjecture is known for ℓ-adic cohomology and for Betti cohomology in degree $\leqslant 1$ (see [Kle68, Theorem 2A9(5)]), meaning in our case that the map $(\iota_* \iota^*)^{-1}$ is induced by a correspondence $\Lambda \in \operatorname{CH}^1(Y \times_K Y)_{\mathbb{Q}}$. Therefore, the composition

$$H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{\iota_*} H^{2d_Y - 1}(Y_{\overline{K}}, \mathbb{Q}_\ell(d_Y - 1)) \xrightarrow[\cong]{\Lambda_*} H^1(Y_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{f_*} H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)), \quad (1.6)$$

which has image $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$, is induced by the required correspondence.

That $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ is a semisimple representation of $\operatorname{Gal}(K)$ follows immediately from the fact that $H^1(C_{\overline{K}}, \mathbb{Q}_\ell)$ is semisimple if K is a field which is finitely generated (over $\mathbb{Q}$ [Fal83, FWGSS92] or a finite field [Tat66, Zar75]).

When $K \subseteq \mathbb{C}$, the comparison isomorphisms (1.3) establish that the image of the induced morphism of Hodge structures $\gamma_* : H^1(C_{\mathbb{C}}, \mathbb{Q}) \to H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$ is $\mathrm{N}^n H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$. $\square$

1.3 Coniveau filtrations and the Abel–Jacobi map

In this section we review the connection between the coniveau filtrations and the Abel–Jacobi map. Recall that, for a smooth complex projective variety X, the intermediate Jacobian is a complex torus defined by

$$J^{2n+1}(X) := F^{n+1} H^{2n+1}(X, \mathbb{C}) \backslash H^{2n+1}(X, \mathbb{C}) / H^{2n+1}(X, \mathbb{Z})$$

and that there is an *Abel–Jacobi map*

$$\operatorname{Ker}(\operatorname{CH}^n(X) \to H^{2n}(X, \mathbb{Z})) \to J^{2n+1}(X_{\mathbb{C}}).$$

By definition, the algebraic intermediate Jacobian $J_{\mathrm{alg}}^{2n+1}(X)$ is the largest subtorus of $J^{2n+1}(X)$ that is projective; it determines the nth piece of the Hodge coniveau filtration:

$$\mathrm{N}_H^n \, H^{2n+1}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = T_0 J_{\mathrm{alg}}^{2n+1}(X) \oplus \overline{T_0 J_{\mathrm{alg}}^{2n+1}(X)},$$
$$\mathrm{N}_H^n \, H^{2n+1}(X, \mathbb{Q}(n)) = H^1(\widehat{J}_{\mathrm{alg}}^{2n+1}(X), \mathbb{Q}).$$

On the other hand, it is standard [Voi07, Theorem 12.17] that the nth piece of the geometric coniveau filtration $\mathrm{N}^n \, H^{2n+1}(X, \mathbb{Q})$ is determined by the image of the Abel–Jacobi map. More precisely, the image $J_a^{2n+1}(X)$ of the Abel–Jacobi map $AJ : \mathrm{A}^{n+1}(X) \to J^{2n+1}(X)$ has tangent space $T_0 J_a^{2n+1}(X) = H^{n,n+1}(X) \cap \mathrm{N}^n \, H^{2n+1}(X, \mathbb{C})$, and

$$\mathrm{N}^n \, H^{2n+1}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = T_0 J_a^{2n+1}(X) \oplus \overline{T_0 J_a^{2n+1}(X)},$$
$$\mathrm{N}^n \, H^{2n+1}(X, \mathbb{Q}(n)) = H^1(\widehat{J}_a^{2n+1}(X), \mathbb{Q}).$$

Thus, the Abel–Jacobi map restricted to algebraically trivial cycles

$$AJ : \mathrm{A}^{n+1}(X) \to J^{2n+1}(X)$$

is surjective if and only if $H^{2n+1}(X, \mathbb{Q})$ has geometric coniveau n.

In light of this discussion, the first goal of this paper is to consider analogous questions for geometric Galois representations.

Question 1.4. Given a smooth projective variety X over a field K finitely generated over the prime field, do there exist abelian varieties $\tilde{J}$ and J defined over K such that for each prime ℓ,

$$\mathrm{N}_\ell^n \, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)) \cong H^1(\tilde{J}_{\overline{K}}, \mathbb{Q}_\ell),$$
$$\mathrm{N}^n \, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)) \cong H^1(J_{\overline{K}}, \mathbb{Q}_\ell)$$

as $\mathrm{Gal}(K)$-modules?

The abelian variety $\tilde{J}/K$ would provide an answer to the special case of Question 1.2 that we are considering in this paper, and consequently to Mazur's question (i.e., the case where $\mathrm{N}_\ell^n \, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)) = H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$). Note that if $\tilde{J}$ exists, then a direct consequence of the Tate conjecture for $X \times_K \tilde{J}$ would be that N_ℓ^n and N^n agree on $H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ (see also [Voi14, p. 34] for the corresponding Hodge-theoretic statement). Our aim in this paper is to construct the abelian variety J/K, under some further hypotheses.

1.4 Mazur's question over finite fields

Honda–Tate theory suggests an approach to Mazur's question over finite fields. Consider a smooth projective variety $X/\mathbb{F}_q$ such that $h^{2n+1-j,j}(X) = 0$ unless $j \in \{n, n+1\}$. By Mazur's 'Newton over Hodge' theorem [Maz72], each eigenvalue of Frobenius, acting on $H^{2n+1}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$, is divisible by q^n, and thus $H^{2n+1}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(n))$ is an effective $\mathrm{Gal}(\mathbb{F}_q)$-representation of weight one. It is then natural to ask if there is an abelian variety $J/\mathbb{F}_q$ such that $H^1(J_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong H^{2n+1}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(n))$ as Galois representations. (More generally, of course, one has Question 1.2.) Honda–Tate theory asserts a bijection

$$\{\text{conjugacy classes of } q\text{-Weil numbers}\} \xrightarrow{\ \sim\ } \{\text{isogeny classes of simple abelian varieties}/\mathbb{F}_q\},$$

which we crudely denote $\varpi \mapsto A_\varpi$. The Galois representation attached to an abelian variety over a finite field is semisimple [Tat66], and one immediately deduces from the above bijection

that if V_ℓ is a semisimple effective weight-one $\mathbb{Q}_\ell$-representation of $\mathrm{Gal}(\mathbb{F}_q)$, then there exist an abelian variety $A/\mathbb{F}_q$ and an inclusion of $\mathrm{Gal}(\mathbb{F}_q)$-representations

$$V_\ell \hookrightarrow H^1(A_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell). \tag{1.7}$$

By starting with a Weil number ϖ for which $[\mathbb{Q}(\varpi) : \mathbb{Q}] < 2 \dim A_\varpi$ (see, e.g., [Tat71, p. 98] for examples), it is easy to construct semisimple representations V_ℓ for which (1.7) is never an isomorphism. Nonetheless, we have the following result.

LEMMA 1.5. *Let* $X/\mathbb{F}_q$ *be a smooth projective variety. Suppose that* X *is of geometric coniveau* n *in degree* $2n + 1$.

(a) *There is an abelian variety* $A/\mathbb{F}_q$ *such that, for all* $\ell \neq \mathrm{char}(\mathbb{F}_q)$, *there is an inclusion of Galois representations*

$$H^{2n+1}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(n)) \hookrightarrow H^1(A_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell). \tag{1.8}$$

(b) *If* X *is ordinary (the Newton and Hodge polygons in degree* $2n + 1$ *coincide), or if the Tate conjecture holds for* $X \times_{\mathbb{F}_q} X$, *then one may choose* A *so that* (1.8) *is an isomorphism.*

Proof. Part (a) follows immediately from Honda–Tate theory and from the semisimplicity of $H^{2n+1}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(n))$ (see Proposition 1.3).

For part (b), first suppose that X is ordinary in degree $2n + 1$. It suffices to verify that for each Weil number ϖ occurring as an eigenvalue of Frobenius in $H^{2n+1}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(n))$, one has $[\mathbb{Q}(\varpi) : \mathbb{Q}] = 2 \dim A_\varpi$. By ordinarity, all slopes of the Newton polygon of $H^{2n+1}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(n))$ are 0 and 1. For each ϖ, the calculation of $\mathrm{End}(A_\varpi)$ in [Tat71, Theorem 1] implies that $[\mathbb{Q}(\varpi) : \mathbb{Q}] = 2 \dim A_\varpi$.

Second, suppose instead that the Tate conjecture holds for $X \times_{\mathbb{F}_q} X$. Then one can calculate [Mil94, Propositions 2.4 and 2.8] the dimension of (the motive corresponding to) $H^{2n+1}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(n))$. The calculation of this dimension, which relies only on the Weil numbers, is the same expression used to calculate the dimension of an abelian variety with these Weil numbers. In particular, (1.8) is an isomorphism. $\square$

Remark 1.6. In a similar spirit (but necessarily more involved fashion), Volkov [Vol05] has characterized the potentially crystalline p-adic representations of $\mathrm{Gal}(\mathbb{Q}_p)$ which arise as summands of the p-adic Tate module of an abelian variety over $\mathbb{Q}_p$. As with finite fields, this lets one conclude that if $X/\mathbb{Q}_p$ is a smooth projective threefold with $h^{3,0}(X) = 0$, and if X acquires good reduction over a tamely ramified extension of $\mathbb{Q}_p$, then $H^3(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p(1))$ is isomorphic to a subrepresentation of $H^1(A_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ for some abelian variety $A/\mathbb{Q}_p$.

2. Phantoms via correspondences and polarizations

In this section we prove the following theorem, which implies both the first part of Theorem A and a partial, affirmative answer to Question 1.4. We will write $\mathrm{CH}^*(-)_\mathbb{Q}$ for $\mathrm{CH}^*(-) \otimes_\mathbb{Z} \mathbb{Q}$.

THEOREM 2.1. *Suppose that* X *is a smooth projective variety of pure dimension* d *over a field* $K \subseteq \mathbb{C}$. *Let* n *be a non-negative integer. Then there exist a smooth (possibly reducible) projective*

curve C over K, and a correspondence $\gamma \in \mathrm{CH}^{n+1}(C \times_K X)_{\mathbb{Q}}$, such that for all prime numbers ℓ the induced map

$$\gamma_* : H^1((J_{C/K})_{\overline{K}}, \mathbb{Q}_\ell) \to H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$$

has image $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$. Assume further any of the following:

(a) $2n + 1 \leqslant 3$;

(b) X satisfies the Lefschetz standard conjecture;

(c) $2n+1 \leqslant d$ and $H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$ is of geometric niveau $n-1$, in the sense of [Via13], i.e., there exist a (possibly reducible) smooth projective threefold Y over K and a correspondence $\Gamma \in \mathrm{CH}^{n+2}(Y \times_K X)_{\mathbb{Q}}$ such that $\Gamma_* : H^3(Y_{\mathbb{C}}, \mathbb{Q}(1)) \to H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$ is surjective;

(d) $2n + 1 \leqslant d$ and $H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$ has geometric coniveau n.

Then there exists a subabelian variety $J \stackrel{\iota}{\hookrightarrow} J_{C/K}$ defined over K such that for all prime numbers ℓ, the correspondence γ composed with the graph of the inclusion Γ_ι induces a split inclusion of $\mathrm{Gal}(K)$-representations

$$H^1(J_{\overline{K}}, \mathbb{Q}_\ell) \overset{\gamma_* \circ \Gamma_{\iota*}}{\lhook\joinrel\longrightarrow} H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)), \tag{2.1}$$

with the splitting induced by a correspondence over K and with image $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$. In particular, if $2n + 1 \leqslant d$ and if $H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$ is of geometric coniveau n, then (2.1) is an isomorphism.

Remark 2.2. Let X/K be a smooth integral projective variety. Let L/K be a finite extension such that X_L is a disjoint union of geometrically irreducible components, and $\mathrm{Gal}(L/K)$ acts transitively on these components. If $\mathrm{N}^n H^{2n+1}(-, \mathbb{Q}_\ell(n))$ of some component of X_L admits a phantom abelian variety A over L, then $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$ admits the Weil restriction $\mathbf{R}_{L/K}(A)$ of A over K as a phantom, since $H^1(\mathbf{R}_{L/K}(A)_{\overline{K}}, \mathbb{Q}_\ell) \cong \mathrm{Ind}_{\mathrm{Gal}(L)}^{\mathrm{Gal}(K)} H^1(A_{\overline{K}}, \mathbb{Q}_\ell)$.

Proof of Theorem 2.1. Proposition 1.3 provides a smooth projective curve C/K and a correspondence $\gamma \in \mathrm{CH}^{n+1}(C \times_K X)_{\mathbb{Q}}$ such that the induced morphism of Hodge structures

$$\gamma_* : H^1(C_{\mathbb{C}}, \mathbb{Q}) \to H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$$

has image $\mathrm{N}^n H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$. This establishes the first claim of Theorem 2.1.

In order to prove the remaining claims of Theorem 2.1, we would like to find an abelian subvariety $J/K \hookrightarrow J_{C/K}$ so that the composition

$$H^1(J_{\overline{K}}, \mathbb{Q}_\ell) \lhook\joinrel\longrightarrow H^1((J_{C/K})_{\overline{K}}, \mathbb{Q}_\ell) \overset{\gamma_*}{\twoheadrightarrow} \mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$$

is an isomorphism. At the level of rational Hodge structures and complex abelian varieties, this is elementary; the issue is to find a suitable abelian variety defined over K. Rather than trying to directly descend an abelian variety from $\mathbb{C}$ to K, our strategy for obtaining J/K is motivic in nature and consists in finding a correspondence $\pi \in \mathrm{CH}^1(C \times_K C)_{\mathbb{Q}}$ which acts idempotently on $H^1(C_{\mathbb{C}}, \mathbb{Q})$ such that $\pi_* H^1(C_{\mathbb{C}}, \mathbb{Q})$ is mapped via γ_* isomorphically onto $\mathrm{N}^n H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$. Such a correspondence π defines a motive (C, π), which determines an abelian variety J/K; more concretely, the correspondence π induces a K-morphism $\Pi : J_{C/K} \to J_{C/K}$, and the image is the desired abelian variety J/K with $H^1(J_{\overline{K}}, \mathbb{Q}_\ell)$ identified with $\pi_* H^1(C_{\overline{K}}, \mathbb{Q}_\ell)$ as $\mathrm{Gal}(K)$-representations, by comparison isomorphisms.

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The approach we take for finding this idempotent is motivated by the following well-known, elementary lemma on polarized Hodge structures; see [Via13, Lemma 1.6] and [Voi04, Lemma 5]. In the notation of Proposition 1.3, we will apply the lemma with $H = H^1(C_{\mathbb{C}}, \mathbb{Q})$, $H' = H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q})$, and $\gamma = \gamma_*$.

LEMMA 2.3. *Let (H, Q) and (H', Q') be polarized rational Hodge structures of weight i and let $\gamma : H \to H'$ be a morphism of Hodge structures. Then*

$$H = \mathrm{Im}(\gamma') \oplus^{\perp_Q} \mathrm{Ker}(\gamma) \quad and \quad \mathrm{Im}(\gamma') = \mathrm{Im}(\gamma' \circ \gamma),$$

where γ' is the transpose of γ with respect to the polarizations; i.e., $\gamma' = Q \circ \gamma^{\vee} \circ Q'$ with $\gamma^{\vee}$ the dual of γ. □

Going forward, we fix the standard principal polarization Q on the rational Hodge structure $H^1(C_{\mathbb{C}}, \mathbb{Q})$. Recall that this is induced by the cup product, and that the identification $H^1(C_{\mathbb{C}}, \mathbb{Q}) = H^1(C_{\mathbb{C}}, \mathbb{Q})^{\vee} \otimes \mathbb{Q}(-1)$ via Q is given by the action of the diagonal $\Delta_C \in \mathrm{CH}^1(C \times C)_{\mathbb{Q}}$ on $H^1(C_{\mathbb{C}}, \mathbb{Q})$. Recall also that the dual morphism

$$(\gamma_*)^{\vee} : H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))^{\vee} \to H^1(C_{\mathbb{C}}, \mathbb{Q})^{\vee}$$

is identified via Poincaré duality with the morphism of Hodge structures

$$^t\gamma_* = \gamma^* : H^{2d-2n-1}(X_{\mathbb{C}}, \mathbb{Q}(d-n)) \to H^1(C_{\mathbb{C}}, \mathbb{Q}(1)),$$

where $^t\gamma$ is the transpose of γ in $\mathrm{CH}^{n+1}(X \times_K C)$. We now show that Theorem 2.1 will follow if we can endow $H' = H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$ with a polarization Q' that is induced by a correspondence defined over K.

LEMMA 2.4. *Let X, C, and γ be as in Proposition 1.3. Let Q be the standard polarization on $H^1(C_{\mathbb{C}}, \mathbb{Q})$. If there are a polarization Q' of $H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$ and a correspondence $\Theta \in \mathrm{CH}^{2d-2n-1}(X \times_K X)_{\mathbb{Q}}$ making the following diagram commute:*

$$
\begin{array}{ccccccc}
H^{2n+1}(X_{\mathbb{C}})(n) & \xrightarrow{Q'} & H^{2n+1}(X_{\mathbb{C}})(n)^{\vee}(-1) & \xrightarrow{(\gamma_*)^{\vee}} & H^1(C_{\mathbb{C}})^{\vee}(-1) & \xrightarrow{Q} & H^1(C_{\mathbb{C}}) \xrightarrow{\gamma_*} H^{2n+1}(X_{\mathbb{C}})(n) \\
\| & & \uparrow{PD} & & \uparrow{PD} & & \\
H^{2n+1}(X_{\mathbb{C}})(n) & \xrightarrow{\Theta_*} & H^{2d-2n-1}(X_{\mathbb{C}})(d-n) & \xrightarrow{^t\gamma_*} & H^1(C_{\mathbb{C}}) & &
\end{array}
\tag{2.2}
$$

then there exist an abelian subvariety $J \subseteq J_{C/K}$ defined over K, and a correspondence defined over K, that induce for all prime numbers ℓ a split inclusion of $\mathrm{Gal}(K)$-representations

$$H^1(J_{\overline{K}}, \mathbb{Q}_{\ell}) \hookrightarrow H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_{\ell}(n)), \tag{2.3}$$

with image $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_{\ell}(n))$. Moreover, there is a correspondence over K inducing for all primes ℓ a splitting for (2.3).

Proof. As discussed above, up to the statement involving a splitting for (2.3), it is enough to exhibit a correspondence $\pi \in \mathrm{CH}^1(C \times_K C)_{\mathbb{Q}}$ which acts idempotently on $H^1(C_{\mathbb{C}}, \mathbb{Q})$ such that $\pi_* H^1(C_{\mathbb{C}}, \mathbb{Q})$ is mapped via γ_* isomorphically onto $\mathrm{Im}(\gamma_*) = \mathrm{N}^n H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$. Following the notation of Lemma 2.3, set $\gamma'_* = Q \circ (\gamma_*)^{\vee} \circ Q'$, and set $p : H^1(C_{\mathbb{C}}, \mathbb{Q}) \to \mathrm{Im}(\gamma'_*)$ to be the orthogonal projection. Consider the endomorphism

$$\gamma'_* \circ \gamma_* = {}^t\gamma_* \circ \Theta_* \circ \gamma_* : H^1(C_{\mathbb{C}}, \mathbb{Q}) \to H^1(C_{\mathbb{C}}, \mathbb{Q}).$$

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From Lemma 2.3, the endomorphism $\gamma'_* \circ \gamma_*$ induces an automorphism of $\mathrm{Im}(\gamma'_*)$. By the Cayley–Hamilton theorem, there is a polynomial P with $\mathbb{Q}$-coefficients such that the inverse of $(\gamma'_* \circ \gamma_*)|_{\mathrm{Im}(\gamma'_*)}$ is $P((\gamma'_* \circ \gamma_*)|_{\mathrm{Im}(\gamma'_*)})$. It is then apparent that p is induced by

$$\pi := {}^t\gamma \circ \Theta \circ \gamma \circ P({}^t\gamma \circ \Theta \circ \gamma) \in \mathrm{CH}^1(C \times_K C)_{\mathbb{Q}}, \tag{2.4}$$

since by construction it acts as the identity on $\mathrm{Im}(\gamma'_*)$ and as zero on $\mathrm{Ker}(\gamma_*)$. That $\mathrm{Im}\,(\gamma \circ \pi)_* = \mathrm{Im}\,(\gamma_*)$ follows from Lemma 2.3.

Finally, concerning the splitting for (2.3), writing the correspondence π of formula (2.4) as $\pi = \varphi \circ \gamma$, we see that a $\mathrm{Gal}(K)$-equivariant splitting to the inclusion

$$\gamma_* : H^1(J_{\overline{K}}, \mathbb{Q}_\ell) \cong \pi_* H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \hookrightarrow H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$$

is given by the action φ_* of φ. $\qquad\square$

Up to proving Lemma 2.6, the proof of Theorem 2.1 is now complete. $\qquad\square$

Remark 2.5. In motivic terms, we have proved the following. With the notation of [And04], denote by $h^1(J)$ the weight-1 direct summand of the homological motive of J, and by $h(X)$ the homological motive of X. We proved that the morphism of motives

$$h^1(J) \cong (C, \pi) \xrightarrow{\gamma \circ \pi} h(X)(n)$$

is split injective with a splitting given by the correspondence $\pi \circ \varphi : h(X)(n) \to (C, \pi)$, and that the cohomology of $\mathrm{Im}\,(\gamma \circ \pi)$ is $\mathrm{N}^n\,H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$.

Lemma 2.6. *Let X be a smooth projective variety of pure dimension d over $K \subseteq \mathbb{C}$. Assume any of (a)–(d) of Theorem 2.1. Then there is a correspondence $\Theta \in \mathrm{CH}^{d-2n-1}(X \times_K X)$ such that the pairing $(x, y) \mapsto \int_{X_{\mathbb{C}}} x \cup \Theta_* y$ defines a polarization Q' on $H^{2n+1}(X_{\mathbb{C}}, \mathbb{Q}(n))$; i.e., such that (2.2) is commutative.*

Proof. Let us first recall how to polarize the Hodge structure $H^n(X_{\mathbb{C}}, \mathbb{Q})$ using the Lefschetz decomposition formula, for any smooth projective variety X of pure dimension d. We fix the class L of a smooth hyperplane section of X. By the Lefschetz hyperplane theorem, the powers of L induce isomorphisms $L^{d-n} : H^n(X_{\mathbb{C}}, \mathbb{Q}) \to H^{2d-n}(X_{\mathbb{C}}, \mathbb{Q}(d-n))$ for $n \leqslant d$. By Poincaré duality, the bilinear form Q_n on $H^n(X_{\mathbb{C}}, \mathbb{Q})$ given by

$$Q_n(x, y) = \int_{X_{\mathbb{C}}} x \cup L^{d-n} y$$

is non-degenerate. By definition, for $n \leqslant d$, the primitive part of $H^n(X_{\mathbb{C}}, \mathbb{Q})$ is

$$H^n(X_{\mathbb{C}}, \mathbb{Q})_{\mathrm{prim}} := \mathrm{Ker}(L^{d-n+1} : H^n(X_{\mathbb{C}}, \mathbb{Q}) \to H^{2d-n+2}(X_{\mathbb{C}}, \mathbb{Q})).$$

We have the Lefschetz decomposition formula

$$H^n(X_{\mathbb{C}}, \mathbb{Q}) = \bigoplus_{0 \leqslant r} L^r H^{n-2r}(X_{\mathbb{C}}, \mathbb{Q}(-r))_{\mathrm{prim}}. \tag{2.5}$$

The Hodge index theorem states that decomposition (2.5) is orthogonal for Q_n and that the sub-Hodge structure $L^r H^{n-2r}(X_{\mathbb{C}}, \mathbb{Q}(-r))_{\mathrm{prim}}$ endowed with the form $(-1)^r Q_n$ is polarized. We let $p^{n,r}$ denote the orthogonal projector

$$H^n(X_{\mathbb{C}}, \mathbb{Q}) \to L^r H^{n-2r}(X_{\mathbb{C}}, \mathbb{Q}(-r))_{\mathrm{prim}} \to H^n(X_{\mathbb{C}}, \mathbb{Q})$$

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and we define

$$s_n := \sum_r (-1)^r p^{n,r} \quad \text{for } n \leqslant d \quad \text{and} \quad s_n := \sum_r (-1)^r \, {}^t p^{2d-n,r} \quad \text{for } n > d.$$

With this notation, the bilinear form Q' on $H^n(X_\mathbb{C}, \mathbb{Q})$ given by $Q'(x, y) = \int_{X_\mathbb{C}} x \cup L^{d-n} s_n y$ is a polarization. (Here, L^r for $r < 0$ is defined to be the inverse of L^{-r}.)

Let us now consider the polarized Hodge structure $H^{2n+1}(X_\mathbb{C}, \mathbb{Q}(n))$ and proceed to the proof of Lemma 2.6.

(a) If $2n + 1 = 1$, then note that $H^1(X_\mathbb{C}, \mathbb{Q}) = H^1(X_\mathbb{C}, \mathbb{Q})_{\text{prim}}$, so that s_1 is induced by the diagonal $\Delta_X \in \mathrm{CH}^d(X \times X)_\mathbb{Q}$. Therefore, $\Theta = L^{n-1}$ is suitable. (In fact, in the case $n = 0$, Theorem 2.1 holds with $J = \mathrm{Pic}^0_X$.)

If $2n + 1 = 3$, let us write $\Lambda \in \mathrm{CH}^1(X \times X)_\mathbb{Q}$ for a correspondence that induces the inverse to the Lefschetz isomorphism $L^{d-1} : H^1(X_\mathbb{C}, \mathbb{Q}) \xrightarrow{\simeq} H^{2d-1}(X_\mathbb{C}, \mathbb{Q}(d-1))$. (Such a correspondence Λ does exist; see for instance the proof of Proposition 1.3.) We define

$$P^{3,1} := L \circ \Lambda \circ L^{d-2} \in \mathrm{CH}^d(X \times X)_\mathbb{Q}.$$

It is clear that $P^{3,1}$ acts as $p^{3,1}$ on $H^3(X_\mathbb{C}, \mathbb{Q}(1))$; we may then define $P^{3,0} = \Delta_X - P^{3,1}$, and $S_3 := P^{3,0} - P^{3,1} \in \mathrm{CH}^d(X \times X)_\mathbb{Q}$ acts on $H^3(X_\mathbb{C}, \mathbb{Q}(1))$ as s_3. Hence, $\Theta := L^{d-2} \circ S_3$ is suitable.

(b) If X satisfies the Lefschetz standard conjecture, then Kleiman [Kle68] has shown that the projectors $p^{2n+1,r}$ are induced by correspondences $P^{2n+1,r} \in \mathrm{CH}^d(X \times X)_\mathbb{Q}$. Therefore, denoting by $\Lambda \in \mathrm{CH}^{2d-2n-1}(X \times X)_\mathbb{Q}$ either the correspondence L^{d-2n-1} if $2n+1 \leqslant d$ or a correspondence inducing the inverse of the action of L^{2n+1-d} on $H^{2d-2n-1}(X_\mathbb{C}, \mathbb{Q}(d-n-1))$ if $2n + 1 > d$, we get that $\Theta := \Lambda \circ \left(\sum_r (-1)^r P^{2n+1,r} \right)$ is suitable.

(c) Let Y be a smooth projective threefold, and let $\Gamma \in \mathrm{CH}^{n+2}(Y \times_K X)_\mathbb{Q}$ be such that

$$\Gamma_* : H^3(Y_\mathbb{C}, \mathbb{Q}(1)) \to H^{2n+1}(X_\mathbb{C}, \mathbb{Q}(n))$$

is surjective. The polarization on $H^3(Y_\mathbb{C}, \mathbb{Q}(1))$ coming from the Lefschetz decomposition induces a polarization on $H^{2n+1}(X_\mathbb{C}, \mathbb{Q}(n))$ and we are going to show that this can be done algebro-geometrically. By (a), there is a correspondence $S_Y \in \mathrm{CH}^3(Y \times Y)_\mathbb{Q}$ such that the pairing $(x, y) \mapsto \int_{Y_\mathbb{C}} x \cup (S_Y)_* y$ defines a polarization on $H^3(Y_\mathbb{C}, \mathbb{Q}(1))$. By Lemma 2.3, we get a decomposition

$$H^3(Y_\mathbb{C}, \mathbb{Q}(1)) = \mathrm{Im}((S_Y)_* \circ \Gamma^* \circ L^{d-2n-1}) \oplus \mathrm{Ker}(\Gamma_*).$$

It follows that $\Gamma \circ S_Y \circ {}^t\Gamma \circ L^{d-2n-1}$ induces an automorphism of $H^{2n+1}(X_\mathbb{C}, \mathbb{Q}(n))$. Consider the correspondence

$$\Gamma' := S_Y \circ {}^t\Gamma \circ L^{d-2n-1} \in \mathrm{CH}_{n+2}(X \times Y)_\mathbb{Q}.$$

The bilinear form

$$Q'(x, y) := \int_{Y_\mathbb{C}} (\Gamma')_* x \cup (S_Y)_* (\Gamma')_* y = \int_{X_\mathbb{C}} x \cup (\Gamma')^* (S_Y)_* (\Gamma')_* y$$

then defines a polarization on $H^{2n+1}(X_\mathbb{C}, \mathbb{Q}(n))$. The correspondence $\Theta := {}^t\Gamma' \circ S_Y \circ \Gamma'$ is then suitable.

(d) is a special case of (c) because if $H^{2n+1}(X_\mathbb{C}, \mathbb{Q}(n))$ is of geometric coniveau n, then, by Proposition 1.3, $H^{2n+1}(X_\mathbb{C}, \mathbb{Q}(n))$ is of geometric niveau n (meaning that there are a smooth projective curve C over K and a correspondence $\gamma \in \mathrm{CH}^{n+1}(C \times_K X)$ such that $\gamma_* : H^1(C_\mathbb{C}, \mathbb{Q}) \to H^{2n+1}(X_\mathbb{C}, \mathbb{Q}(n))$ is surjective) and hence of geometric niveau $n - 1$. $\qquad\square$

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3. Algebraic representatives over algebraically closed fields

In this section we review Murre's work [Mur85] on algebraic representatives over algebraically closed fields. The theory of algebraic representatives is indeed the starting point to showing in Theorem 5.1 that the image of the Abel–Jacobi map $AJ : \mathrm{A}^2(X_{\mathbb{C}}) \to J^3(X_{\mathbb{C}})$ descends to a field of definition of X. In § 3.3, we recall the functorial properties of Chow and Albanese schemes under field extension, and similar results for homomorphisms of abelian varieties. This is used in § 3.4 to establish algebraically closed base change for algebraic representatives; see our main theorem (Theorem 3.7).

3.1 Preliminaries on Chow groups and algebraic representatives

Fix an algebraically closed field k. Recall, following Samuel (see [Mur85, Definition 1.6.1] or [Sam60, 2.5]), that given an abelian variety A over k, a homomorphism of groups

$$\mathrm{A}^i(X) \xrightarrow{\ \phi\ } A(k)$$

is said to be *regular* if for every pair (T, Z) with T a pointed smooth integral variety, and $Z \in \mathrm{CH}^i(T \times X)$, the composition

$$T(k) \xrightarrow{\ w_Z\ } \mathrm{A}^i(X) \xrightarrow{\ \phi\ } A(k)$$

is induced by a morphism of varieties $\psi_Z : T \to A$ over k, where, if $t_0 \in T(k)$ is the base point of T, $w_Z : T(k) \to \mathrm{A}^i(X)$ is given by $t \mapsto Z_t - Z_{t_0}$; here Z_t is the refined Gysin fiber (see, e.g., [Ful98, ch. 10]).

Given a *surjective* regular homomorphism $\phi : \mathrm{A}^i(X) \to A(k)$, there exists a correspondence $Z \in \mathrm{CH}^i(A \times X)$ such that $\phi \circ w_Z$ is an isogeny (e.g., [Mur85, Corollary 1.6.3]; see also Lemma 4.9).

DEFINITION 3.1. An *algebraic representative* for $\mathrm{A}^i(X)$ is an abelian variety that is initial for regular homomorphisms. In other words, it is a pair $(\mathrm{Ab}^i(X), \phi^i_X)$ with $\mathrm{Ab}^i(X)$ an abelian variety over k and

$$\mathrm{A}^i(X) \xrightarrow{\ \phi^i_X\ } \mathrm{Ab}^i(X)(k)$$

a regular homomorphism of groups such that for every pair (A, ϕ) consisting of an abelian variety A over k and a regular homomorphism of groups $\phi : \mathrm{A}^i(X) \to A(k)$, there exists a unique morphism $f : \mathrm{Ab}^i(X) \to A$ of varieties over k such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{A}^i(X) & \xrightarrow{\ \phi^i_X\ } & \mathrm{Ab}^i(X)(k) \\
& \phi \searrow & \ \downarrow{\scriptstyle \exists!}\ {\scriptstyle f(k)} \\
& & A(k)
\end{array}
$$

Algebraic representatives, if they exist, are unique up to unique isomorphism. For an algebraic representative, the homomorphism ϕ^i_X is surjective (e.g., [Mur85, Remark (a), p. 225]). It is well known that algebraic representatives of $\mathrm{A}^1(X)$ and $\mathrm{A}^d(X)$ (where d is the dimension of X) do exist: the Abel–Jacobi map taking a divisor to its associated line bundle $A^1(X) \to \mathrm{Pic}^0(X)(k)$ and the Albanese map $A^d(X) \to \mathrm{Alb}(X)(k)$ are universal surjective regular homomorphisms.

We will frequently use the following necessary and sufficient condition for the existence of an algebraic representative.

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PROPOSITION 3.2 ([Sai79, Theorem 2.2], [Mur85, Proposition 2.1]). *An algebraic representative for $\mathrm{A}^i(X)$ exists if and only if there exists a constant M such that for every surjective regular homomorphism $\phi : \mathrm{A}^i(X) \twoheadrightarrow A(k)$, we have $\dim_k A \leqslant M$.*

Remark 3.3. Note that the notions of regular homomorphism and of algebraic representative are usually defined for smooth projective *irreducible* varieties over an algebraically closed field k. However, it is harmless to consider possibly disconnected smooth projective varieties: for instance, a smooth projective variety X over k admits an algebraic representative $(\mathrm{Ab}^i(X), \phi_X^i)$ for $\mathrm{A}^i(X)$ if and only if each of its irreducible components $X_1, \ldots, X_n$ admits an algebraic representative $(\mathrm{Ab}^i(X_j), \phi_{X_j}^i)$ for $\mathrm{A}^i(X_j)$, $j = 1, \ldots, n$. In this case, $(\mathrm{Ab}^i(X), \phi_X^i) \cong \bigoplus_{j=1}^n (\mathrm{Ab}^i(X_j), \phi_{X_j}^i)$.

3.2 Results of Murre on codimension-two cycles

We recall Murre's main result concerning algebraic representatives for codimension-two cycles.

THEOREM 3.4 [Mur85, Theorem A, p. 226, Theorem C, p. 229, Theorem 10.3, p. 258]. *Let X be a smooth projective variety over an algebraically closed field k.*

(a) *There exists an algebraic representative $(\mathrm{Ab}^2(X), \phi_X^2)$ of $\mathrm{A}^2(X)$.*

(b) *For all but finitely many primes $\ell \neq \mathrm{char}\, k$, there is a natural inclusion*

$$H^1(\widehat{\mathrm{Ab}^2(X)}, \mathbb{Z}_\ell) \hookrightarrow H^3(X, \mathbb{Z}_\ell(1)). \tag{3.1}$$

(c) *For all primes $\ell \neq \mathrm{char}\, k$, there is a natural inclusion*

$$H^1(\widehat{\mathrm{Ab}^2(X)}, \mathbb{Q}_\ell) \hookrightarrow H^3(X, \mathbb{Q}_\ell(1)). \tag{3.2}$$

(d) *If $k = \mathbb{C}$, then an algebraic representative is given by the image of the Abel–Jacobi map. In other words, the pair $(J_{\mathrm{a}}^3(X), \psi^2)$, where*

$$\psi^2 : \mathrm{A}^2(X) \to J^3(X)$$

is the Abel–Jacobi map, and $J_{\mathrm{a}}^3(X) = \mathrm{Im}(\psi^2)$, is an algebraic representative of $\mathrm{A}^2(X)$. For each ℓ, the Abel–Jacobi map ψ^2 is an injection on torsion, so that there is an isomorphism:

$$T_\ell \mathrm{A}^2(X) \xrightarrow[\cong]{T_\ell \psi^2} T_\ell J_{\mathrm{a}}^3(X). \tag{3.3}$$

Remark 3.5. The inclusions (3.1) and (3.2) are not stated explicitly in [Mur85]; the argument is contained in Murre's argument that the second Bloch map is injective [Mur85, Proposition 9.2, p. 256]. More precisely, [Mur85, (18), p. 250] establishes that for almost all ℓ there is an inclusion

$$H^1(\widehat{\mathrm{Ab}^2(X)}, \mathbb{Z}_\ell(1)) = H^1(\mathrm{Ab}^2(X), \mathbb{Z}_\ell)^\vee = T_\ell \mathrm{Ab}^2(X) \subseteq T_\ell \mathrm{A}^2(X) \subseteq T_\ell \mathrm{CH}^2(X). \tag{3.4}$$

(For the finitely many exceptional $\ell \neq \mathrm{char}\, k$, Murre's argument shows that there is still an inclusion of *rational* Tate modules $H^1(\widehat{\mathrm{Ab}^2(X)}, \mathbb{Q}_\ell(1)) = V_\ell \mathrm{Ab}^2(X) \subseteq V_\ell \mathrm{CH}^2(X)$.) Then Murre's argument that the second Bloch map is injective shows that for almost all ℓ there is also an inclusion:

$$T_\ell \mathrm{CH}^2(X) \hookrightarrow H^3(X, \mathbb{Z}_\ell(2)). \tag{3.5}$$

(This may fail to be an inclusion at the primes for which $H^3(X, \mathbb{Z}_\ell(2))$ has torsion; for cohomology with $\mathbb{Q}_\ell$-coefficients, this does not pose a problem, and one obtains the inclusion for all ℓ.)

The inclusion (3.5) is not explicitly stated in [Mur85] but follows immediately from [Mur85, (19)] and [Blo79, Lemma 2.4], together with the fact that $H^3(X, \mathbb{Z}_\ell(2))$ is torsion-free for almost all ℓ by a theorem of Gabber [Gab83].

Remark 3.6. We can be more precise about the primes ℓ for which the argument above does not establish the inclusion (3.1). By [Mur85, Lemma 1.6.2, p. 224], there exists a cycle Z on $X \times \mathrm{Ab}^2(X)$ such that the composition $\psi_Z := \phi_X \circ w_Z : \mathrm{Ab}^2(X) \to \mathrm{A}^2(X) \to \mathrm{Ab}^2(X)$ is an isogeny. For each $\ell \nmid \deg(\psi_Z)$, Murre [Mur85, p. 250] established (3.4). Therefore, (3.1) holds for all primes ℓ such that $\ell \nmid \deg(\psi_Z)$ and such that $H^3(X, \mathbb{Z}_\ell(2))$ is torsion-free. Over $\mathbb{C}$, a result of Voisin [Voi13, Theorems 1.12 and 1.13], [Voi15, Theorem 1.7] establishes that for threefolds, if the diagonal admits an *integral* cohomological decomposition, then there exists a cycle Z so that ψ_Z is of degree one (see also [Voi13, Remark 1.9]).

3.3 Algebraic representatives, Chow varieties, and traces for primary field extensions

If an algebraic representative exists, then by definition it is dominated by the Albanese varieties of components of the Chow scheme. Since Albanese varieties and Chow schemes respect base change of fields, this gives an approach for studying the problem of base change for algebraic representatives. A key tool for studying base change of abelian varieties for primary extensions of fields is the so-called L/K-trace. We review in this section some of the results in the literature, and how we will employ them in the context of algebraic representatives.

3.3.1 *Chow rigidity and L/K-trace for primary field extensions.* We exploit (Chow) rigidity and L/K-traces for abelian varieties and primary field extensions. See [Con06] for a modern treatment; here, we summarize what we need. Let L/K be a primary extension of fields, i.e., an extension such that the algebraic closure of K in L is purely inseparable over K. If L/K is primary and separable (e.g., if K is algebraically closed), then L/K is said to be regular.

If A is an abelian variety over L, an L/K-*trace* is a final object $(\mathrm{tr}_{L/K}(A), \tau_A)$ in the category of pairs $(\underline{B}, f)$, where $\underline{B}$ is an abelian variety over K and $f : \underline{B}_L \to A$ is a map of abelian varieties over L. An L/K-trace exists [Con06, Theorem 6.2] and is unique up to unique isomorphism. For brevity, we will frequently denote the L/K-trace by the pair $(\underline{A}, \tau)$.

The kernel $\ker(\tau)$ is always a finite group scheme [Con06, Theorem 6.4(4)]. If $\mathrm{char}(K) = 0$, then $\ker(\tau)$ is trivial [Con06, p. 72]; more generally, if L/K is regular, then $\ker(\tau)$ is a connected group scheme with connected dual [Con06, Theorems 6.4 and 6.12] and thus τ is purely inseparable. In particular [Con06, p. 76], if L/K is regular, then on L-points τ induces an injection

$$\underline{A}(L) \overset{\tau(L)}{\hookrightarrow} A(L). \tag{3.6}$$

The image of τ in A descends to K if and only if $\mathrm{Ker}(\tau)$ is the trivial group scheme [Con06, p. 72]. As we have seen, this is automatic in characteristic zero, but need not hold in positive characteristic.

Chow's rigidity theorem (see [Con06, Theorem 3.19]) states that if $\underline{B}$ and $\underline{C}$ are both abelian varieties over K (and L/K is still primary), then the natural inclusion

$$\mathrm{Hom}_K(\underline{B}, \underline{C}) \to \mathrm{Hom}_L(\underline{B}_L, \underline{C}_L)$$

is a bijection. In particular, if A/L and $\underline{B}/K$ are abelian varieties, then there are canonical bijections

$$\mathrm{Hom}_L(\underline{B}_L, A) = \mathrm{Hom}_L(\underline{B}_L, \underline{\underline{A}}_L) = \mathrm{Hom}_K(\underline{B}, \underline{A}), \tag{3.7}$$

where the first equality is the universal property of the L/K-trace, and the second equality comes from rigidity.

3.3.2 *Chow varieties, Albaneses, and algebraic representatives.* If X/K is a projective variety over a perfect field K, then for each i there is a Chow scheme $\mathrm{Chow}^i(X)$ over K, a union of countably many connected components, whose points parametrize cycles of codimension i. If L/K is an extension of fields and $X_L := X_K \times_K L$, then $\mathrm{Chow}^i(X_L) \cong \mathrm{Chow}^i(X)_L := \mathrm{Chow}^i(X) \times_K L$ [Fri91, Proposition 1.1]; see also [Kol96].

If $(Y, y)/K$ is a pointed geometrically integral variety over a perfect field, an Albanese variety for Y is an abelian variety $\mathrm{Alb}(Y)$ over K equipped with a pointed map $(Y, y) \to (\mathrm{Alb}(Y), 0)$ that is initial with respect to pointed maps from (Y, y) to abelian varieties. If Y/K is smooth and proper, then $\mathrm{Alb}(Y) = \mathrm{Pic}^0_{(\mathrm{Pic}^0_{Y/K})_{\mathrm{red}}/K}$; since the Picard functor behaves well under base change, we have $\mathrm{Alb}(Y_L) \cong \mathrm{Alb}(Y) \times_K L$ for any extension of perfect fields L/K. For geometrically integral varieties Y/K that are not smooth and proper, either using (a compactification and) resolution of singularities in characteristic zero, or an argument via de Jong's alterations as in [Moc12, Corollary A.11], one may again show that Y admits an Albanese, and that $\mathrm{Alb}(Y_L) \cong \mathrm{Alb}(Y) \times_K L$.

One setting where we will use Albaneses extensively (e.g., the proof of Theorem 3.7) is the following. Let X be a smooth projective variety defined over an algebraically closed field k, and assume that there is an algebraic representative $(\mathrm{Ab}^i(X), \phi^i_X)$. From the discussion above, we see that $\mathrm{Ab}^i(X)$ is the colimit of the inverse system of abelian varieties obtained from the Albanese varieties of components of the Chow scheme $\mathrm{Chow}^i(X)$. In fact, one should work with Albaneses of smooth alterations of the reduction of products of components of the Hilbert scheme, but we avoid this technical point in this discussion. In particular, $\mathrm{Ab}^i(X)$ admits a surjection from a finite product of these abelian varieties. Moreover, if $K \subseteq k$ is a perfect field, and $\underline{X}$ is defined over K with $X = \underline{X} \times_K k$, then $\mathrm{Ab}^i(X)$ admits a morphism from the pullback to k of the inverse system of torsors under abelian varieties defined over K obtained from the Albanese torsors of components of the Chow scheme $\mathrm{Chow}^i(\underline{X})$. In particular, since every component of $\mathrm{Chow}^i(X)$ is a component of a pullback to k of a component of $\mathrm{Chow}^i(\underline{X})$, we have that $\mathrm{Ab}^i(X)$ admits a surjection from a torsor under an abelian variety defined over K. (In fact, it will follow from Theorems 3.7 and 4.4 that if $k = \overline{K}$, or $\mathrm{char}(k) = 0$, then $\mathrm{Ab}^i(X)$, if it exists, descends to an abelian variety over K.)

3.4 Algebraically closed base change for algebraic representatives

The aim of this subsection is to establish our main addition to Murre's work on algebraic representatives over algebraically closed fields.

THEOREM 3.7. *Let Ω/k be an extension of algebraically closed fields. Let X/k be a smooth projective variety. Assume that either $\mathrm{A}^i(X)$ or $\mathrm{A}^i(X_\Omega)$ admits an algebraic representative. Then both $\mathrm{A}^i(X)$ and $\mathrm{A}^i(X_\Omega)$ admit an algebraic representative.*

Moreover, let $(\mathrm{Ab}^i(X), \phi^i_X)$ and $(\mathrm{Ab}^i(X_\Omega), \phi^i_{X_\Omega})$ be algebraic representatives for $\mathrm{A}^i(X)$ and $\mathrm{A}^i(X_\Omega)$, respectively. Then there are natural maps

$$\mathrm{Ab}^i(X) \xrightarrow{\ b'\ } \underline{\mathrm{Ab}}^i(X_\Omega) \quad and \quad \underline{\mathrm{Ab}}^i(X_\Omega)_\Omega \xrightarrow{\ \tau\ } \mathrm{Ab}^i(X_\Omega)$$

over k and Ω, respectively, where $(\underline{\mathrm{Ab}}^i(X_\Omega), \tau)$ is the Ω/k-trace of $\mathrm{Ab}^i(X_\Omega)$ (§3.3.1). In characteristic zero, b' and τ are isomorphisms; in positive characteristic, they are purely

inseparable isogenies. In particular, in characteristic zero, there is a natural isomorphism $\mathrm{Ab}^i(X)_\Omega \cong \mathrm{Ab}^i(X_\Omega)$.

We start by recalling a rigidity result for torsion in Chow groups.

THEOREM 3.8 (Lecomte [Lec86, Theorem 3.11]). *Let* Ω/k *be an extension of algebraically closed fields, and let* X *be a projective variety over* k. *Base change on cycles induces an injective homomorphism of Chow groups* $b_{\Omega/k} : \mathrm{CH}^i(X) \to \mathrm{CH}^i(X_\Omega)$. *For each positive integer* N, $b_{\Omega/k}$ *induces isomorphisms on the* N-*torsion groups:*

(a) $b_{\Omega/k}[N] : \mathrm{CH}^i(X)[N] \xrightarrow{\sim} \mathrm{CH}^i(X_\Omega)[N]$;

(b) $b_{\Omega/k}[N] : \mathrm{A}^i(X)[N] \xrightarrow{\sim} \mathrm{A}^i(X_\Omega)[N]$.

Proof. That the base-change homomorphism $b_{\Omega/k} : \mathrm{CH}^i(X) \to \mathrm{CH}^i(X_\Omega)$ is injective is standard and is proved using a spreading out argument followed by a specialization argument (e.g., the argument of [Blo10, Lemma 1A.3, p. 22]). Precisely, denoting by $r_{\Omega/k} : \mathrm{CH}^i(X_\Omega) \to \mathrm{CH}^i(X)$ the specialization homomorphism (see [Blo10, Lemma 1A.3, p. 22] and [Ful98, §20.3]), we have that $r_{\Omega/k} \circ b_{\Omega/k}$ is the identity on $\mathrm{CH}^i(X)$, giving the injectivity of $b_{\Omega/k}$. Part (a) is the content of [Lec86, Theorem 3.11]. As a consequence of (a), $r_{\Omega/k}$ and $b_{\Omega/k}$ are in fact inverses of each other on N-torsion. Part (b) then follows from the fact that $b_{\Omega/k}$ and $r_{\Omega/k}$ respect algebraic equivalence. $\square$

Proof of Theorem 3.7. We proceed in four steps.

Step 1. Let X be a smooth projective variety defined over k, let A/k be an abelian variety, and let $\phi : \mathrm{A}^i(X) \to A(k)$ be a regular homomorphism. Then there is a regular homomorphism $\phi_\Omega : \mathrm{A}^i(X_\Omega) \to A_\Omega(\Omega)$, which is surjective if ϕ is surjective, making the following diagram commute:

$$\begin{array}{ccc} \mathrm{A}^i(X) & \xrightarrow{\phi} & A(k) \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \text{base change}} \\ \mathrm{A}^i(X_\Omega) & \xrightarrow{\phi_\Omega} & A_\Omega(\Omega) \end{array} \qquad (3.8)$$

Note that by the criterion of [Sai79, Theorem 2.2], [Mur85, Proposition 2.1] (see Proposition 3.2), it follows from Step 1 that if $\mathrm{A}^i(X_\Omega)$ has an algebraic representative, then so does $\mathrm{A}^i(X)$.

To establish Step 1, we first construct ϕ_Ω; we thank the referee for providing such a construction. Denote by $\mathrm{A}^i_\natural(X)$ the contravariant functor on the category of smooth integral varieties over k given by families of algebraically trivial cycles on X. More precisely, the functor $\mathrm{A}^i_\natural(X)$ sends a variety T as above to the group of cycles $Z \in \mathrm{CH}^i(X \times_k T)$ such that for some (equivalently, for any) geometric point $x : \mathrm{Spec}(\Omega') \to T$ with Ω' being an algebraically closed field over k, we have that the cycle $x^!(Z) \in \mathrm{CH}^i(X_{\Omega'})$ is algebraically trivial. By definition of a regular homomorphism, we have a morphism of contravariant functors

$$\phi_\natural : \mathrm{A}^i_\natural(X) \to A$$

on the category of smooth integral varieties over k. Moreover, both functors $\mathrm{A}^i_\natural(X)$ and A send inverse limits of varieties to direct limits of abelian groups. Therefore, the morphism of functors $\phi_\natural$ extends in a canonical way to the category of schemes over k that can be obtained as inverse

limits of smooth integral varieties over k. In particular, evaluating on $\operatorname{Spec}\Omega$, we obtain the desired regular homomorphism

$$\mathrm{A}^i(X_\Omega) = \mathrm{A}^i_\sharp(X)(\Omega) \xrightarrow{\ \phi_\sharp(\Omega)\ } A(\Omega),$$

which we denote by ϕ_Ω. Note that this homomorphism is surjective when ϕ is surjective due to the existence of the cycle and isogeny from [Mur85, Corollary 1.6.3] (see Lemma 4.9(c)).

Step 2. Let X be a smooth projective variety defined over k, let A/Ω be an abelian variety, and let $\phi_{X_\Omega} : \mathrm{A}^i(X_\Omega) \to A(\Omega)$ be a regular homomorphism. Then there is a regular homomorphism $\underline{\underline{\phi}} : \mathrm{A}^i(X) \to \underline{\underline{A}}(k)$ to the k-points of the Ω/k-trace of A (see §3.3.1), making the following diagram commute:

$$
\begin{array}{ccc}
\mathrm{A}^i(X) & \xrightarrow{\ \underline{\underline{\phi}}\ } & \underline{\underline{A}}(k) \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \text{base change}} \\
& & \underline{\underline{A}}(\Omega) \\
& & \downarrow{\scriptstyle \tau(\Omega)} \\
\mathrm{A}^i(X_\Omega) & \xrightarrow{\ \phi_{X_\Omega}\ } & A(\Omega)
\end{array}
\tag{3.9}
$$

Here, $b = b_{\Omega/k} : \mathrm{A}^i(X) \to \mathrm{A}^i(X_\Omega)$ is the base-change map on cycles, and $\tau : \underline{\underline{A}}_\Omega \to A$ is the canonical map (see §3.3.1); recall that τ is an isomorphism onto its image in characteristic zero, and is at worst a purely inseparable isogeny onto its image in positive characteristic. In particular, τ induces an inclusion on Ω-points (3.6).

To establish Step 2, let us define $\underline{\underline{\phi}}$. For this, assume that $\alpha \in \mathrm{A}^i(X)$. Then there are a smooth projective irreducible curve T over k, with k-points t_1, t_0, and a cycle $Z \in \mathrm{CH}^i(T \times_k X)$ such that $\alpha = w_{Z,t_0}(t_1)$ (e.g., [Ful98, Example 10.3.2]). Since ϕ_{X_Ω} is regular, there is an Ω-morphism $\gamma : T_\Omega \to A$ that induces $\phi_{X_\Omega} \circ w_{Z_\Omega}$. The morphism γ necessarily factors through the Albanese map $a_\Omega : T_\Omega \to \mathrm{Alb}(T_\Omega)$, via a morphism, say, $\delta : \mathrm{Alb}(T_\Omega) \to A$. Since $(\mathrm{Alb}(T_\Omega), a_\Omega) \cong (\mathrm{Alb}(T), a)_\Omega$, the definition of the Ω/k-trace and rigidity (see §3.3.1) together imply that there is a k-morphism $\epsilon : \mathrm{Alb}(T) \to \underline{\underline{A}}$ so that $\delta = \tau \circ \epsilon_\Omega$. In other words, there is a commutative diagram

We define $\underline{\underline{\phi}}(\alpha) := \epsilon(a(t_1) - a(t_0))$. Since the base-change map on points is injective and since $\mathrm{Ker}(\tau)(\Omega)$ is trivial (see (3.6)), $\underline{\underline{\phi}}(\alpha)$ is well defined, i.e., it is independent of the choice of T. It is clear from our construction of $\underline{\underline{\phi}}$ that (3.9) commutes. In fact, starting with a general smooth pointed connected variety T, the same argument shows that $\underline{\underline{\phi}}$ is regular (since $\epsilon \circ a$ is defined over k).

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Step 3. In the notation of Step 2, if ϕ_{X_Ω} is surjective, then $\underline{\underline{\phi}} : A^i(X) \to \underline{\underline{A}}(k)$ is surjective and $\tau : \underline{\underline{A}}_\Omega \to A$ is a purely inseparable isogeny (and an isomorphism in characteristic zero).

Note that by the criterion of [Sai79, Theorem 2.2], [Mur85, Proposition 2.1] (see Proposition 3.2), it follows from Step 3 and (3.9) that if $\mathrm{A}^i(X)$ has an algebraic representative, then so does $\mathrm{A}^i(X_\Omega)$.

To establish Step 3, recall that if ϕ_{X_Ω} is surjective, then A/Ω is dominated by an inverse system of abelian varieties defined over k (see the discussion in § 3.3.2); in fact, there is an abelian variety B/k (namely the Albanese of a finite union of smooth alterations of components of the Chow scheme) such that there is a surjection $\alpha : B_\Omega \to A$. It follows (see § 3.3.1) that τ is a purely inseparable isogeny (and an isomorphism in characteristic zero).

Let us now show that $\underline{\underline{\phi}}$ is surjective provided ϕ_{X_Ω} is surjective. Let $\ell \in \mathbb{N}$ be a prime $\neq \mathrm{char}(k)$, and let us consider ℓ-torsion in the commutative diagram (3.9). The morphism τ induces an isomorphism (3.6) $T_\ell\tau : T_\ell\underline{\underline{A}}(\Omega) \to T_\ell A(\Omega)$. Lecomte's rigidity theorem (Theorem 3.8) says that the homomorphism $T_\ell b$ is an isomorphism. The base-change map $T_\ell\underline{\underline{A}}(k) \to T_\ell\underline{\underline{A}}(\Omega)$ on torsion is also an isomorphism. Therefore, a diagram chase in (3.9) shows the surjectivity of $T_\ell\underline{\underline{\phi}}$, and thus that of $\underline{\underline{\phi}}$ ([Mur85, Lemma 1.6.2(i)]; see Lemma 4.9(a)).

Step 4. Let $(\mathrm{Ab}^i(X), \phi^i_X)$ and $(\mathrm{Ab}^i(X_\Omega), \phi^i_{X_\Omega})$ be algebraic representatives for $\mathrm{A}^i(X)$ and $\mathrm{A}^i(X_\Omega)$, respectively. The k-homomorphism $b' : \mathrm{Ab}^i(X) \to \underline{\underline{\mathrm{Ab}}}^i(X_\Omega)$ induced by the regular homomorphism $\underline{\underline{\phi}} : A^i(X) \to \underline{\underline{\mathrm{Ab}}}^i(X_\Omega)(k)$ of Step 2 is an isomorphism in characteristic zero, and a purely inseparable isogeny in positive characteristic.

It suffices to show that $b'(k)$ is an isomorphism. We have the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{A}^i(X) & \xrightarrow{\phi^i_X} & \mathrm{Ab}^i(X)(k) \\
\| & & \downarrow{b'(k)} \\
\mathrm{A}^i(X) & \xrightarrow{\underline{\underline{\phi}}} & \underline{\underline{\mathrm{Ab}}}^i(X_\Omega)(k)
\end{array}
\tag{3.10}
$$

We already know from Step 3 that $\underline{\underline{\phi}}$ is surjective. Therefore, it remains only to prove that b' is injective on k-points. Let $\zeta \in \mathrm{Ker}\, b'(k)$ and let $z \in \mathrm{A}^i(X)$ be a cycle mapping to ζ (i.e., $\phi^i_X(z) = \zeta$). Then $z_\Omega := b(z) \in \mathrm{A}^i(X_\Omega)$ is abelian equivalent to 0 (i.e., z_Ω is sent to 0 under any regular morphism), since $\phi^i_{X_\Omega}(z_\Omega) = 0$ by the commutativity of (3.9) and (3.10). In particular, considering the regular homomorphism $(\phi^i_X)_\Omega$ of Step 1, we have $(\phi^i_X)_\Omega(z_\Omega) = 0$. By virtue of the commutativity of (3.8), and the injectivity of the base-change map, we have that $\phi^i_X(z) = 0$; i.e., $\zeta = 0$, and we have that b' is injective on geometric points. □

4. Algebraic representatives and Galois actions

In this section we extend the notion of algebraic representatives to the setting of Galois field extensions. The main result of the section is Theorem 4.4, which says that algebraic representatives descend from the algebraic closure of a perfect field. We also establish several basic results on regular homomorphisms in this setting following [Mur85].

Throughout this section we take k to be an algebraically closed field, and $K \subseteq k$ to be a perfect subfield such that k/K is algebraic. (In particular, K is the fixed field of $\mathrm{Gal}(k/K)$.)

4.1 Preliminaries on Galois descent

If X/k is a variety and $\sigma \in \mathrm{Gal}(k/K)$, let X^σ be the fiber product $X^\sigma = X \times_{\mathrm{Spec}\,k,\sigma} \mathrm{Spec}\,k$; by definition, it comes equipped with a map $\tilde{\sigma} : X^\sigma \to X$ over σ. A k/K descent datum on X is a system of isomorphisms $\{\sigma_X : X \xrightarrow{\sim} X \text{ over } \sigma : \sigma \in \mathrm{Gal}(k/K)\}$ satisfying the cocycle condition $(\sigma\tau)_X = \sigma_X \circ \tau_X : X \to X$. There is a similar notion of a descent datum on a morphism of varieties (i.e., a descent datum on its graph), and the category of quasi-projective k-schemes equipped with k/K descent data is equivalent to the category of quasi-projective schemes over K (e.g., [BLR90, Theorem 6, pp. 135 and 141], [GW10, Theorem 14.84, p. 457]). We will often denote the effect of this functor by $(X, \{\sigma\}) \mapsto \underline{X}$; the inverse functor is $\underline{Y} \mapsto Y := \underline{Y} \times_{\mathrm{Spec}\,K} \mathrm{Spec}\,k$. Similar notation will be employed for cycles.

Note that a k/K descent datum on X (or, equivalently, a choice of K-model $\underline{X}$) determines an action of $\mathrm{Gal}(k/K)$ on the k-points $X(k)$; a morphism $X \to Y$ of k-varieties with descent data descends to K if and only if its action on k-points is $\mathrm{Gal}(k/K)$-equivariant (e.g., [Mil14, Proposition 16.9]).

4.2 Action of the Galois group on cycles

Let X/k be a quasi-projective variety equipped with a k/K descent datum $\{\sigma_X\}$. Pullback by the flat morphism of schemes σ_X induces [Ful98, §1.7] a homomorphism

$$\sigma_X^* : \mathrm{CH}^i(X) \to \mathrm{CH}^i(X)$$

and we have $\sigma_X^*[Z] = [\sigma_X^* Z]$. The cocycle condition for descent data then yields an action of $\mathrm{Gal}(k/K)$ on $\mathrm{CH}^i(X)$, which restricts to an action on $\mathrm{A}^i(X)$.

Now let T/k be a smooth, connected variety over k, let Z be a codimension-i cycle on $T \times_k X$, and let $t_0 : \mathrm{Spec}\,k \to T$ be a k-point. If T, Z, and t_0 are equipped with compatible k/K descent data, we obtain a commutative diagram

$$
\begin{array}{ccc}
T(k) & \xrightarrow{w_{Z,t_0}} & \mathrm{A}^i(X) \\
\sigma_T^* \downarrow & & \downarrow \sigma_X^* \\
T(k) & \xrightarrow{w_{Z,t_0}} & \mathrm{A}^i(X)
\end{array}
$$

and we see that w_Z is $\mathrm{Gal}(k/K)$-equivariant.

Even in the absence of a distinguished K-model for T and Z one can show, using the refined Gysin map [Ful98, §6.3], that

$$\sigma_X^* Z_{t_0} = ((\tilde{\sigma} \times_\sigma \sigma_X)^* Z)_{\tilde{\sigma}^* t_0} \tag{4.1}$$

and we thus have a commutative diagram

$$
\begin{array}{ccc}
T(k) & \xrightarrow{\quad w_{Z,t_0} \quad} & \mathrm{A}^i(X) \\
\tilde{\sigma}^* \downarrow & & \downarrow \sigma_X^* \\
T^\sigma(k) & \xrightarrow{w_{(\tilde{\sigma} \times_\sigma \sigma_X)^* Z, \tilde{\sigma}^* t_0}} & \mathrm{A}^i(X)
\end{array}
\tag{4.2}
$$

4.3 Galois equivariant algebraic representatives

Let k be an algebraically closed field, and $K \subseteq k$ a perfect subfield such that k/K is algebraic. Let X be a smooth projective variety over k, equipped with k/K descent data. Let A be an abelian variety over k that is also equipped with k/K descent data. We say that a regular homomorphism

$$\phi : \mathrm{A}^i(X) \to A(k)$$

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is $\mathrm{Gal}(k/K)$-*equivariant* if, for each $\sigma \in \mathrm{Gal}(k/K)$, the following diagram is commutative:

$$
\begin{array}{ccc}
\mathrm{A}^i(X) & \xrightarrow{\phi} & A(k) \\
\downarrow{\scriptstyle \sigma_X^*} & & \downarrow{\scriptstyle \sigma_A^*} \\
\mathrm{A}^i(X) & \xrightarrow{\phi} & A(k)
\end{array}
$$

Remark 4.1. Given such data $(A, \{\sigma_A\}, \phi)$, the class $[0] \in \mathrm{A}^i(X)$ is fixed by $\mathrm{Gal}(k/K)$, and so $\phi([0])$ descends to K; $A(K)$ is non-empty. By uniqueness of group structures on abelian varieties, we see that the group law on A also descends to $\underline{A}$; A descends to K as an abelian variety.

Definition 4.2. Let k be an algebraically closed field, and $K \subseteq k$ a perfect subfield such that k/K is algebraic. Let X be a smooth projective variety over k, equipped with k/K descent data. A *Galois equivariant algebraic representative* for $\mathrm{A}^i(X)$ is a pair $(\mathrm{Ab}^i_{\mathrm{Gal}(k/K)}(X), \phi_{X, \mathrm{Gal}(k/K)})$ which is initial among all pairs (A, ϕ) consisting of an abelian variety A/k with a k/K descent datum and a Galois-equivariant regular homomorphism $\phi : \mathrm{A}^i(X) \to A(k)$.

Remark 4.3. Note that for a pointed smooth integral k-variety T, and a codimension-i cycle Z on $T \times_k X$, the following diagram is commutative for a Galois equivariant regular homomorphism $\phi : \mathrm{A}^i(X) \to A(k)$:

$$
\begin{array}{ccccc}
T(k) & \xrightarrow{w_{Z,t_0}} & \mathrm{A}^i(X) & \xrightarrow{\phi} & A(k) \\
\downarrow{\scriptstyle \tilde{\sigma}^*} & & \downarrow{\scriptstyle \sigma_X^*} & & \downarrow{\scriptstyle \sigma_A^*} \\
T^\sigma(k) & \xrightarrow{w_{(\tilde{\sigma} \times \sigma \sigma_X)^* Z, \tilde{\sigma}^* t_0}} & \mathrm{A}^i(X) & \xrightarrow{\phi} & A(k)
\end{array}
$$

Moreover, for a smooth integral k-variety T with descent datum, together with a $\mathrm{Gal}(k/K)$-invariant marked point t_0, and a codimension-i cycle Z on $T \times_k X$ such that the descent data for T and X induces descent data for Z, the following diagram is commutative for a Galois invariant regular homomorphism $\phi : \mathrm{A}^i(X) \to A(k)$:

$$
\begin{array}{ccccc}
T(k) & \xrightarrow{w_{Z,t_0}} & \mathrm{A}^i(X) & \xrightarrow{\phi} & A(k) \\
\downarrow{\scriptstyle \sigma_T^*} & & \downarrow{\scriptstyle \sigma_X^*} & & \downarrow{\scriptstyle \sigma_A^*} \\
T(k) & \xrightarrow{w_{Z,t_0}} & \mathrm{A}^i(X) & \xrightarrow{\phi} & A(k)
\end{array}
$$

This says that the morphism of k-varieties $T \to A$ inducing $\phi \circ w_Z$ descends to K.

We now prove some basic results about Galois equivariant regular morphisms. The main result we want is that for smooth projective varieties over a perfect field K, algebraic representatives of the varieties over the algebraic closure give Galois equivariant algebraic representatives; in particular, they descend to K.

Theorem 4.4. *Let X/k be a smooth projective variety over an algebraically closed field k, and suppose that $\mathrm{A}^i(X)$ has an algebraic representative $(\mathrm{Ab}^i(X), \phi_X^i)$. Let $K \subseteq k$ be a perfect field such that k/K is algebraic. Then each k/K descent datum on X induces a k/K descent datum on $\mathrm{Ab}^i(X)$. Moreover, with respect to these descent data, ϕ_X^i is $\mathrm{Gal}(k/K)$-equivariant, and the pair $(\mathrm{Ab}^i(X), \phi_X^i)$ is a Galois equivariant algebraic representative.*

Remark 4.5. Note that by Remark 4.1, $\mathrm{Ab}^i(X)$ descends to an abelian variety $\underline{\mathrm{Ab}}^i(X)$ over K.

The proof uses the universal property of $\mathrm{Ab}^i(X)$ to transfer descent data for X to descent data for $\mathrm{Ab}^i(X)$. Before giving the proof, we introduce one more notion that we will utilize.

DEFINITION 4.6. Let X be a smooth projective variety over an algebraically closed field k, and let $\sigma \in \mathrm{Aut}(k)$. Given an abelian variety A over k, a homomorphism of groups

$$\mathrm{A}^i(X) \xrightarrow{\phi} A(k)$$

is said to be σ-*regular* if for every pair (T, Z) with T a pointed smooth integral k-variety, and $Z \subseteq T \times_k X$ a relative codimension-i cycle over T, the composition

$$T(k) \xrightarrow{w_Z} \mathrm{A}^i(X) \xrightarrow{\phi} A(k)$$

is induced by a morphism $g : T \to A$ over σ; by this (i.e., $\phi \circ w_Z$ being induced by the morphism g over σ), we mean that there is a commutative diagram of morphisms of schemes

$$
\begin{array}{ccc}
T & \xrightarrow{g} & A \\
\downarrow & & \downarrow \\
k & \xrightarrow{\sigma} & k
\end{array}
$$

such that for each k-point $t : \mathrm{Spec}\, k \to T$, we have $(\phi \circ w_Z)(t) = g_*(t) := g \circ t \circ \sigma^{-1}$.

LEMMA 4.7. *Let X be a smooth projective variety over an algebraically closed field k, and let $\sigma \in \mathrm{Aut}(k)$. Suppose that $(\mathrm{Ab}^i(X), \phi_X^i)$ is an algebraic representative for $\mathrm{A}^i(X)$. Given an abelian variety A/k and a σ-regular homomorphism $\phi : \mathrm{A}^i(X) \to A(k)$, there exists a unique morphism $f : \mathrm{Ab}^i(X) \to A$ over σ such that $\phi = f_* \circ \phi_X^i$.*

Proof. Let $\phi : \mathrm{A}^i(X) \to A(k)$ be a σ-regular homomorphism. Consider the morphisms

$$A(k) \xrightarrow{\tilde{\sigma}^*} A^\sigma(k), \quad A \xrightarrow{\tilde{\sigma}^{-1}} A^\sigma,$$

with the latter over $\sigma^{-1} \in \mathrm{Aut}(k)$, and where given a k-point $a : \mathrm{Spec}\, k \to A$, we define $\tilde{\sigma}^*(a) := (\tilde{\sigma}^{-1})_*(a) = \tilde{\sigma}^{-1} a \sigma$.

The composition $\tilde{\sigma}^* \circ \phi : \mathrm{A}^i(X) \to A^\sigma(k)$ is a regular homomorphism. Indeed, given a triple (T, Z, t_0) with T a smooth integral k-variety, Z a codimension-i cycle on $T \times_k X$, and $t_0 : \mathrm{Spec}\, k \to T$ a k-point, there is by definition a morphism $g : T \to A$ over σ such that for each k-point $t : \mathrm{Spec}\, k \to T$ we have $(\phi w_Z)(t) = g t \sigma^{-1}$. Therefore, $(\tilde{\sigma}^* \phi w_Z)(t) = \tilde{\sigma}^*(g t \sigma^{-1}) = \tilde{\sigma}^{-1}(g t \sigma^{-1})\sigma = \tilde{\sigma}^{-1} g t$, so that $\tilde{\sigma}^* \phi w_Z$ is induced by $\tilde{\sigma}^{-1} g : T \to A^\sigma$, and we have established that $\tilde{\sigma}^* \phi$ is a regular homomorphism.

It follows from the universal property of the algebraic representative that there is a unique morphism $\tilde{f} : \mathrm{Ab}^i(X) \to A^\sigma$ of abelian varieties over k such that $\tilde{\sigma}^* \circ \phi = \tilde{f} \circ \phi_X^i$ on k-points. The morphism $f := \tilde{\sigma} \circ \tilde{f}$ over σ satisfies $\phi = \tilde{\sigma}_* \tilde{\sigma}^* \phi = \tilde{\sigma}_* \tilde{f} \phi_X^i = f_* \phi_X^i$. Here, we are using that $\tilde{\sigma}_* \tilde{f} = f_*$ on k-points; indeed, given a k-point $a : \mathrm{Spec}\, k \to \mathrm{Ab}^i(X)$, we have $\tilde{\sigma}_* \tilde{f}(a) = \tilde{\sigma}_*(\tilde{f}a) = \tilde{\sigma} \tilde{f} a \sigma^{-1} = f a \sigma^{-1} = f_*(a)$. The uniqueness of f follows again from the universal property of the algebraic representative. $\square$

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Proof of Theorem 4.4. We start by showing that k/K descent data for X induces k/K descent data for $\mathrm{Ab}^i(X)$. The first claim is that the composition $\phi_X^i \circ \sigma_X^* : \mathrm{A}^i(X) \to \mathrm{Ab}^i(X)(k)$ is σ^{-1}-regular. Indeed, suppose that we are given T/k, a smooth integral pointed variety, and Z a codimension-i cycle on $T \times_k X$. It was shown in (4.2) that we have a commutative diagram

$$
\begin{array}{ccccc}
T(k) & \xrightarrow{w_Z} & \mathrm{A}^i(X) & \xrightarrow{\phi_X^i} & \mathrm{Ab}^i(X)(k) \\
\downarrow{\tilde{\sigma}^*} & & \downarrow{\sigma_X^*} & & \\
T^\sigma(k) & \xrightarrow{w_{(\tilde{\sigma} \times \sigma_X)^* Z}} & \mathrm{A}^i(X) & \xrightarrow{\phi_X^i} & \mathrm{Ab}^i(X)(k)
\end{array}
$$

Let $g' : T^\sigma \to \mathrm{Ab}^i(X)$ be the morphism of k-varieties inducing the bottom row of the diagram, and set $g := g' \circ \tilde{\sigma}^{-1} : T \to \mathrm{Ab}^i(X)$ to be the induced morphism over σ^{-1}. For a k-point $t : \mathrm{Spec}\, k \to T$, we have

$$(\phi_X^i \sigma_X^* w_Z)(t) = (\phi_X^i w_{(\tilde{\sigma} \times \sigma_X)^* Z} \tilde{\sigma}^*)(t) = g'(\tilde{\sigma}^* t) = g' \tilde{\sigma}^{-1} t \sigma = g t \sigma = g_*(t).$$

Therefore, we have established that $\phi_X^i \sigma_X^*$ is σ^{-1}-regular.

By virtue of the previous lemma, there is a unique morphism $(\sigma_{\mathrm{Ab}})^{-1} : \mathrm{Ab}^i(X) \to \mathrm{Ab}^i(X)$ over σ^{-1} making the following diagram commute:

$$
\begin{array}{ccccc}
T(k) & \xrightarrow{w_Z} & \mathrm{A}^i(X) & \xrightarrow{\phi_X^i} & \mathrm{Ab}^i(X)(k) \\
\downarrow{\tilde{\sigma}^*} & & \downarrow{\sigma_X^*} & & \downarrow{((\sigma_{\mathrm{Ab}})^{-1})_*} \\
T^\sigma(k) & \xrightarrow{w_{(\tilde{\sigma} \times \sigma_X)^* Z}} & \mathrm{A}^i(X) & \xrightarrow{\phi_X^i} & \mathrm{Ab}^i(X)(k)
\end{array}
$$

Using the universal property of the algebraic representative, one can see that the morphisms $(\sigma_{\mathrm{Ab}})^{-1} : \mathrm{Ab}^i(X) \to \mathrm{Ab}^i(X)$ are isomorphisms, and that the morphisms $\sigma_{\mathrm{Ab}} := ((\sigma_{\mathrm{Ab}})^{-1})^{-1}$ define a lift of the action of $\mathrm{Gal}(k/K)$ to $\mathrm{Ab}^i(X)$. In particular, the morphisms σ_{Ab} provide a k/K descent datum for $\mathrm{Ab}^i(X)$. Finally, the commutativity of the diagram

$$
\begin{array}{ccccc}
T(k) & \xrightarrow{w_Z} & \mathrm{A}^i(X) & \xrightarrow{\phi_X^i} & \mathrm{Ab}^i(X)(k) \\
\downarrow{\tilde{\sigma}^*} & & \downarrow{\sigma_X^*} & & \downarrow{\sigma_{\mathrm{Ab}}^*} \\
T^\sigma(k) & \xrightarrow{w_{(\tilde{\sigma} \times \sigma_X)^* Z}} & \mathrm{A}^i(X) & \xrightarrow{\phi_X^i} & \mathrm{Ab}^i(X)(k)
\end{array}
$$

shows that ϕ_X^i is $\mathrm{Gal}(k/K)$-equivariant, and therefore that $(\mathrm{Ab}^i(X), \phi_X^i)$ is a Galois equivariant algebraic representative. □

4.4 Some structure results for algebraic representatives

We would also like to extend some basic structure results from Murre [Mur85] to the case of Galois equivariant regular homomorphisms; a crucial ingredient is the following technical, yet general, proposition extending a well-known result of Weil [Wei54, Lemma 9] over an algebraically closed field to the setting of arbitrary perfect fields.

PROPOSITION 4.8 [ACV16]. *Let $\underline{X}/K$ be a scheme of finite type over a perfect field K and let X be the base change of $\underline{X}$ to an algebraic closure k of K. If $\alpha \in \mathrm{CH}^i(X)$ is algebraically trivial, then there exist an abelian variety $\underline{A}/K$, a cycle $\underline{Z}$ on $\underline{A} \times_K \underline{X}$, and a pair of k-points t_1, t_0 on $\underline{A}$ such that $\alpha = Z_{t_1} - Z_{t_0}$.* □

Lemma 4.9 [Mur85, Lemma 1.6.2 and Corollary 1.6.3]. *Let k be an algebraically closed field, let $K \subseteq k$ be a perfect subfield such that k/K is algebraic, let X be a smooth projective variety over k, let A be an abelian variety over k, and assume that each is equipped with k/K descent data. Now let $\phi : \mathrm{A}^i(X) \to A(k)$ be a $\mathrm{Gal}(k/K)$-equivariant regular homomorphism. Then:*

(a) *there is an abelian subvariety $\underline{A}' \subseteq \underline{A}$ defined over K such that $\mathrm{Im}(\phi) = A'(k)$;*

(b) *there are an abelian variety $\underline{D}$ defined over K and a cycle $\underline{Z} \in \mathrm{CH}^i(\underline{D} \times_K \underline{X})$ such that the composition $D(k) \xrightarrow{w_Z} \mathrm{A}^i(X) \xrightarrow{\phi} A(k)$ surjects onto $\mathrm{Im}(\phi)$ and is induced by a K-morphism of abelian varieties $\underline{D} \to \underline{A}$;*

(c) *if ϕ is surjective, then there are an abelian variety $\underline{B}$ defined over K and a cycle $\underline{Z} \in \mathrm{CH}^i(\underline{B} \times_K \underline{X})$ such that the composition $B(k) \xrightarrow{w_Z} \mathrm{A}^i(X) \xrightarrow{\phi} A(k)$ is induced by a K-isogeny of abelian varieties $\underline{B} \to \underline{A}$;*

(d) *if ϕ is surjective, then there exists a cycle $\underline{Z} \in \mathrm{CH}^i(\underline{A} \times_K \underline{X})$ such that the composition*

$$A(k) \xrightarrow{w_Z} \mathrm{A}^i(X) \xrightarrow{\phi} A(k)$$

is induced by the K-morphism $r \cdot \mathrm{Id}_{\underline{A}} : \underline{A} \to \underline{A}$ for some non-zero integer r.

Proof. This is [Mur85, Lemma 1.6.2 and Corollary 1.6.3] in the case $K = k$. We will see that, with the arithmetic input of Proposition 4.8, the same proof establishes the present lemma. As (a) follows from the proof of (b), we begin by proving (b).

(b) The key point is that given $\alpha \in \mathrm{Im}(\phi)$, there exist an abelian variety $\underline{D}$ defined over K and a cycle $\underline{Y} \in \mathrm{CH}^i(\underline{D} \times_K \underline{X})$ such that $\alpha \in \mathrm{Im}(\phi \circ w_Y)$. In the case $k = K$, this is [Wei54, Lemma 9]; in the case $k \neq K$, this is Proposition 4.8.

Now, for all such pairs $(\underline{D}, \underline{Y})$, consider the abelian subvarieties $\mathrm{Im}(\phi \circ w_Y)$. Note that, since we are assuming that ϕ is $\mathrm{Gal}(k/K)$-equivariant, Remark 4.3 says that $\phi \circ w_Y$ descends to K. Clearly we can take a pair $(\underline{D}', \underline{Y}')$ such that $\mathrm{Im}(\phi \circ w_{Y'})$ is of maximal dimension. The claim is that $\mathrm{Im}(\phi \circ w_{Y'}) = \mathrm{Im}(\phi)$. If not, let $\phi(\alpha) \notin \mathrm{Im}(\phi \circ w_{Y'})$; then, by the assertion of the first paragraph, there exists a pair $(\underline{D}'', \underline{Y}'')$ such that $\phi(\alpha) \in \mathrm{Im}(\phi \circ w_{Y''})$. Now consider $\underline{D}''' := \underline{D}' \times_K \underline{D}''$ and the projections $\pi' : \underline{D}' \times_K \underline{D}'' \times_K \underline{X} \to \underline{D}' \times_K \underline{X}$ and $\pi'' : \underline{D}' \times_K \underline{D}'' \times_K \underline{X} \to \underline{D}'' \times_K \underline{X}$. Take

$$\underline{Y}''' := \pi'^* \underline{Y}' + \pi''^* \underline{Y}''.$$

Then $\mathrm{Im}(\phi \circ w_{Y'''}) \supset \mathrm{Im}(\phi \circ w_{Y'})$ and $\mathrm{Im}(\phi \circ w_{Y'''}) \supset \mathrm{Im}(\phi \circ w_{Y''})$, which is a contradiction, proving the claim, and thus (b).

(c) We have a surjection $\underline{D} \to \underline{A}$, and there exists a K-subabelian variety $\underline{B} \subseteq \underline{D}$ such that the induced map $\underline{B} \to \underline{A}$ is an isogeny. The restricted cycle $\underline{Z}|_{\underline{B} \times_K \underline{X}}$ provides the desired cycle.

(d) This follows from (c). The dual isogeny $g : \underline{A} \to \underline{B}$ has the property that there is a non-zero integer $r \in \mathbb{Z}$ such that the composition $\underline{A} \to \underline{B} \to \underline{A}$ is given by $r \cdot \mathrm{Id}_{\underline{A}}$. The cycle $(g \times_K \mathrm{Id}_{\underline{X}})^* \underline{Z}$ gives the desired cycle. $\qquad\square$

In light of Murre's results [Mur85] (see Theorem 3.4), we have the following consequence (see also Corollary 5.2 and Remark 5.4, which establishes a stronger statement for cohomology with $\mathbb{Q}_\ell$-coefficients, under the hypothesis $K \subseteq \mathbb{C}$).

Corollary 4.10. *Let X be a smooth projective variety over a perfect field K. Let $\underline{\mathrm{Ab}}^2(X)/K$ be the model of $\mathrm{Ab}^2(X)$ from Theorem 4.4. There is a natural inclusion of $\mathrm{Gal}(k/K)$-representations*

$$H^1(\underline{\widehat{\mathrm{Ab}^2(X)}}_k, \mathbb{Q}_\ell) \hookrightarrow H^3(X, \mathbb{Q}_\ell(1)).$$

For almost all ℓ, there is an inclusion $H^1(\underline{\widehat{\mathrm{Ab}^2(X)}}_k, \mathbb{Z}_\ell) \hookrightarrow H^3(X, \mathbb{Z}_\ell(1))$ of $\mathrm{Gal}(k/K)$-modules.

Proof. We aim to show that the two inclusions (3.4), i.e., $T_\ell \operatorname{Ab}^2(X) \hookrightarrow T_\ell \operatorname{A}^2(X)$, and (3.5), i.e., $T_\ell \operatorname{CH}^2(X) \hookrightarrow H^3(X, \mathbb{Z}_\ell(2))$, constructed by Murre are $\operatorname{Gal}(k/K)$-equivariant. The second inclusion (3.5) can be seen to be equivariant since the map is constructed via natural maps of sheaves, all of which have natural $\operatorname{Gal}(k/K)$-actions. Let us now show that the first inclusion (3.4) can be taken to be equivariant. Using the modification of [Mur85, Corollary 1.6.3] given in Lemma 4.9, there is a cycle $\underline{Z} \in \operatorname{CH}^2(\underline{\operatorname{Ab}}^2(X) \times_K \underline{X})$ such that

$$\phi_X^2 \circ w_Z : \operatorname{Ab}^2(X)(k) \to \operatorname{A}^2(X) \to \operatorname{Ab}^2(X)(k)$$

is induced by a K-isogeny $f : \underline{\operatorname{Ab}}^2(X) \to \underline{\operatorname{Ab}}^2(X)$. We also saw in the proof of Theorem 4.4 that both w_Z and ϕ_X^2 are Galois equivariant. Now take a prime number ℓ such that $(\ell, \deg(f)) = 1$ and ℓ is invertible in k; on points of order ℓ^ν, we have

$$f : \operatorname{Ab}^2(X)[\ell^\nu] \xrightarrow{\sim} \operatorname{Ab}^2(X)[\ell^\nu],$$

as Galois representations, for all $\nu > 0$. Therefore, $\operatorname{Ab}^2(X)[\ell^\nu]$ is a direct summand of the Galois representation $\operatorname{A}^2(X)[\ell^\nu]$ and we obtain a Galois equivariant inclusion

$$\operatorname{Ab}^2(X)[\ell^\nu] \hookrightarrow \operatorname{A}^2(X)[\ell^\nu].$$

Taking the associated Tate modules completes the proof. $\qquad\square$

5. Phantoms via algebraic representatives

We now prove Theorem B and complete the proof of Theorem A, by showing in Theorem 5.1 that the image of the Abel–Jacobi map in the intermediate Jacobian $J^3(X_\mathbb{C})$ descends to a field of definition of X.

THEOREM 5.1. *Suppose that X is a smooth projective variety over a field $K \subseteq \mathbb{C}$, and n is a non-negative integer. Assume that $\operatorname{A}^{n+1}(X_\mathbb{C})$ admits $(J_a^{2n+1}(X_\mathbb{C}), AJ)$ as an algebraic representative, where $J_a^{2n+1}(X_\mathbb{C})$ is the image of the Abel–Jacobi map $AJ : \operatorname{A}^{n+1}(X_\mathbb{C}) \to J^{2n+1}(X_\mathbb{C})$. Then $J_a^{2n+1}(X_\mathbb{C})$ has a distinguished model J over K making $AJ : \operatorname{A}^{n+1}(X_\mathbb{C}) \to J_a^{2n+1}(X_\mathbb{C})$ an $\operatorname{Aut}(\mathbb{C}/K)$-equivariant homomorphism, and there is a correspondence γ on $\widehat{J} \times_K X$ inducing for each prime number ℓ an inclusion of $\operatorname{Gal}(K)$-representations*

$$H^1(\widehat{J}_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{\gamma_*} H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)), \tag{5.1}$$

with image $\operatorname{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$. Consequently, if in addition $H^{2n+1}(X_\mathbb{C}, \mathbb{Q})$ is of geometric coniveau n, then (5.1) is an isomorphism.

Proof. Let X be a smooth projective variety defined over $K \subseteq \mathbb{C}$ and assume that $\operatorname{A}^{n+1}(X_\mathbb{C})$ admits an algebraic representative $(\operatorname{Ab}^{n+1}(X_\mathbb{C}), \phi_{X_\mathbb{C}}^{n+1})$. We have seen in Theorem 3.7 that $\operatorname{A}^{n+1}(X_{\overline{K}})$ admits an algebraic representative $(\operatorname{Ab}^{n+1}(X_{\overline{K}}), \phi_{X_{\overline{K}}}^{n+1})$ with $\operatorname{Ab}^{n+1}(X_{\overline{K}})_\mathbb{C} \cong \operatorname{Ab}^{n+1}(X_\mathbb{C})$, and in Theorem 4.4 that $\operatorname{Ab}^{n+1}(X_{\overline{K}})$ admits a distinguished model J over K. (Note that the distinguished model J over K is indeed an abelian variety; see Remark 4.1.) We also saw in Theorem 3.7 that $(\operatorname{Ab}^{n+1}(X_{\overline{K}}), \phi_{X_{\overline{K}}}^{n+1})$ is the $\mathbb{C}/\overline{K}$-trace of $(\operatorname{Ab}^{n+1}(X_\mathbb{C}), \phi_{X_\mathbb{C}}^{n+1})$ and in Theorem 4.4 that $\operatorname{Ab}^{n+1}(X_{\overline{K}})$ is equipped with a Galois descent datum making $\phi_{X_{\overline{K}}}^{n+1}$ Galois equivariant. Since the $\mathbb{C}/\overline{K}$-trace is constructed by fpqc descent [Con06], we see that $\phi_{X_\mathbb{C}}^{n+1} : \operatorname{A}^{n+1}(X_\mathbb{C}) \to \operatorname{Ab}^{n+1}(X_\mathbb{C})$ is $\operatorname{Aut}(\mathbb{C}/K)$-equivariant. Moreover, since the regular

homomorphism $\phi_{X_\mathbb{C}}^{n+1}$ is surjective, the model J is *distinguished* in the sense that it is uniquely determined by the natural Galois action on $\mathrm{A}^{n+1}(X_\mathbb{C})$.

We now make a general observation. Let $X_\mathbb{C}$ be a smooth projective variety defined over $\mathbb{C}$. Given an abelian variety $A_\mathbb{C}$ over $\mathbb{C}$ and a correspondence $Z_\mathbb{C} \in \mathrm{CH}^{n+1}(A_\mathbb{C} \times_\mathbb{C} X_\mathbb{C})$, then, using the functoriality of the Abel–Jacobi map with respect to the action of correspondences, we have a commutative diagram

$$
\begin{array}{ccc}
& \xrightarrow{\quad w_{Z_\mathbb{C}} \quad} & \\
A_\mathbb{C}(\mathbb{C}) \longrightarrow \mathrm{A}_0(A_\mathbb{C}) & \xrightarrow{Z_*} & \mathrm{A}^{n+1}(X_\mathbb{C}) \\
\downarrow{\scriptstyle AJ} & & \downarrow{\scriptstyle AJ} \\
\mathrm{Alb}(A_\mathbb{C})(\mathbb{C}) & \longrightarrow & J^{2n+1}(X_\mathbb{C})(\mathbb{C})
\end{array}
\tag{5.2}
$$

The differential of the map $\mathrm{Alb}(A_\mathbb{C}) \to J^{2n+1}(X_\mathbb{C})$ in the bottom row is the complexification of the map

$$
Z_* : H^{2\dim A - 1}(A_\mathbb{C}, \mathbb{Q}(\dim A_\mathbb{C} - 1)) \to H^{2n+1}(X_\mathbb{C}, \mathbb{Q}(n))
\tag{5.3}
$$

induced by the correspondence $Z_\mathbb{C}$ (e.g., [Voi07, Theorem 12.17]).

The natural identification $H^1(\widehat{A}_\mathbb{C}, \mathbb{Q}) \to H^{2\dim A - 1}(A_\mathbb{C}, \mathbb{Q}(\dim A_\mathbb{C} - 1))$ can be obtained via correspondences as the composition of isomorphisms $\Theta_* \circ \Gamma_{\lambda*}$, where Γ_λ is the graph of a polarization $\lambda : \widehat{A}_\mathbb{C} \to A_\mathbb{C}$, and $\Theta_* : H^1(A_\mathbb{C}, \mathbb{Q}) \to H^{2\dim A - 1}(A_\mathbb{C}, \mathbb{Q}(\dim A_\mathbb{C} - 1))$ is induced by the $(\dim A_\mathbb{C} - 1)$-fold intersection Θ of the class of the dual polarization on $A_\mathbb{C}$. In conclusion, we have a morphism

$$
H^1(\widehat{A}_\mathbb{C}, \mathbb{Q}) \xrightarrow{Z_* \circ \Theta_* \circ \Gamma_{\lambda*}} H^{2n+1}(X_\mathbb{C}, \mathbb{Q}(n)).
\tag{5.4}
$$

Finally, consider the special case where $(A_\mathbb{C}, \phi) = (J_a^{2n+1}, AJ)$, and where the cycle $Z_\mathbb{C}$ is taken as in [Mur85, Corollary 1.6.3] (see Lemma 4.9(d)), so that $AJ \circ w_{Z_\mathbb{C}}$ is given by $r \cdot \mathrm{Id}_{J_a^{2n+1}}$ for some non-zero integer r. The diagram (5.2) then shows that (5.4) is injective, with image $\mathrm{N}^n H^{2n+1}(X_\mathbb{C}, \mathbb{Q})$.

With this set-up, Lemma 4.9 implies that there exists a cycle $Z \in \mathrm{CH}^{n+1}(J \times_K X)$ such that the composition

$$
\mathrm{Ab}^{n+1}(X_{\overline{K}}) \xrightarrow{w_{Z_{\overline{K}}}} \mathrm{A}^{n+1}(X_{\overline{K}}) \xrightarrow{\phi_{X_{\overline{K}}}^{n+1}} \mathrm{Ab}^{n+1}(X_{\overline{K}})
$$

is induced by the K-morphism $r \cdot \mathrm{Id}_J : J \to J$ for some non-zero integer r, and the maps $w_{Z_{\overline{K}}}$ and $\phi_{X_{\overline{K}}}^{n+1}$ are $\mathrm{Gal}(K)$-equivariant. Let $\lambda : \widehat{J} \to J$ be a polarization and let Θ be the $(\dim J - 1)$-fold self-intersection of the class of the polarization on J. The correspondence $Z \circ \Theta \circ \Gamma_\lambda$, defined over K, induces a morphism of $\mathrm{Gal}(K)$-representations

$$
H^1(\widehat{J}_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{Z_* \circ \Theta_* \circ \Gamma_{\lambda*}} H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)).
\tag{5.5}
$$

Finally, if we assume further that $(J_a^{2n+1}(X_\mathbb{C}), AJ) \cong (\mathrm{Ab}^{n+1}(X_\mathbb{C}), \phi_{X_\mathbb{C}}^{n+1})$, then (5.4) and the comparison isomorphisms (1.3) imply that the map (5.5) is an inclusion, with image $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$. The comparison isomorphisms (1.3) also show that $H^{2n+1}(X_\mathbb{C}, \mathbb{Q})$ is of geometric coniveau n if and only if $H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is of geometric coniveau n; therefore, (5.5) is an isomorphism under the hypotheses given in the theorem. $\qquad \square$

Corollary 5.2. *Suppose that X is a smooth projective variety over a field $K \subseteq \mathbb{C}$. The abelian variety $J_a^3(X_{\mathbb{C}})$ has a distinguished model J over K. There is a correspondence γ on $\widehat{J} \times_K X$ such that for each prime number ℓ the correspondence induces an inclusion of $\mathrm{Gal}(K)$-representations*

$$H^1(\widehat{J}_{\overline{K}}, \mathbb{Q}_\ell) \xhookrightarrow{\gamma_*} H^3(X_{\overline{K}}, \mathbb{Q}_\ell(1)), \tag{5.6}$$

with image $\mathrm{N}^1 H^3(X_{\overline{K}}, \mathbb{Q}_\ell(1))$. Consequently, if $H^3(X_{\mathbb{C}}, \mathbb{Q})$ is of geometric coniveau 1 (e.g., if $X_{\mathbb{C}}$ is a uni-ruled threefold), then (5.6) is an isomorphism.

Proof. This follows from Theorem 5.1 and [Mur85] (see Theorem 3.4). That a uni-ruled threefold $X_{\mathbb{C}}$ has $H^3(X_{\mathbb{C}}, \mathbb{Q})$ of geometric coniveau 1 follows, via a decomposition of the diagonal argument [BS83], from the fact that $\mathrm{CH}_0(X_{\mathbb{C}})$ is supported on a surface. $\qquad\square$

Remark 5.3. Unlike the approach taken to prove Theorem 2.1, the approach to proving Theorem 5.1 using Murre's results [Mur85] does not seem to provide a splitting for the inclusion (5.6). However, since we are taking $2n + 1 = 3$, the arguments in the proof of Theorem 2.1 do provide a splitting for (5.6), induced by an algebraic correspondence if K is finitely generated. Indeed, let J' be the abelian variety over K and γ' be the correspondence provided by Theorem 2.1. The inclusion (5.6) and the inclusion $\gamma'_* : H^1(J'_{\overline{K}}, \mathbb{Q}_\ell) \hookrightarrow H^3(X_{\overline{K}}, \mathbb{Q}_\ell(1))$ have the same image, namely $\mathrm{N}^1 H^3(X_{\overline{K}}, \mathbb{Q}_\ell(1))$. Therefore, $H^1(J_{\overline{K}}, \mathbb{Q}_\ell)$ and $H^1(J'_{\overline{K}}, \mathbb{Q}_\ell)$ are isomorphic as $\mathrm{Gal}(K)$-representations. The splitting of γ'_* from Theorem 2.1 then provides a splitting of (5.6). If moreover K is finitely generated, then J and J' are K-isogenous, in which case composing the graph of such an isogeny with the correspondence from Theorem 2.1 that gives the splitting of γ'_* yields a correspondence splitting (5.6).

Remark 5.4. Comparing Corollary 5.2 to Corollary 4.10, we see that by assuming that $K \subseteq \mathbb{C}$ and by using the functoriality of the Abel–Jacobi map with respect to the action of correspondences, we are able to obtain the inclusion of $\mathrm{Gal}(K)$-representations via a correspondence, and to characterize its image in terms of the geometric coniveau filtration.

6. Complements on phantoms and algebraic representatives

In this section we discuss specialization of phantom abelian varieties and algebraic representatives. This has particular relevance to the question of phantoms from the perspective of Honda–Tate theory. Motivated by Theorem 5.1, we also establish some results on algebraic representatives for higher codimension cycles.

6.1 Phantoms, algebraic representatives, and specializations

Lemma 6.1. *Let $S = \mathrm{Spec}\, R$ be a discrete valuation ring with generic point $\eta = \mathrm{Spec}\, K$ and special point $0 = \mathrm{Spec}\, \kappa$, and let ℓ be a prime number invertible in κ. Let X/S be a smooth, projective scheme, and let A/η be an abelian variety equipped with a $\mathrm{Gal}(K)$-equivariant inclusion*

$$H^1(A_{\overline{\eta}}, \mathbb{Q}_\ell) \lhook\joinrel\longrightarrow H^{2n+1}(X_{\overline{\eta}}, \mathbb{Q}_\ell(n)). \tag{6.1}$$

(a) *Then A extends to an abelian scheme A/S.*

(b) *Suppose that the inclusion (6.1) has image $\mathrm{N}^n H^{2n+1}(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ and $\dim \mathrm{N}^n H^{2n+1}(X_{\overline{0}}, \mathbb{Q}_\ell) = \dim \mathrm{N}^n H^{2n+1}(X_{\overline{\eta}}, \mathbb{Q}_\ell)$. Then specialization induces an inclusion*

$$H^1(A_{\overline{0}}, \mathbb{Q}_\ell) \lhook\joinrel\longrightarrow H^{2n+1}(X_{\overline{0}}, \mathbb{Q}_\ell(n)), \tag{6.2}$$

with image $\mathrm{N}^n\, H^{2n+1}(X_{\overline{0}}, \mathbb{Q}_\ell)$. Moreover, if (6.1) is realized by a correspondence on $A \times_\eta X_\eta$, then (6.2) is realized by the specialization of the correspondence on $A_0 \times_0 X_0$.

(c) If A_η is a phantom for X_η (i.e., (6.1) is an isomorphism), then A_0 is a phantom for X_0 (i.e., (6.2) is an isomorphism).

Proof. (a) Since X_η has good reduction, it follows that $H^{2n+1}(X_{\overline{\eta}}, \mathbb{Q}_\ell)$ is unramified as a representation of $\mathrm{Gal}(K)$, and thus so is $H^1(A_{\overline{\eta}}, \mathbb{Q}_\ell)$. The Néron–Ogg–Shafarevich criterion shows that A extends to S.

(b) This follows from the canonical isomorphism $H^r(Y_{\overline{\eta}}, \mathbb{Q}_\ell) \cong H^r(Y_{\overline{0}}, \mathbb{Q}_\ell)$ provided by proper base change for any smooth projective Y/S. More precisely, since specialization is compatible with intersection [Ful98, 20.3.5], the specialization isomorphism gives an inclusion $\mathrm{N}^n\, H^{2n+1}(X_{\overline{\eta}}, \mathbb{Q}_\ell) \subseteq \mathrm{N}^n\, H^{2n+1}(X_{\overline{0}}, \mathbb{Q}_\ell)$. Thus, we have a commutative diagram

$$
\begin{array}{ccccc}
H^1(A_{\overline{\eta}}, \mathbb{Q}_\ell) & \xrightarrow{\;\sim\;} & \mathrm{N}^n\, H^{2n+1}(X_{\overline{\eta}}, \mathbb{Q}_\ell(n)) & \lhook\joinrel\longrightarrow & H^{2n+1}(X_{\overline{\eta}}, \mathbb{Q}_\ell(n)) \\
\downarrow{\scriptstyle\sim} & & \uparrow & & \downarrow{\scriptstyle\sim} \\
H^1(A_{\overline{0}}, \mathbb{Q}_\ell) & \dashrightarrow & \mathrm{N}^n\, H^{2n+1}(X_{\overline{0}}, \mathbb{Q}_\ell(n)) & \lhook\joinrel\longrightarrow & H^{2n+1}(X_{\overline{0}}, \mathbb{Q}_\ell(n))
\end{array}
$$

A dimension count implies that the induced dashed arrow is an isomorphism. Finally, since specialization is compatible with cup product in cohomology, the dashed arrow is in fact induced by the specialization of the correspondence inducing (6.1).

(c) follows from (b). □

As an immediate consequence, we find the following result.

COROLLARY 6.2. *With X/S as in Lemma 6.1, suppose further that* $\mathrm{char}(K) = \mathrm{char}(\kappa) = 0$ *and that* $\dim \mathrm{N}^1\, H^3(X_{\overline{\eta}}, \mathbb{Q}_\ell) = \dim \mathrm{N}^1\, H^3(X_{\overline{0}}, \mathbb{Q}_\ell)$. *Then* $\underline{\mathrm{Ab}}^2(X_{\overline{\eta}})_0$ *(Theorem 4.4 and Remark 4.5) and* $\underline{\mathrm{Ab}}^2(X_{\overline{0}})$ *are isogenous abelian varieties over* κ.

Proof. Use Lemma 6.1 and Corollary 5.2. □

Remark 6.3. If one assumes further in the corollary that $\phi^2_{X_{\overline{\eta}}}$ and $\phi^2_{X_{\overline{0}}}$ are isomorphisms (e.g., by [BS83, Theorem 1(i)], if $\mathrm{CH}_0(X_{\overline{\eta}})$ and $\mathrm{CH}_0(X_{\overline{0}})$ are supported on curves after any base change to an algebraically closed field), then one may conclude in fact that $\underline{\mathrm{Ab}}^2(X_{\overline{\eta}})_0$ and $\underline{\mathrm{Ab}}^2(X_{\overline{0}})$ are isomorphic.

In the setting of Mazur's original question (over a number field K), the following corollary shows that under the geometric coniveau hypothesis there is an abelian variety over K which interpolates the isogeny classes described by Honda–Tate theory.

COROLLARY 6.4. *Let S be a Dedekind scheme with generic point η. Let $X \to S$ be a smooth projective morphism, and let A_η/η be a phantom abelian variety for X_η in degree $2n+1$ (i.e., (6.1) is an isomorphism). Then A_η extends to an abelian scheme A/S such that, for each point $s \in S$, A_s is a phantom abelian variety for X_s in degree $2n+1$.*

Proof. Let A be the Néron model of A over S. Lemma 6.1 shows that every fiber is an abelian variety (and thus A is an abelian scheme), and that each fiber is a phantom for the corresponding fiber of X. □

However, like the Néron–Severi group, the rank of the geometric coniveau-n piece in degree $2n+1$ can vary in families; this behavior rules out the existence, in general, of a relative phantom abelian scheme modeling the coniveau-n piece in degree $2n+1$ for families of projective varieties. In particular, algebraic representatives need not specialize to algebraic representatives. The following provides such an example.

Example 6.5. Let $S = \operatorname{Spec} R$ be the spectrum of a discrete valuation ring as in Lemma 6.1, and let E be a CM field such that $[E : \mathbb{Q}] = 6$ and $[\widetilde{E} : \widetilde{E^{(+)}}] = 8$, where $\widetilde{E}$ and $\widetilde{E^{(+)}}$ are the normal closures over $\mathbb{Q}$ of E and its maximal totally real subfield, respectively. Let X/S be an abelian threefold such that $\operatorname{End}(X_{\overline{\eta}}) \cong \mathbb{Z}$ and $\operatorname{End}(X_0) \cong \operatorname{End}(X_{\overline{0}}) \supseteq \mathcal{O}_E$. On one hand, [Tan93, Theorem 1] implies that $\dim \mathrm{N}^1 H^3(X_{\overline{\eta}}, \mathbb{Q}_\ell) = 6$. On the other hand, [Tan93, Theorem 2] implies that if $\operatorname{char}(\kappa) = 0$, then $\dim \mathrm{N}^1 H^3(X_{\overline{0}}, \mathbb{Q}_\ell) = 18$, while if $\operatorname{char}(\kappa) > 0$ a further specialization argument guarantees that $\dim \mathrm{N}^1 H^3(X_{\overline{0}}, \mathbb{Q}_\ell) \geqslant 18$. Thus, there is no abelian scheme J/S so that for each $s \in S$, there is an isomorphism $H^1(J_{\overline{s}}, \mathbb{Q}_\ell) \cong \mathrm{N}^1 H^3(X_{\overline{s}}, \mathbb{Q}_\ell)$. Moreover, if $\operatorname{char}(\kappa) = 0$, this gives an example where $\underline{\mathrm{Ab}}^2(X_{\overline{\eta}})_0$ is not isogenous to $\underline{\mathrm{Ab}}^2(X_{\overline{0}})$.

6.2 Algebraic representatives for higher codimension cycles

In order to employ Theorem 5.1, one must first have an algebraic representative. However, outside of the cases of codimension-one, codimension-two, and codimension-d ($= \dim X$) cycles, there are not many results on their existence. Saito's result [Sai79, Theorem 2.2] (see Proposition 3.2) provides a criterion for establishing the existence of algebraic representatives. In this section we use this to provide some further criteria for when algebraic representatives exist. To begin, Proposition 6.6, suggested by the referee, asserts that if a variety is dominated by a second variety that admits an algebraic representative, then the first variety also admits an algebraic representative. Next, Proposition 6.7 (see also Remark 6.8) shows that the existence of algebraic representatives for codimension-3 cycles is a birational invariant.

PROPOSITION 6.6. *Let K be a field, n be a natural number, and $f : Y \to X$ be a dominant morphism of smooth projective varieties over K. Assume that there exists an algebraic representative for $\mathrm{A}^n(Y_{\overline{K}})$. Then there exists an algebraic representative for $\mathrm{A}^n(X_{\overline{K}})$.*

Proof. Let $\iota : Z \hookrightarrow Y$ be a closed subvariety of pure dimension $\dim X$ cut out by hyperplanes. The morphism $f|_Z : Z \to X$ is then generically finite, of degree r, say. Assume that there exists an algebraic representative $\mathrm{Ab}^n(Y_{\overline{K}})$ for $\mathrm{A}^n(Y_{\overline{K}})$. By the projection formula, the induced map

$$(f|_Z)_*(f|_Z)^* = (f|_Z)_* \iota^* f^* : \mathrm{A}^n(X_{\overline{K}}) \to \mathrm{A}^n(X_{\overline{K}})$$

is multiplication by r. This map is surjective because $\mathrm{A}^n(X_{\overline{K}})$ is a divisible group. It follows that the map

$$(f|_Z)_* \iota^* : \mathrm{A}^n(Y_{\overline{K}}) \to \mathrm{A}^n(X_{\overline{K}})$$

is surjective. Consider then a surjective regular homomorphism $\mathrm{A}^n(X_{\overline{K}}) \to A(\overline{K})$ to an abelian variety A defined over $\overline{K}$. Then the composition

$$\mathrm{A}^n(Y_{\overline{K}}) \xrightarrow{(f|_Z)_* \iota^*} \mathrm{A}^n(X_{\overline{K}}) \longrightarrow A(\overline{K})$$

is a surjective regular homomorphism. By the universal property of the algebraic representative $\mathrm{Ab}^n(Y_{\overline{K}})$ for $\mathrm{A}^n(Y_{\overline{K}})$, we find that $\dim A \leqslant \dim \mathrm{Ab}^n(Y_{\overline{K}})$. By [Sai79, Theorem 2.2] (see Proposition 3.2), it follows that $\mathrm{A}^n(X_{\overline{K}})$ admits an algebraic representative. $\square$

PROPOSITION 6.7. *Let $K \subseteq \Omega$ be a subfield of a universal domain Ω and let X and Y be smooth projective varieties over K with a correspondence $\Gamma \in \mathrm{CH}^d(Y \times X)$ such that $(\Gamma_\Omega)_* : \mathrm{CH}_0(Y_\Omega)_\mathbb{Q} \to \mathrm{CH}_0(X_\Omega)_\mathbb{Q}$ is surjective. Assume that $\mathrm{A}^3(Y_{\overline{K}})$ admits an algebraic representative (e.g., if $\dim Y \leqslant 3$). Then $\mathrm{A}^3(X_{\overline{K}})$ admits an algebraic representative.*

Proof. The assumption that $\mathrm{CH}_0(X_\Omega)$ is spanned by $\mathrm{CH}_0(Y_\Omega)$, via the action of the correspondence Γ, implies by a decomposition of the diagonal argument [BS83, Proposition 1] (see also [Via, Proposition 3.5] for the factorization assertion below) the existence of a positive integer N such that $N\Delta_X = Z_1 + Z_2$ in $\mathrm{CH}^d(X \times X)$, with Z_1 supported on $D \times X$ for some divisor $D \subset X$, and Z_2, when seen as a correspondence from X to X, factoring as $\Gamma \circ \Gamma'$ for some correspondence $\Gamma' \in \mathrm{CH}_d(X \times Y)$. Let us denote by $\widetilde{D}$ an alteration of D and $\iota : \widetilde{D} \to D \to X$ the natural morphism.

Consider a surjective regular homomorphism $\mathrm{A}^3(X_{\overline{K}}) \twoheadrightarrow A(\overline{K})$. We see from the divisibility of $\mathrm{A}^3(X_{\overline{K}})$ and from the decomposition of the diagonal that

$$\mathrm{A}^3(X_{\overline{K}}) = (N\Delta_X)^* \, \mathrm{A}^3(X_{\overline{K}}) = (\Gamma')^* \, \mathrm{A}^3(Y_{\overline{K}}) + \iota_* \, \mathrm{A}^2(\widetilde{D}_{\overline{K}}).$$

It follows that the regular homomorphism

$$\mathrm{A}^3(Y_{\overline{K}}) \oplus \mathrm{A}^2(\widetilde{D}_{\overline{K}}) \xrightarrow{(\Gamma')^* \oplus \iota_*} \mathrm{A}^3(X_{\overline{K}}) \longrightarrow A(\overline{K})$$

is surjective. Since both $\mathrm{A}^3(Y_{\overline{K}})$ and $\mathrm{A}^2(\widetilde{D}_{\overline{K}})$ admit an algebraic representative (by assumption and by Murre's theorem (Theorem 3.4), respectively), we get by the criterion of [Sai79, Theorem 2.2] (see Proposition 3.2) that there exists an integer M independent of the surjective regular homomorphism $\mathrm{A}^3(X_{\overline{K}}) \twoheadrightarrow A(\overline{K})$ such that $\dim A \leqslant M$. We conclude, by the same criterion, that $\mathrm{A}^3(X_{\overline{K}})$ has an algebraic representative. $\qquad\square$

Remark 6.8. The existence of an algebraic representative for codimension-3 algebraically trivial cycles is a birational invariant of smooth projective varieties over $\overline{K}$. Indeed, let $f : Y \dashrightarrow X$ be a dominant map of smooth projective varieties over $\overline{K}$. Then $f_* : \mathrm{CH}_0(Y_\Omega) \to \mathrm{CH}_0(X_\Omega)$ is surjective, and it follows from Proposition 6.7 that, if $\mathrm{A}^3(Y)$ has an algebraic representative, then $\mathrm{A}^3(X)$ has an algebraic representative.

Remark 6.9. Proposition 6.7 can be generalized by repeated use of the decomposition of the diagonal argument. Specifically, one can show that if X is a smooth projective variety over K such that $\mathrm{CH}_0(X_\Omega)_\mathbb{Q}, \ldots, \mathrm{CH}_l(X_\Omega)_\mathbb{Q}$ are spanned by the CH_0 of threefolds via the action of correspondences, then $\mathrm{A}^0(X_{\overline{K}}), \ldots, \mathrm{A}^{l+3}(X_{\overline{K}})$ admit algebraic representatives.

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Jeffrey D. Achter   j.achter@colostate.edu
Department of Mathematics, Colorado State University, Fort Collins,
CO 80523, USA

Sebastian Casalaina-Martin   casa@math.colorado.edu
Department of Mathematics, University of Colorado, Campus Box 395,
Boulder, CO 80309, USA

Charles Vial   vial@math.uni-bielefeld.de
DPMMS, University of Cambridge, Cambridge CB3 0WB, UK

Current address: Fakultät für Mathematik, Universität Bielefeld, Postfach 100131,
D-33501 Bielefeld, Germany