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MIXED NORM DECAY FOR THE KLEIN-GORDON EQUATION WITH INITIAL DATA IN L^p

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ABSTRACT. This paper gives necessary conditions for mixed norm estimates from L^p to $L^r(L^q)$ for solutions of the Klein-Gordon equation

$$u_{tt} = \Delta u + u = 0 \qquad u(x,0) = 0 \qquad u_t(x,0) = f(x).$$

These conditions are best possible if $p = 2$ or $r = \infty$ or $\frac{1}{p} + \frac{1}{q} \geq 1$.

The purpose of this paper is to examine estimates of the form

$$\|(1 + t)^\alpha u\|_{q,r} \equiv \left(\int_0^\infty ((1 + t)^\alpha \|u(\cdot, t)\|_q)^r \, dt\right)^{1/r} \leq C\|f\|_p \tag{1}$$

where $u(x, t)$ is the solution of the following Cauchy problem for the Klein-Gordon equation

$$\begin{cases} u_{tt} - \Delta_x u + u = 0 \\ u(x,0) = 0 \quad u_t(x,0) = f(x) \end{cases}$$

where $x \in \mathbb{R}^n$, $t > 0$. The expression $\alpha - 1/r$ gives a measure of the decay of the solution u. The operator $T_t f(x) = u(x, t)$ is a Fourier multiplier transformation: $\widehat{T_t f}(\xi) = \sin(t\sqrt{1 + |\xi|^2})\,(1 + |\xi|^2)^{-1/2} \hat{f}(\xi)$.

Define

$$\alpha_0(x, y, z) = \max\left\{nx + (n - 2)y + z - n,\ ny + z - \frac{n}{2}, \frac{n}{2} - nx,\right.$$
$$-(n - 2)x - ny - z + (n - 2),\ -\frac{n}{2}x + \frac{n}{2}y + z,$$
$$\left. -nx + z + \frac{n - 1}{2}\right\}.$$

Define $\mathcal{R}$ to be the region

$$\mathcal{R} = \left\{(x, y, z) : x \geq y,\ nx \leq y + \frac{n + 1}{2},\ ny + z \geq x + \frac{n - 3}{2},\ y \geq \frac{1}{2} - \frac{3}{2n}\right\}.$$

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THEOREM. *There can exist an estimate of the form* (1) *only if* $(1/p, 1/q, 1/r)$ *is in the region* $\mathcal{R}$. *In addition, if* $(1/p, 1/q, 1/r)$ *is in* $\mathcal{R}$ *and* (1) *holds then* $\alpha \geq \alpha_0(1/p, 1/q, 1/r)$.

This theorem gives necessary conditions on $(1/p, 1/q, 1/r)$ and α for there to be an estimate of the form (1). Except possibly for some boundary points $(1/p, 1/q, 1/r) \in \partial\mathcal{R}$ or $\alpha = \alpha_0$, the conditions of the theorem are both necessary and sufficient if either $p = 2$ or $r = \infty$ or $1/p + 1/q \geq 1$. At the boundary points, T_t may or may not be bounded if $\alpha = \alpha_0$. If $r = \infty$, T_t is bounded with $\alpha = \alpha_0$ but condition 6 of the proof shows that (1) does not hold for $\alpha = \alpha_0$ in the region where $a_0(1/p, 1/q, 1/r) = n/q + 1/r - n/2$ and $r < \infty$.

Estimates of the type (1) were needed to prove the existence of the scattering operator for the nonlinear Klien-Gordon equation $u_{tt} - \Delta_x u + u = |u|^{\gamma-1}u$ with small initial data. The estimates used in [5] by Strauss were of the form

$$\|T_t f\|_q \leq C t^{\alpha}\|f\|_p. \tag{2}$$

This is the special case $r = \infty$, which was examined completely in [2]. Estimates of the form (1) when $p = 2$ were used in ([5], II) to obtain the scattering operator for nonlinearities of the type $(V * u^2)u$. The case $p = 2$ has been considered in [3] and [6]. Similar $p = 2$ estimates, composed with differentiation operators in x and t, were used by Brenner in [1] to obtain the scattering operator for arbitrary data when the nonlinear term is $|u|^{\gamma-1}u$. Estimates with $p = 2$ are also used in Pecher [4].

The application of these estimates follow the argument of Strauss in [5]. Suppose the nonlinear term is Pu. To construct the solution of the nonlinear equation using (1) an auxiliary space Z, defined using norms of the form $L^{r_1}(\mathbb{R}, L^{q_1})$, is chosen so that the perturbation

$$\mathcal{P}u(x, t) = \int_s^t T_{t-\tau}Pu(x, \tau)\, d\tau$$

is a contraction mapping on Z. The parameters p, q, r and α are chosen to suit the nonlinear term Pu. It is useful to have as much freedom as possible in the choice of these parameters. The theorem also suggests estimates for T_t. These are given in the second section.

1. **Proof of the Theorem**. Many of the estimates given are modifications of arguments in [2], [3], or [6]. The calculations are therefore brief.

CONDITION 1. $(1/p \geq 1/q)$ See [3], p. 618. If $f_s(x) = s^{n/p}f(sx)$ and $u(x, t)$ is the solution with initial velocity $u_t(x, 0) = \cos(\sqrt{\ell^2 - 1}\, x_1)f_s(x)$ then $\|f_s\|_p = \|f\|_p$ but by [3]

$$\|t^{-\alpha}u\|_{q,r} \geq C s^{n(1/p-1/q)} \text{ for } s < \frac{1}{2t}.$$

Let s approach zero. Thus $\|t^{-\alpha}u\|_{q,r}$ can be bounded only if $1/p \geq 1/q$.

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CONDITION 2. ($n/p \le 1/q + (n+1)/2$) See [2], p. 434. Again let $f_s(x) = s^{n/p} f(sx)$. By [2] there exists a function f and constants a, b, c such that

$$|T_t f_s(x)| \ge c\, s^{n/p-(n+1)/2} \tag{3}$$

for $t - \frac{b}{s} < |x| < t - \frac{a}{s}$, $1 \le t \le 2$, and $s \ge 1$. Then if $u_s = T_t f_s$,

$$\|t^{-\alpha} u_s\|_{q,r} \ge \left(\int_1^2 \left(\int_{\mathbb{R}^n} |t^{-\alpha} T_t f_s(x)|^q \,dx\right)^{r/q} dt\right)^{1/r} \ge C\, s^{n/p-(n+1)/2-1/q}.$$

For this to be bounded as s tends to infinity we need $n/p \le 1/q + (n+1)/2$.

CONDITION 3. ($n/q + 1/r \ge 1/p + (n-3)/2$). We will use (3) and a duality argument as in [3], p. 619. Let $A_1 = \{x: 1 - b_1/s \le |x| \le 1 - a_1/s\}$ and $A_2 = \{t: 1 - b_2/s < t < 1 - a_2/s\}$. Suppose that f is real-valued and g is the characteristic function of the set A_1. Define also $v(x,t)$ to be $f_s(x)$ times the characteristic function of A_2. Then it follows from (3) that

$$\left|\int_0^\infty \int_{\mathbb{R}^n} t^{-\alpha} T_t g(x)\, v(x,t)\,dx\,dt\right| = \int_{A_2}\int_{A_1} t^{-\alpha} |T_t f_s|\,dx\,dt \tag{4}$$

$$\ge C\, s^{-2+n/p-(n+1)/2} \quad \text{for } s \ge 1.$$

Note that for f real-valued $T_t f_s$ is always positive or always negative in the region $A_1 \times A_2$. This is why the absolute value could be brought inside the integral in (4). Since

$$\|v\|_{q',r'} = C\, s^{n/p-n/q'-1/r'}$$

and $\|g\|_2 = C s^{-1/p}$ this shows that

$$\|t^{-\alpha} T_t\| \ge C s^{-2+n/p-(n+1)/2} / \|v\|_{q',r'} \|g\|_q = C\, s^{(n-3)/2+1/p-n/q-1/r}.$$

Here condition 3 follows by letting $s \to \infty$.

CONDITION 4. ($n/q \ge (n-3)/2$) See [3], p. 619. Let $f(x) = \sin\{(|x| - 1)/\sigma\}$ for $1 \le |x| \le M$ and $f(x) = 0$ otherwise. Then $\|f\|_p \le CM^{n/p}$. It follows from the calculations in [3] that

$$\|t^{-\alpha} T_t f\|_{q,r} \ge C\, \sigma^{n/q-(n-3)/2}(1 + M^{-\alpha+(n-1)/2+1/r}).$$

Let $\sigma \to 0$ or $M \to \infty$. Then for $t^{-\alpha} T_t$ to be bounded we need both

$$\frac{n}{q} - \frac{n-3}{2} \ge 0 \quad \text{and} \quad -\alpha + \frac{n-1}{2} + \frac{1}{r} - \frac{n}{p} \le 0. \tag{5}$$

Conditions 1–4 show that estimates of the form $\|(1+t)^{-\alpha} T_t f\|_{q,r} \le C\|f\|_p$ can occur only in the region $\mathfrak{R}$.

CONDITION 5. ($\alpha \ge n/p + (n-2)/q + 1/r - n$). A modification of Lemma 7 in [2] shows that there is a function f and constants a_1, a_2, b_1, b_2 such that

$$|T_t f_s(x)| > C\, t^{-n/p'} \tag{6}$$

for $t - b_1/s < |x| < t - a_1/s$, $a_2 t < s < b_2 t$, $s \ge 1$. Here as usual $f_s(x) = s^{n/p} f(sx)$.

As a result, if $u_s = t^{-\alpha}T_t f_s$ then

$$\|u_s\|_{q,r} \geq C\left(\int_{s/b_2}^{s/a_2}\left(\int_{t-b_1/s}^{t-a_1/s}|t^{-\alpha-n/p'}|^q R^{n-1}\,\mathrm{d}R\right)^{r/q}\mathrm{d}t\right)^{1/r}$$
$$= C\,s^{-\alpha-n/p'+(n+2)/q+1/r}.$$

Letting $s \to \infty$ completes the proof of this condition.

CONDITION 6. ($\alpha > n/q + 1/r - n/2$ if $r < \infty$ and $\alpha \geq n/q - n/2$ if $r = \infty$). Lemma 8 of [2] p. 434 shows that there exist a function f and constants a, b, c such that for any t ≥ 1 the subset of $\{x\colon at < |x| < bt\}$ where

(7) $$|T_t f(x)| \geq c\,t^{-n/2}$$

has measure at least ct^n. It follows therefore that

$$\|t^{-\alpha}T_t f\|_{q,r} \geq C\left(\int_1^\infty t^{r(-\alpha+n/q-n/2)}\,\mathrm{d}t\right)^{1/r} = \infty$$

if $r(-\alpha + n/q - n/2) \geq -1$ and $r < \infty$. If $r = \infty$ then the conclusion still follows easily from (7): $t^{-\alpha}T_t$ is not bounded if $-\alpha + n/q - n/2 < 0$.

CONDITION 7. ($\alpha \geq -(n-2)/p - n/q - 1/r + (n-2)$). Let $A_1 = \{x\colon s - b_1/s < |x| < s - a_1/s\}$ and $A_2 = \{t\colon s - b_2/s < t < s - a_2/s\}$. Suppose that f is real-valued. If g is the characteristic function of A_1 and $v(x, t)$ is $f_s(x) = s^{n/p}f(sx)$ times the characteristic function of A_2 then by (6),

$$\left|\int_0^\infty\int_{\mathbb{R}^n} t^{-\alpha}T_t g(x)v(x,t)\,\mathrm{d}x\,\mathrm{d}t\right| \geq C\int_{A_1}\int_{A_2} t^{-\alpha}t^{-n/p'}\,\mathrm{d}x\,\mathrm{d}t$$
$$\geq C\,s^{n-3-\alpha-n/p'}.$$

Since $\|v\|_{q',r'} = C\,s^{n/p-n/q'-1/r'}$ then

$$\|t^{-\alpha}T_t g\|_{q',r} \geq C\,s^{n-2-\alpha-n/q-1/r}.$$

Also $\|g\|_p = C\,s^{(n-2)/p}$, and so

$$\|t^{-\alpha}T_t\| \geq C\,s^{(n-2)/p'-n/q-1/r-\alpha} \quad \text{for} \quad s \geq 1.$$

Again, letting $s \to \infty$ completes the proof.

CONDITION 8. ($\alpha \geq -n/2p + n/2q + 1/r$) See [6] and [3], p. 620. Suppose that f is the inverse Fourier transform of the characteristic function of the set $A = \{x \in \mathbb{R}^n\colon |x_j| \leq \sqrt{\epsilon}, j = 1, , , . n\}$ and v is the inverse Fourier transform of the characteristic function of the set $Q = \{(x, t)\colon x \in A, |t| \leq \epsilon\}$. Then by the calculation in [3],

$$\|f\|_p \geq C\,\epsilon^{n/2p'}, \quad \|t^\alpha v\|_{q',r'} \leq C\,\epsilon^{n/2q+1/r-\alpha}$$

and

$$\left|\int_0^\infty\int_{\mathbb{R}^n} T_t f(x)v(x,t)\,\mathrm{d}x\,\mathrm{d}t\right| \geq C\,\epsilon^{n/2}.$$

Table

Vertex	decay $= \alpha_0 - 1/r$	Conditions involved
$P_0 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$	0	1, 6, 8, 9
$P_1 = \left(\frac{1}{2} + \frac{1}{n+1}, \frac{1}{2} - \frac{1}{n+1}, 0\right)$	$-\frac{n-1}{n+1}$	2, 3, 5, 7
$P_2 = \left(\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2} - \frac{1}{n-1}, 0\right)$	$\frac{n}{n-1}$	1, 3, 9
$P_3 = \left(\frac{1}{2} + \frac{1}{n-1}, \frac{1}{2} + \frac{1}{n-1}, 0\right)$	$\frac{n}{n-1}$	1, 2, 6
$P_4 = \left(\frac{1}{2}, \frac{1}{2} - \frac{1}{n}, 0\right)$	0	3, 7, 9
$P_5 = \left(\frac{1}{2} + \frac{1}{n}, \frac{1}{2}, 0\right)$	0	2, 5, 6
$P_6 = \left(\frac{1}{2} + \frac{1}{n+2}, \frac{1}{2} - \frac{1}{n+2}, 0\right)$	$-\frac{n}{n+2}$	5, 6, 7, 8, 9
$P_{11} = \left(\frac{1}{2} + \frac{n-2}{n(n-1)}, \frac{1}{2} - \frac{1}{n-1}, \frac{2}{n}\right)$	-1	2, 3, 5, 7, 8
$P_{21} = \left(\frac{n-3}{2n}, \frac{n-3}{2n}, \frac{n-3}{2n}\right)$	$1 + \frac{3}{2n}$	1, 3, 4, 9
$P_{01} = \left(\frac{n-1}{2n}, \frac{n-1}{2n}, \frac{1}{2}\right)$	0	1, 8, 9, 10
$P_{22} = \left(\frac{n-3}{2n}, \frac{n-3}{2n}, \frac{1}{2}\right)$	1	1, 4, 9, 10
$P_{41} = \left(\frac{1}{2}, \frac{n-3}{2n}, \frac{1}{2}\right)$	$-\frac{1}{2}$	3, 4, 7, 9, 10
$P_{61} = \left(\frac{n+3}{2(n+2)}, \frac{n^2-n-4}{2n(n+2)}, \frac{1}{2}\right)$	$-\frac{n+1}{n+2}$	7, 8, 9, 10
$P_{62} = \left(\frac{n+1}{2n}, \frac{n-3}{2n}, \frac{n+1}{2n}\right)$	-1	3, 4, 8, 10
$P_{63} = \left(\frac{(n+3)(n-1)}{2n^2}, \frac{n-3}{2n}, \frac{(n+3)(n-1)}{2n^2}\right)$	$-\frac{5n+3}{4n}$	2, 3, 4, 8
$P'_0 = \left(\frac{1}{2}, \frac{1}{2}, 1\right)$	0	1, 6, 8
$P'_{01} = \left(\frac{n-1}{2n}, \frac{n-1}{2n}, 1\right)$	0	1, 8, 10
$P'_3 = \left(\frac{1}{2} + \frac{1}{n-1}, \frac{1}{2} + \frac{1}{n-1}, 1\right)$	$\frac{n}{n-1}$	1, 2, 8

Table (*Concluded*)

Vertex	decay = $\alpha_0 - 1/r$	Conditions involved
$P'_5 = \left(\frac{1}{2} + \frac{1}{n}, \frac{1}{2}, 1\right)$	0	2, 5, 6
$P'_6 = \left(\frac{1}{2} + \frac{1}{n+2}, \frac{1}{2} - \frac{1}{n+2}, 1\right)$	$-\frac{n}{n+2}$	5, 6, 8
$P'_{11} = \left(\frac{1}{2} + \frac{n-2}{n(n-1)}, \frac{1}{2} - \frac{1}{n-1}, 1\right)$	-1	2, 5, 8
$P'_{21} = \left(\frac{n-3}{2n}, \frac{n-3}{2n}, 1\right)$	1	1, 4, 10
$P'_{62} = \left(\frac{n+1}{2n}, \frac{n-3}{2n}, 1\right)$	-1	4, 8, 10
$P'_{63} = \left(\frac{(n+3)(n-1)}{2n^2}, \frac{n-3}{2n}, 1\right)$	$-\frac{5n+3}{4n}$	2, 4, 8

Therefore

$$\|t^{-\alpha}T\| \geq C\epsilon^{n/2}/\|f\|_p\|t^{\alpha}v\|_{q',r'} \geq C\epsilon^{n/2p-n/2q-1/r+\alpha}.$$

Since ϵ can be arbitrarily small this shows that $\|t^{-\alpha}T\|$ is not bounded if $n/2p - n/2q - 1/r + \alpha < 0$.

CONDITION 9. ($\alpha \geq n/2 - n/p$). Let f be the function in (7). Suppose that g is the characteristic function of $A = \{x: s + as < |x| < s + bs\}$ and $v(x, t)$ is $f(x)$ multiplied by the characteristic function of $A_2 = \{t: s + d < t < s + c\}$. Then

$$\left|\int_0^\infty \int_{\mathbb{R}^n} t^{-\alpha}T_t g(x)v(x,t)\,dx\,dt\right| \geq \int_{A_2}\int_{A_1} |T_t f|t^{-\alpha}\,dt \geq C s^{-\alpha+n/2}.$$

Hence

$$\|t^{-\alpha}T_t g\|_{q,r} \geq C s^{-\alpha+n/2}/\|v\|_{q',r'} = C s^{-\alpha+n/2}$$
$$= C s^{-\alpha+n/2-n/p}\|g\|_p \quad \text{for} \quad s \geq 1.$$

Now let $s \to \infty$.

CONDITION 10. ($\alpha \geq -n/p + 1/r + (n-1)/2$). This condition was obtained already in (5).

The proof of the theorem is now complete.

2. **Sufficient conditions**. To examine the boundedness of $(1+t)^{-\alpha}T_t$ we now look at the "vertices" determined by $\mathcal{R}$ and α_0. Those are the points $P = (1/p, 1/q, 1/r)$ and values of $\alpha_0 = \alpha_0(P)$ such that if estimates $\|(1+t)^{-\alpha_0}T_t f\|_{q,r} \leq C\|f\|_p$ can be

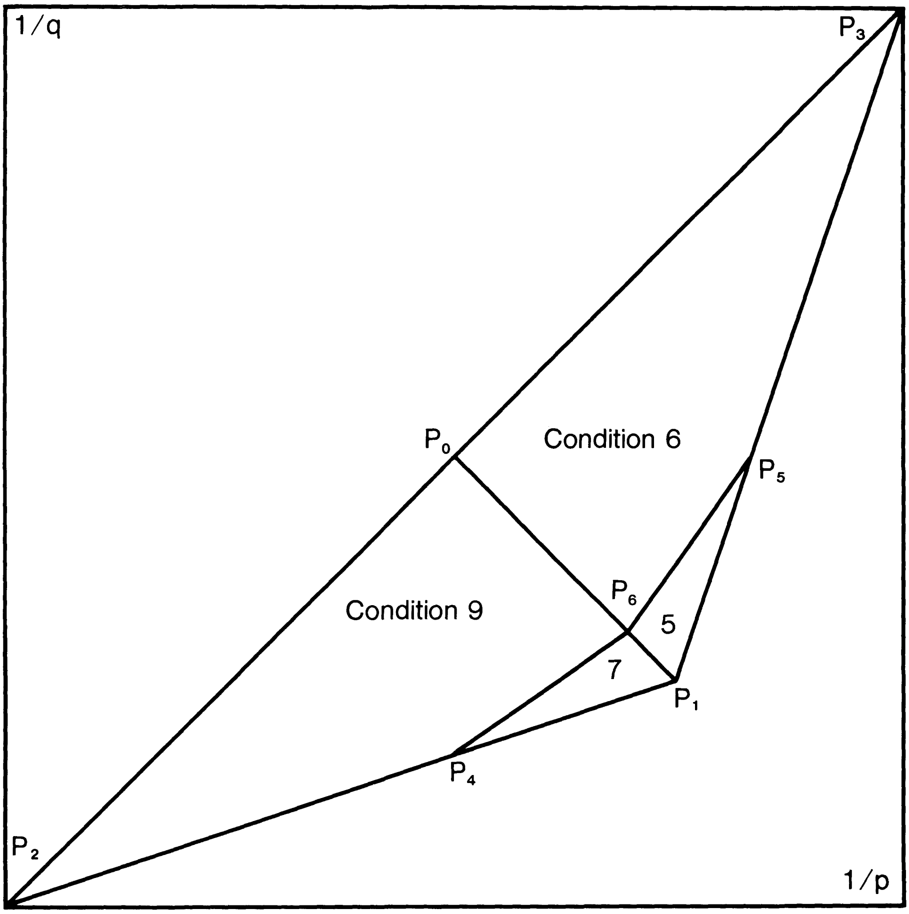

Figure 1 n = 3, 1/r = 0

obtained at the vertices then all the other estimates will follow by interpolation. The vertices suggested by the theorem are given in the table.

The cross-sections $1/r = 0$, $n = 3$ and $1/r = 1/2$, $n = 3$ are drawn in diagrams 1 and 2. As $1/r$ increases the region determined by condition 8 grows from the line segment P_0P_6. For $1/r \geq 1/2$ condition 10 replaces condition 9.

As pointed out earlier condition 6 implies that in some cases the estimate at P with $\alpha = \alpha_0$ might not be possible (for example, at P_0', P_2', P_4', or P_6'). We will therefore be willing to settle for estimates arbitrarily close to P, in the interior of $\mathcal{R}$ with $\alpha > \alpha_0$. For example, the estimate near P_{41} was obtained in [3]. Specifically it was shown that $\|T_t f\|_{q,r} \leq C\|f\|_2$ holds for a set of points $(1/2, 1/q, 1/r)$ which contains P_{41} in its closure.

The estimates at $P_0, P_1, P_2, P_3, P_4, P_5$, and P_6 are all contained in [2]. Estimates near the primed vertices can be obtained from those at or near the unprimed vertices by using

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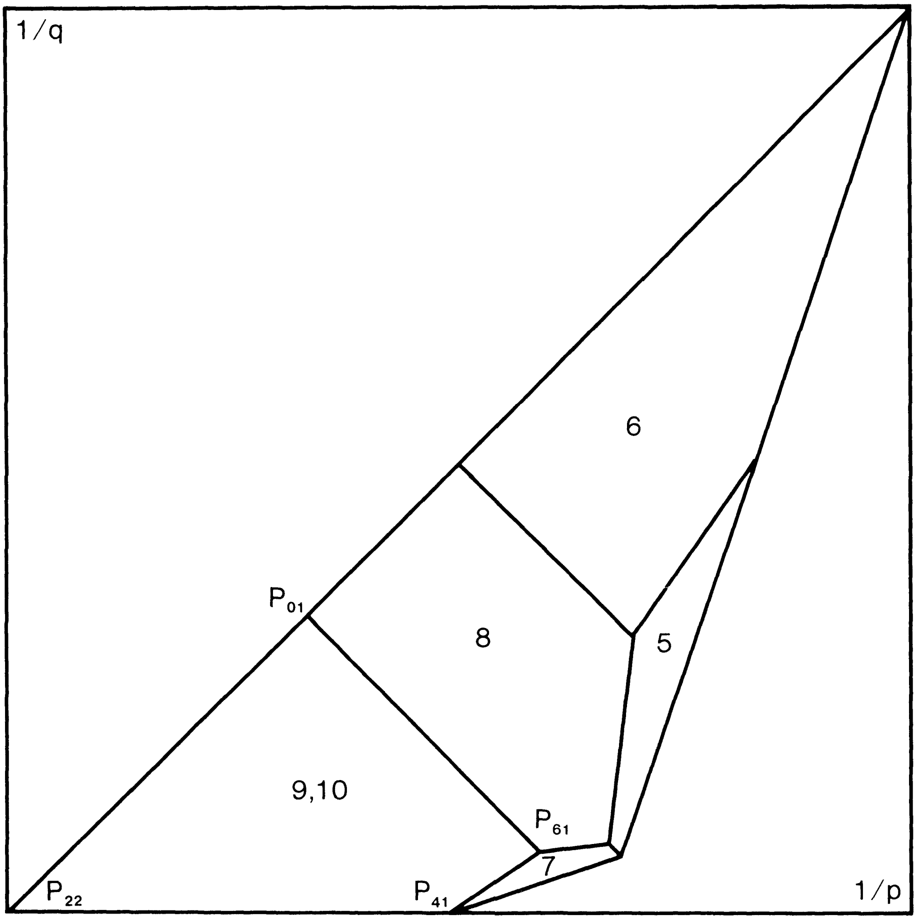

Figure 2 n = 3, 1/r = 1/2

Hölder's inequality since

$$\int_0^\infty \|(1+t)^{-\alpha}T_t f\|_q \frac{dt}{(1+t)} \le C\left(\int_0^\infty \|(1+t)^{-\beta}T_t f\|_q^r \frac{dt}{(1+r)}\right)^{1/r}$$

whenever $\beta < \alpha$. The vertices that still require estimates are therefore $P_{11}, P_{21}, P_{01}, P_{22}$, P_{61}, P_{62}, and P_{63}.

When $n = 3$ this simplifies because $P_{21} = P_2$, $P_{11} = P_{62} = P_{63} = (\frac{2}{3}, 0, \frac{2}{3})$, $P_{01} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$, $P_{22} = (0, 0, \frac{1}{2})$, $P_{61} = (\frac{3}{5}, \frac{1}{15}, \frac{1}{2})$. Of these the most interesting for scattering are P_{62} and P_{61}.

When $n = 2$ the unresolved vertices are $P_{11} = (\frac{3}{4}, 0, \frac{1}{4})$, $P_{6*} = (\frac{2}{3}, 0, \frac{1}{3})$, $P_{01} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, and $P_{22} = (0, 0, \frac{1}{2})$. Here $\alpha_0(P_{11}) = -\frac{1}{4}$ and $\alpha_0(P_{6*}) = -\frac{1}{3}$.

In the case $n = 1$ the remaining points are $P_{01} = (0, 0, \frac{1}{2})$ and $P_{6*} = (\frac{2}{3}, 0, \frac{1}{6})$. In this case $\alpha(P_{01}) = \frac{1}{2}$ and $\alpha(P_{6*}) = -\frac{1}{6}$.

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