ISOMORPHISMS OF SIMPLE RINGS

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1. Let A be a simple ring with minimum condition, and B_1, B_2, and C be regular subrings of A such that $B_i > C$, $i = 1, 2$. A pair of isomorphisms σ_i of B_i into A such that $\sigma_i|C$ is the identity, and that $B_i\sigma_i$ are regular subrings of A $(i = 1, 2)$, is called *compatible* if $\sigma_1|B_1 \cap B_2 = \sigma_2|B_1 \cap B_2$. Here $\sigma|X$ means the restriction of σ to X. Bialynicki-Birula has proved some necessary and sufficient conditions that every compatible pair (σ_1, σ_2) has a common extention to an automorphism σ of A **(1)**. When A is a division ring, he shows that the linear disjointness of the division subrings B_1 and B_2 is necessary and almost sufficient for the existence of a common extension of any compatible pair. In this note we extend this result to simple rings with minimum condition and point out that his general necessary and sufficient conditions are valid for continuous transformation rings.

2. Let A be a continuous transformation ring, that is, $A = L(M, N)$, where M is a (right) vector space over a division ring D and N is a (left) total subspace of the full dual of M. If s is a unit in A we denote by I_s the inner automorphism $x \to s^{-1}xs$ of A. Let G be an automorphism group of A. We denote by A^G the fixed subring of G, G^0 the normal subgroup of all inner automorphisms belonging to G, and by T_G the subring generated by all linear transformations s such that $I_s \in G^0$. The group G is called *complete* if for every unit $s \in T_G$, $I_s \in G^0$. The product of the index $[G : G^0]$ and the dimension $[T_G : Z]$ is called the *reduced order* of G. An automorphism group G is called *regular* if it is complete, having finite reduced order, and if T_G is simple. When G is regular, T_G is the centralizer in A of the fixed subring $A^G : T^G = Z(A, A^G)$. Here $Z(A, X)$ denotes the centraliser of X in A; $Z = Z(A, A)$. A subring C of the ring A is called *regular* if it is the fixed subring of a regular automorphism group. We have the following fundamental theorems of Galois theory due to Rosenberg and Zelinsky **(4)**.

(a) *Let A be a continuous transformation ring, G_0 a regular group of automorphisms of A, and C_0 the fixed ring under G_0. Then we have the usual fixed ring – Galois group correspondence as a* 1–1 *correspondence between regular subgroups G of G_0 and continuous transformation subrings C of A that contain C_0 and have simple centralizers $Z(A, C)$ in A.*

(b) *Let A, C, C_0 be as in* (a). *Then every isomorphism σ of C into A which is the identity on C_0 can be extended to an automorphism of A if $C\sigma$ has simple centralizer $Z(A, C\sigma)$ in A.*

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If C is a subring of A, we denote by $G(A/C)$ the group of C-automorphisms of A. If C is regular and X is a subset of C, we denote by $G(C, A/X)$ the set of X-isomorphisms σ of C into A such that $C\sigma$ has simple centralizer in A.

THEOREM 1. *Let C be a Galois subring of the continuous transformation ring A, and let B_1, B_2 be regular subrings between C and A. Then the following five statements are equivalent:*

(1) $G(A/B_2)|B_1 = G(B_1, A/B_1 \cap B_2)$.

(2) *Every compatible pair* (σ_1, σ_2), *where* $\sigma_i \in G(B_i, A/C)$, $i = 1, 2$, *has a common extension* $\sigma \in G(A/C)$.

(3) $G(A/B_1 \cap B_2) = G(A/B_2)\cdot G(A/B_1)$.

(4) $G(A/B_1)|B_2 = G(B_2, A/B_1 \cap B_2)$.

(5) $G(A/B_1 \cap B_2) = G(A/B_1)\cdot G(A/B_2)$.

If the five conditions are satisfied and if the centre Z of A is infinite, then the subring $B_1 \cap B_2$ is regular whose centralizer $Z(A, B_1 \cap B_2)$ is either $Z(A, B_1)$ or $Z(A, B_2)$.

Proof. When A is a simple ring with minimum condition, Bialynicki-Birula proves the equivalence of (1)–(5) and their consequence that $Z(A, B_1 \cap B_2)$ is either $Z(A, B_1)$ or $Z(A, B_2)$. His proof, which depends only on the fundamental theorems of Galois theory and the simplicity of T_G, can be applied verbatim to the case of continuous transformation rings. It follows from the simplicity of $Z(A, B_1 \cap B_2)$ that the group $G(A/B_1 \cap B_2)$ is regular. Then

$$B_1 \cap B_2 \leqslant A^{G(A/B_1-B_2)} = A^{G(A/B_1)\cdot G(A/B_2)} \leqslant A^{G(A/B_1)} \cap A^{G(A/B_2)} = B_1 \cap B_2$$

is the fixed subring of the regular group $G(A/B_1 \cap B_2)$ and is regular.

3. Now, let A be a simple ring with minimum condition, and C a regular subring with the Galois group $G = G(A/C)$. We consider A as a left C-module. Then G and the ring A_r of right multiplications are C-endomorphisms of A. Hence, GA_r is a subring of the ring $E_C(A)$ of C-endomorphisms of A. In (**1**; (i), (ii), and (iv)), Bialynicki-Birula shows that $E_C(A)$ and GA_r have the same finite dimension over A_r. Hence we have the following lemma.

LEMMA 1. $E_C(A) = GA_r$.

We can also consider A as a right A-module. The centralizer of A_r is the ring of left multiplications A_l. Therefore, the centralizer of the (C, A)-module A is $Z(A, C)_l$. Since C is a regular subring, $Z(A, C)$ is a simple ring with minimum condition. Hence, the left $Z(A, C)$-module A is free of finite dimension. The (C, A)-submodules of A are right ideals; therefore, the (C, A)-module A satisfies the minimum condition. Since the counter-module is of finite type, the ring $C' \otimes A$ satisfies the minimum condition, where C' is the opposite of C and the tensor product is taken over the field

$$F = Z(C') \cap Z = Z(C) \cap Z$$

(cf. **2**; Prop. 8 p. 26). The subfield F is Galois in Z, for it is the fixed field of the finite automorphism group $G|Z$. Hence $Z(C') \otimes Z$ is a direct sum of a finite number of fields and the ring $C' \otimes A$ is semi-simple with minimum condition.

Let u be an irreducible idempotent in $Z(A, C)$. Then uA is an irreducible (C, A)-module, for its centralizer is isomorphic with the division ring $uZ(A, C)u$. If u_1, u_2 are irreducible idempotents in $Z(A, C)$, then there are matrix units u_{ij} in $Z(A, C)$ such that $u_{ij}u_{ji} = u_i$, $i, j = 1, 2$. Then $u_1 \to u_{21}$ induces a (C, A)-isomorphism of u_1A with u_2A. To summarize, we have the following lemma.

LEMMA 2. *The ring A is a homogeneous completely reducible (C, A)-module.*

Let B_1, B_2 be regular subrings between A and C. Form the groups $B_1 \otimes_{B_1 \cap B_2} B_2$ and B_1B_2. We call the map $\sum x_i \otimes y_i \to \sum x_i y_i$ the natural homomorphism of $B_1 \otimes_{B_1 \cap B_2} B_2$ onto B_1B_2. We say that B_1 is *linearly disjoint from* B_2 if the natural homomorphism of $B_1 \otimes_{B_1 \cap B_2} B_2$ onto B_1B_2 is an isomorphism. The subrings B_1 and B_2 are said to be *linearly disjoint* if each is linearly disjoint from the other.

THEOREM 2. *Let A be a simple ring with minimum condition, C a Galois subring, and let B_1, B_2 be regular subrings between A and C. Suppose that the centre Z of A is infinite. Then every compatible pair (σ_1, σ_2), where $\sigma_i \in G(B_i, A/C)$, $i = 1, 2$, has a common extension $\sigma \in G(A/C)$ if and only if the subrings B_1, B_2 are linearly disjoint and either $Z(A, B_1 \cap B_2) = Z(A, B_1)$ or $Z(A, B_1 \cap B_2) = Z(A, B_2)$.*

Proof. (i) *Necessity.* Suppose that (B_1, B_2) satisfies Condition (2) of Theorem 1. Then, by Theorem 1, $G(A/B_1 \cap B_2) = G(A/B_1)\cdot G(A/B_2)$ and $B_1 \cap B_2$ is a regular subring of A whose centralizer is either $Z(A, B_1)$ or $Z(A, B_2)$. Hence A is a completely reducible left $B_1 \cap B_2$-module and B_1 is a direct summand of A. Then every $B_1 \cap B_2$-homomorphism of B_1 into A can be extended to a $B_1 \cap B_2$-endomorphism of A. It follows from Lemma 1 and Theorem 1 that the set of $B_1 \cap B_2$-homomorphisms of B_1 into A is

$$G(A/B_1 \cap B_2)A_r|B_1 = G(A/B_1)\cdot G(A/B_2)A_r|B_1 = G(A/B_2)A_r|B_1.$$

Therefore every $B_1 \cap B_2$-homomorphism of B_1 into A is the restriction to B_1 of a B_2-endomorphism of A. It follows that B_2 is linearly disjoint from B_1, and, by symmetry, B_1, B_2 are linearly disjoint.

(ii) *Sufficiency.* Suppose that B_2 is linearly disjoint from B_1 and that $Z(A, B_1 \cap B_2) = Z(A, B_2)$. Then any $B_1 \cap B_2$-homomorphism of B_1 into A can be extended to a B_2-homomorphism of B_2B_1 into A. Since A is a completely reducible left B_2-module, a B_2-homomorphism of B_2B_1 into A is the restriction to B_2B_1 of a B_2-endomorphism of A. Then the group $E_{B_1 \cap B_2}(B_1, A)$ of $B_1 \cap B_2$-homomorphisms of B_1 into A is $G(A/B_2)A_r|B_1$.

The group $E_{B_1 \cap B_2}(B_1, A)$ can be considered as a left B_1-module and a right A-module. As a (B_1, A)-module, it is completely reducible, for

$$E_{B_1 \cap B_2}(B_1, A) = \sum_{\sigma \in G(A/B_2)} \sigma A_r|B_1$$

and each $\sigma A_r|B_1$ is a completely reducible (B_1, A)-module by Lemma 2. Let σ be an isomorphism belonging to $G(B_1, A/B_1 \cap B_2)$. Then there are $\sigma_1, \ldots, \sigma_m$ in $G(A/B_2)$ such that

$$\sigma A_r \subseteq \sum_{i=1}^{m} \sigma_i A_r|B_1.$$

By the complete reducibility of $E_{B_1 \cap B_2}(B_1, A)$, an irreducible component of σA_r is (B_1, A)-isomorphic with, say, an irreducible component of $\sigma_1 A_r|B_1$. Since σA_r and $\sigma_1 A_r|B_1$ are homogeneous of equal dimension over $B_1' \otimes A$ (B_1' is the opposite of B_1 and the tensor product is taken over $Z(B_1') \cap Z$), σA_r is (B_1, A)-isomorphic with $\sigma_1 A_r|B_1$. Let h be a (B_1, A)-isomorphism of σA_r onto $\sigma_1 A_r|B_1$ and let $\sigma h = \sigma_1 a_r|B_1$. Then, for any b in B_1, we have

$$(b_r\sigma)h = b_r(\sigma h) = b_r\sigma_1 a_r|B_1 = \sigma_1[(b\sigma_1)a]_r|B_1$$

and

$$(b_r\sigma)h = [\sigma(b\sigma)_r]h = (\sigma h)(b\sigma)_r = \sigma_1[a(b\sigma)]_r|B_1.$$

It follows that $(b\sigma_1)a = a(b\sigma)$ for any $b \in B_1$. If $b \in B_1 \cap B_2$, then $b = b\sigma_1 = b\sigma$ and, consequently, a is a unit in $Z(A, B_1 \cap B_2)$. Since it is assumed that $Z(A, B_1 \cap B_2) = Z(A, B_2)$, we have $\sigma = \sigma_1 I_a|B_1 \in G(A/B_2)|B_1$. Therefore $G(B_1, A/B_1 \cap B_2) = G(A/B_2)|B_1$.

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