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HOMOTOPY EQUIVALENCE OF A COFIBRE FIBRE COMPOSITE

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0. Introduction. Consider the following commutative diagram in Top, the category of topological spaces

$$\begin{array}{ccc} A & \xrightarrow{f_0} & A' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ E & \xrightarrow{f_1} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f_2} & B' \end{array}$$

in which j and j' are cofibrations, p and p' are (Hurewicz) fibrations and f_0, f_1 and f_2 are homotopy equivalences. It is well known that the pairs (f_0, f_1) and (f_1, f_2) are homotopy equivalences in the category whose objects are maps in Top and whose morphisms are commutative squares (c.f. [**1**, 7.4.1], respectively [**2**, 3.2]).

In this note, we give some conditions under which the triple (f_0, f_1, f_2) is a homotopy equivalence in the category $\mathscr{T}$ whose objects are cofibration-fibration composites and whose morphisms are commutative diagrams of the type shown above. We consider variations on this theme, applications of which include relative and a base pointed version of a theorem of Dold, and a comparison theorem for ex-spaces.

The following example testifies to the fact that some extra condition is indeed necessary.

Example. Let $B = B' = I$ the closed unit interval, $E = E' = I \times I$ with $p = p'$, the projection on the first factor, suppose further that $A = I \times \{0\}$, $A' = I \times \{0, 1\} \cup \{0\} \times I$ with f_1, f_2 identities and f_0 the inclusion, then (f_0, f_1, f_2) is not a homotopy equivalence in $\mathscr{T}$.

The paper is divided into three sections; section one gives the main theorem and results, section two considers conditions for the triple to be a homotopy equivalence in subcategories of $\mathscr{T}$ (in which f_2 or f_0 are identities) to give the promised applications; finally, section three gives a sketch proof of the main theorem.

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I would like to thank Ronnie Brown for his motivating interest and for helpful comments on an early manuscript.

1. Homotopy equivalence of maps of pairs. Our technique in answering the question posed in the introduction is to transfer the problem to the category Toppair of topological pairs and their maps. By abuse of notation, we regard the composite

$$A \xrightarrow{j} E \xrightarrow{p} B,$$

for example, as a map $p : (E, A) \to (B, p(A))$ in Toppair. This is possible since every cofibration is an embedding [**8**]. We generalize some of the techniques and results of [**2**] to the category Toppair.

Consider the following commutative diagram

$$\begin{array}{ccc} (E, E_0) & \xrightarrow{f} & (E', E_0') \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ (B, B_0) & \xrightarrow[g]{} & (B', B_0') \end{array}$$

in Toppair. Let p and p' have the *Covering Homotopy Property* (*CHP*) in Toppair with respect to the pairs (E, E_0), (E', E_0'), $(E, E_0) \times I$ and $(E', E_0') \times I$. (The concept of fibration in a category with homotopy is defined for example in [**5**]; the more general notion of *CHP* in such a category is defined in the obvious way.)

THEOREM 1.1. (Homotopy equivalence of maps of pairs). *In the situation described above, if f and g are homotopy equivalences in* Toppair, *then the pair* (f, g) *is a homotopy equivalence in the category of commutative squares in* Toppair. *More precisely, let g' be a homotopy inverse of g and let $H : g'g \simeq 1$ and $H' : gg' \simeq 1$ be homotopies of pairs which are constant on the first half of the homotopy, then there is a map $f' : (E', E_0') \to (E, E_0)$ over g'* (i.e. $pf' = g'p'$) *and a homotopy $H : f'f \simeq 1$ over H; moreover, ff' is homotopic to the identity over the conjugate homotopy*

$$H' + gH(g' \times 1) - H'(gg' \times 1) : gg' \simeq 1.$$

An outline of the proof is removed to the Appendix.

Remark. The Toppair maps p and p' are seen to have the required *CHP* if (a) p, p' and their restrictions are fibrations in Top and the pairs (E, E_0) and (E', E_0') are closed with the *H.E.P.* (this is a simple consequence of [**8**; **4**]) or again (b) if p, p' and their restrictions are Serre fibrations and (E, E_0) and (E', E_0') are polyhedral pairs or (c) if p and p' are fibrations in Toppair or again (d) if E_0 and E_0' are base points and p and p' are fibrations in Top_*—the category of pointed topological spaces and pointed maps.

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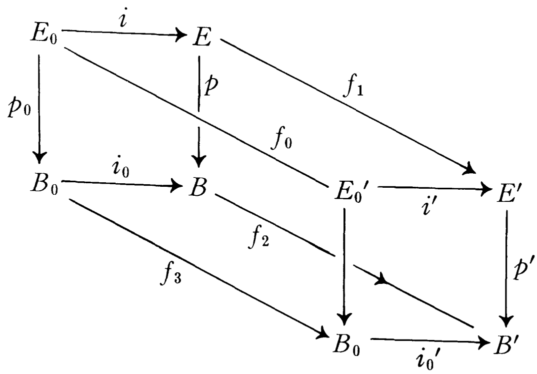
The remainder of this section gives some consequences of 1.1, the above remark and the technique discussed earlier. We need only recall conditions under which a map in Toppair is a homotopy equivalence, namely, that the pairs involved have the *WHEP* (weak homotopy extension property) and the map and its restriction are ordinary (Top) homotopy equivalences. (c.f. for example, [**1**; 7.4.1], [**5**; 6.1]).

Consider the following commutative diagram in Top,



in which i and i' are cofibrations with $i(E_0)$ and $i'(E_0')$ closed, in which i_0 and i_0' are weak cofibrations ((B, B_0) and (B', B_0') have the *WHEP*) and in which the four vertical maps are fibrations. The following corollary is an analogue of 1.2 and 3.9 of [**2**].

COROLLARY 1.2. *Under the above assumptions, if f_0, f_1, f_2 and f_3 are homotopy equivalences, then the quadruple $(f_0, f_1, f_2$ and $f_3)$ is a homotopy equivalence in the category whose objects are commutative squares in* Top.

COROLLARY 1.3. *The triple (f_0, f_1, f_2)* (c.f. Section 0) *is a homotopy equivalence in $\mathscr{T}$ under the further condition that $j(A)$ and $j'(A')$ are closed $p(A) = B$, $p'(A') = B'$, and the restrictions of p and p' to A and A' respectively are also fibrations.*

COROLLARY 1.4. *The triple (f_0, f_1, f_2)* (c.f. Section 0) *is a homotopy equivalence in $\mathscr{T}$ under the further conditions that the A and A' are base points of E respectively E' and either*

(a) *A and A' are closed and $p(A)$ and $p'(A')$ are non-degenerate, or*

(b) *f_1 and f_2 are pointed homotopy equivalences and p and p' are fibrations in* Top_* *the category of pointed topological spaces.*

2. Relative fibre homotopy equivalence and ex-spaces. Dold has shown in [**3**] that any fibre map between fibrations with the same base is a fibre homotopy equivalence if and only if it is an ordinary homotopy equivalence. These concepts generalize in the obvious way to the category Toppair. Let $p : (E, E_0) \to (B, B_0)$ and $p' : (E', E_0') \to (B, B_0)$ be maps in which the four corresponding Top maps are fibrations and let (E, E_0) and (E', E_0') be closed with the *HEP*. (Note we do not require any condition on the pair (B, B_0).) The following theorem generalizes [**3**, 6.1].

THEOREM 2.1. (Relative Fibre Homotopy Equivalence Theorem). *Let* $f : (E, E_0) \to (E', E_0')$ *be a fibre map in* Toppair (i.e. $p'f = p$). *Under the above conditions* f *is a* Toppair *fibre homotopy equivalence if and only if* $f : E \to E'$ *and its restriction* $f|E_0 : E_0 \to E_0'$ *are ordinary homotopy equivalences in* Top. *In particular, if* E_0 *and* E_0' *are closed nondegenerate base points, and* B_0 *is the base point of* B, *then* f *is a pointed* (i.e. Top_*) *fibre homotopy equivalence if and only if* f *is an ordinary homotopy equivalence.*

Remark. There is, of course, an analogue of the latter part of this theorem for fibrations and homotopy equivalences in Top_*.

Proof. We chose g and g' in 1.1 to be identities and H and H' to be constant. We observe that in this case the conjugate homotopy is also constant at the identity on (B, B_0).

We close this section with an application to ex-spaces.

Recall [**7**] that a triple (X, σ, p) is an *ex-space* over B if X and B are spaces and $\sigma : B \to X$, $p : X \to B$ are maps such that $p\sigma = 1_B$. An *ex-map* $f : (X_0, \sigma_0, p_0) \to (X_1, \sigma_1, p_1)$ is an ordinary map $f : X_0 \to X_1$ such that $f\sigma_0 = \sigma_1$ and $p_0 f = p_0$.

The following comparison theorem generalizes one due to James [**6**, 6.1] proved there for ex-complexes.

THEOREM 2.2. *Let* $f : (X_0, \sigma_0, p_0) \to (X_1, \sigma_1, p_1)$ *be an ex-map in which the* p_i *are fibrations and* $(X_i, \sigma_i(B))$ *are closed with the HEP* $(i = 0, 1)$. *Then* f *is an ex-homotopy equivalence* [**7**] *if and only if* $f : X_0 \to X_1$ *is an ordinary* (Top) *homotopy equivalence.*

Proof. By abuse of notation, we identify $\sigma_i(B)$ with B $(i = 0, 1)$. Putting $B = B_0 = E_0 = E_0'$ and $f|E_0(=B)$ equal to the identity, we observe that any Toppair fibre homotopy inverse $f' : (E', B) \to (E, B)$ must cover the identity on (B, B). Clearly, $f'|B$ is the identity. One easily checks that homotopies also behave nicely, and we are finished.

3. Appendix.

In order to prove 1.1, we outline two lemmas which enable us to mimic the proof of 3.2 in [**2**]. Let $p : (E, E_0) \to (B, B_0)$ and $f : (X, X_0) \to (B, B_0)$ be maps in Toppair. We denote by $[f, p]$ the set of vertical homotopy classes of lifts of f over p; thus $\tilde{f}$ and $\tilde{f}' : (X, X_0) \to (E, E_0)$ are in the same class in $[f, p]$ if there exists a homotopy $H_t : \tilde{f} \simeq \tilde{f}' : (X, X_0) \times I \to (E, E_0)$ with $pH_t = f, t \in I$.

LEMMA A.1. *If* p *as above has the covering homotopy property CHP in* Toppair *with respect to* (X, X_0) *and* $(X, X_0) \times I$ *and if* $G_t : f \simeq g : (X, X_0) \to (B, B_0)$ *is a homotopy, then there exists a bijection* $G_\# : [f, p] \to [g, p]$.

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Proof. The bijection is defined on a class $[\tilde{f}]$ by picking a representative $\tilde{f} \in [\tilde{f}]$ and using the *CHP* to lift G at $\tilde{f}$. The class of the end point of this lift gives an element of $[g, p]$. Suppose $\tilde{f}' \in [\tilde{f}]$ and $\tilde{G} : \tilde{f} \simeq \tilde{G}(\ ,1)$ and $\tilde{G}' : \tilde{f} \simeq \tilde{G}'(\ ,1)$ are homotopies lifting G at $\tilde{f}$ and $\tilde{f}'$ respectively. We need to show $\tilde{G}'(\ ,1) \in [\tilde{G}(\ ,1)]$. Let $H : \tilde{f}' \simeq \tilde{f}$ be a vertical homotopy, then $K = -p\tilde{G}' + pH + p\tilde{G}$ is homotopic rel. end maps to the constant homotopy 0_f at f. We now define a map $L : (X, X_0) \times (I \times \{0\} \cup \{0, 1\} \times I) \to (E, E_0)$ to be $-\tilde{G} + H + G$ on $(X, X_0) \times I \times \{0\}$, and the constant homotopy at $\tilde{G}(\ ,1)$ and $\tilde{G}'(\ ,1)$ on $(X, X_0) \times \{0\} \times I$ and $(X, X_0) \times \{1\} \times I$ respectively. Since the pair $(I \times I, I \times \{0\} \cup \{0, 1\} \times I)$ is homeomorphic to $(I \times I, I \times \{0\})$, we see that $((X, X_0) \times I \times I, (X, X_0) \times (I \times \{0\} \cup \{0, 1\} \times I \cup X_0 \times I \times I)$ is homeomorphic to $((X, X_0) \times I \times I, (X, X_0) \times I \times \{0\} \cup X_0 \times I \times I)$, (I am grateful to K. H. Kamps for this comment) and the method of [**2**, 2.4] allows us to deduce the existence of a map M extending L and lifting $K \simeq 0_f$. The restriction of M to $(X, X_0) \times I \times \{1\}$ is the required vertical homotopy $\tilde{G}(\ ,1) \simeq \tilde{G}(\ ,1)$. The inverse of $G_\#$ is now easily seen to be $(-G)_\#$.

LEMMA A.2. *If $k : (X, X_0) \to (Y, Y_0)$ is a homotopy equivalence in* Toppair *and $p : (E, E_0) \to (B, B_0)$ has the CHP with respect to (X, X_0), $(X, X_0) \times I$, (Y, Y_0) and $(Y, Y_0) \times I$, then for any $f : (X, X_0) \to (B, B_0)$ the function $k_\# : [f, p] \to [kf, p]$ induced by composition is a bijection.*

Essentially, one proves that if l is a homotopy inverse of k, then since kl and lh are homotopic to the identity $k_\# l_\#$ and $l_\# k_\#$ are bijections. The result follows.

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