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A Note on a Conjecture of S. Stahl

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Abstract. S. Stahl (Canad. J. Math. **49**(1997), no. 3, 617–640) conjectured that the zeros of genus polynomial are real. L. Liu and Y. Wang disproved this conjecture on the basis of Example 6.7. In this note, it is pointed out that there is an error in this example and a new generating matrix and initial vector are provided.

The reader is referred to [2] for the explanation of all terms not defined here.

By the *genus polynomial* of a graph G, we shall mean the polynomial

$$g_G(x) = \sum_{i=0}^{\infty} g_i x^i,$$

where g_i is the number of embeddings on the orientable surface with genus i. We refer the reader to [2] for the techniques of calculating graph embeddings.

Many papers have been written in the last twenty years concerning the genus distribution of the embeddings of graphs. Most of the papers supported the following well-known conjecture in topological graph theory.

Conjecture 1 *The genus distributions $g_0, g_1, \dots,$ of the family of all graphs is log concave.*

Stahl [2] posed the following stronger conjecture.

Conjecture 2 *The zeros of a genus polynomial $g_G(x)$ are real and negative.*

In particular, Stahl considered the H-linear family of graphs obtained by consistently amalgamating additional copies of a graph H, and verified the conjecture for several infinite families of such graphs. For such a family G_n, there is a square matrix M and a vector V with entries in $Z[x]$ such that the genus polynomial of G_n is the first entry of $M^{n-1}V$ [2, Proposition 5.2].

The following are genus generating matrices and initial vectors of W_4-linear graphs G_n as given by Stahl:

$$M = \begin{pmatrix} 8 + 260x + 216x^2 & 4 + 88x \\ 64x + 416x^2 & 32x + 64x^2 \end{pmatrix}, \quad V = \begin{pmatrix} 2 + 58x + 36x^2 \\ 16 + 80x \end{pmatrix}.$$

L. Liu and Y. Wang [1] disproved Conjecture 2 by noting that the zeros of the genus polynomial that constitutes the first entry of the vector MV are not real.

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By using the method of [2], the correct genus polynomials of W_4-linear graphs are given by the following recurrence formula.

$$\begin{aligned} g_{G_n}(x) &= (14x + 36x^2)g_{G'_{n-1}}(x) + (2 + 44x)g_{G''_{n-1}}(x), \\ g_{G'_{n-1}}(x) &= 6g_{G_{n-1}}(x), \\ g_{G''_{n-1}}(x) &= 4g_{G_{n-1}}(x) + 96x^2 g_{G'_{n-2}}(x) + (64x^2 + 32x)g_{G''_{n-2}}(x), \\ g_{G'_0}(x) &= g_{G''_0}(x) = 1. \end{aligned}$$

Let

$$V_n = \begin{pmatrix} g_{G_n}(x) \\ g_{G'_{n-1}}(x) \\ g_{G''_{n-1}}(x) \end{pmatrix},$$

$$M = \begin{pmatrix} 8 + 260x + 216x^2 & 192x^2 + 4224x^3 & 64x + 1536x^2 + 2816x^3 \\ 6 & 0 & 0 \\ 4 & 96x^2 & 32x + 64x^2 \end{pmatrix}.$$

Then

$$V_n = M^{n-1} \begin{pmatrix} 2 + 58x + 36x^2 \\ 1 \\ 1 \end{pmatrix}.$$

It is a routine task to compute that

$$g_{G_1}(x) = 2 + 58x + 36x^2,$$

$$g_{G_2}(x) = 16 + 1048x + 17528x^2 + 28928x^3 + 7776x^4.$$

Using a computer program, we verified our calculation for $g_{G_1}(x)$ and $g_{G_2}(x)$. Using Mathematica, the approximations of four zeros of $g_{G_2}(x)$ are $x_1 = -2.97848$, $x_2 = -0.676668$, $x_3 = -0.0385049$, $x_4 = -0.0265141$.

We also verified, with the aid of a computer, that the zeros of genus polynomials of graphs with maximum genus 2 are real and negative. Thus the conjecture of Stahl is still open.

References

[1] L. Liu and Y. Wang, *A unified approach to polynomial sequences with only real zeros.* Adv. Appl. Math. **38**(2007), no. 4, 542–560.

[2] S. Stahl, *On the zeros of some genus polynomials.* Canad. J. Math. **49**(1997), no. 3, 617–640.

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