A THEOREM ON FIXED POINTS OF INVOLUTIONS IN S^3

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1. Introduction. Let T be an orientation-preserving homeomorphism of period two of the 3-sphere S^3 onto itself; further let T be different from the identity and have at least one fixed point. It has been shown by Smith **(8**, p. 162**)** that the set F of all fixed points of T is a simple closed curve. However, very little is known about the position of F in S^3. There is an example **(4)**, a slight variation of an example given by Bing, which shows that F may be wildly imbedded. Even if we assume F to be tame, as we shall do in this paper, it is not known whether T is equivalent to an orthogonal transformation, or even whether F is necessarily unknotted. Our purpose is to show, assuming T semi-linear, that at any rate F cannot belong to a certain class of knots including the ordinary cloverleaf.

We consider S^3 as the unit sphere

$$\sum_1^4 x_i^2 = 1$$

in Euclidean 4-space E^4. "Polyhedral" will always be understood in the sense of spherical geometry; all simplicial decompositions used are obtained by dividing S^3 into simplices by means of planes through the origin of E^4, that is, into spherical simplices; this is equivalent to the decompositions considered in **(5)**, where S^3 is considered as boundary of a Euclidean 4-simplex. The orientation-preserving homeomorphism T of period two is assumed to be semi-linear, which by definition means that T is (spherically-) affine on the simplices of a certain simplicial decomposition Σ of S^3; because of the periodicity of T we may assume that in addition T leaves the decomposition Σ invariant.

The set F of fixed points of T is then a simple closed (spherical) polygon **(8)**; by going to a subdivision, if necessary, F may be assumed to be a subcomplex of Σ, that is, F is made up of one-cells of Σ.

Let C_1 and C_2 be two simple closed polygons in S^3, disjoint from each other; C_1 is called a parallel knot, of order 2, of C_2, if there exists a polyhedral Möbius band M in S^3 such that C_1 is the boundary of M, and such that C_2 is what we shall call a middle line of M, i.e., such that C_2 represents a generator of the fundamental group $\pi_1(M)$ of M. (By Möbius band we mean the customary figure obtained by identifying two opposite sides of a rectangle, with a twist.) The theorem we propose to prove is as follows:

Received May 3, 1954. The work of the second author was supported by a grant from the National Science Foundation.

THEOREM I. *Suppose the fixed point curve F of the semi-linear involution T of S^3 is a parallel knot, of order 2, of the simple closed polygon C. Then the knot group of C, i.e., the fundamental group $\pi_1(S^3 - C)$ of the complement of C, is infinite cyclic, and the linking number of F and C is ± 1.*

A standard torus W in S^3 is the figure obtained by first constructing a standard torus in E^3 (a circle D in a plane $E^2 \subset E^3$ is revolved around a line $E^1 \subset E^2$ which does not meet D) and then projecting E^3 stereographically onto S^3. The surface W is parametrized by using an angular coordinate r_1 on D and an angular coordinate r_2 for the rotation around E^1. If p, q are relatively prime natural numbers then the curve in S^3 given by

$$r_1 = 2\pi pt, \quad r_2 = 2\pi qt, \qquad 0 \leqslant t \leqslant 1$$

is called the standard torus knot (p, q). A simplicial standard torus knot (p, q) is defined to be a sufficiently close polygonal approximation (obtained by selecting values $0 = t_0 < t_1 < \ldots < t_n = 1$ with max $|t_i - t_{i-1}|$ sufficiently small in an obvious sense and replacing the segment corresponding to $[t_{i-1}, t_i]$ of the original torus knot by the spherical segment connecting the end points).

COROLLARY. *If the fixed point set F of T is a simplicial standard torus knot $(p, 2)$, then $p = 1$, which implies that F is not knotted.*

This follows from Theorem I since the torus knot $(p, 2)$ is a parallel knot, of order 2, of the center line of the torus. It is necessary to observe that all requirements are satisfied simplicially but this is not very difficult to show. Note that the torus knot (3, 2) is the cloverleaf.

The method of proof of Theorem I has as a by-product the following Theorem which (we have learned) has also been proved by E. E. Moise:

THEOREM II. *If F is unknotted in the sense of bounding a polyhedral 2-cell, then T is equivalent to a rotation of S^3.*

In the proofs we make use of a theorem of Alexander **(1)** which states that a polyhedral 2-sphere S^2 in S^3 separates S^3 into two domains A, A' such that $A \cup S^2$ and $A' \cup S^2$ are closed 3-cells (each with boundary S^2). We also use a second theorem by Alexander **(1)** which states that a polyhedral torus B in S^3 separates S^3 into two domains D, D' such that at least one of $B \cup D$, $B \cup D'$ is a solid torus, that is, homeomorphic with the product of a closed disk and a circumference, with B of course corresponding to the boundary. Actually Alexander's proof needs a small modification to become applicable to the spherical polyhedra used here. However Schubert **(6)** has given a proof which is directly applicable to our case.

2. First simplification of $M \cap T(M)$. We now begin the proof of Theorem I and recall that F is boundary of a Möbius band M with middle line C. We can assume that M and C are subcomplexes of the T-invariant simplicial decomposition Σ of S^3. Then $T(M)$ is another simplicial Möbius band and

$T(M)$ also has F as boundary, since F is pointwise fixed. The set $M \cap T(M)$ includes F but must include further points, for otherwise $M \cup T(M)$ would be a Klein bottle imbedded in S^3 without singularity and this is impossible.

We shall show how M may be modified so that $M \cap T(M)$ is made simpler. In the process M remains polyhedral, F remains the boundary, and C is not changed essentially. More specifically, all the Möbius bands considered in the modification process will satisfy the following condition.

CONDITION (m): *There exists a semi-linear homeomorphism of S^3, which sends some middle line of the band into the curve C (of Theorem I) and leaves F pointwise fixed.* (It is immaterial which middle line is selected because if C_1 and C_2 are two polygonal middle lines of M there is a semi-linear homeomorphism of S^3 onto itself which maps M onto itself, leaves F pointwise fixed, and takes C_1 to C_2. This can be shown by methods indicated in **(6)** and in §4. This fact is implicitly involved in a number of our arguments.)

In the process, subdivisions of Σ may be needed; they will be introduced without explicit mention. The ultimate aim of the modifications is to prove the following:

LEMMA A. *Under the hypothesis of Theorem I, there exists a simplicial Möbius band M_1 with boundary F such that*

$$M_1 \cap T(M_1) = F \cup C_1$$

where C_1 is a middle line of M_1 and where condition (m) *is satisfied for C_1.*

The proof is in several parts. The remainder of this section and the next section are devoted to selecting M so that M and $T(M)$ are in general position.

LEMMA 1. *Let V be any neighborhood of F. Then there exists a simplicial Möbius band M' such that:*

(1) *the boundary of M' is F;*

(2) $M \cap (S^3 - V) = M' \cap (S^3 - V)$, *that is, M and M' coincide outside of V;*

(3) *there is a neighborhood V' of F such that*

$$M' \cap T(M') - F \subset S^3 - V',$$

that is, the intersection, except for F, is outside of V'. Note that in consequence of (2) *and with V small, a middle line of M is also one of M', so that condition* (m) *is satisfied.*

Proof. By replacing Σ, if necessary, by a subdivision we can assume the following:

(a) F is a normal subcomplex, meaning that every simplex of Σ, whose vertices belong to F, itself belongs to F;

(*b*) the barycentric star St (F) is contained inside V;

(*c*) St(F) is a simplicially divided solid torus whose boundary is a torus surface Q.

https://doi.org/10.4153/CJM-1955-027-1 Published online by Cambridge University Press

As a matter of fact, St(F) is the union of the 3-cells, corresponding to the vertices of Σ on F in the dual subdivision Σ^* of S^3. Two such 3-cells, corresponding to neighboring vertices of F, have in common a 2-cell, the cell dual to the 1-cell of Σ joining the two vertices, whereas two such 3-cells that correspond to non-neighboring vertices on F are disjoint. We may also assume that M is a normal subcomplex of Σ. It is true that $M \cap \mathrm{St}(F)$ is the barycentric star of F on M. It follows that $M \cap Q$ is a simple closed curve which we shall call D, and that D and F form the boundary of the annulus $M \cap \mathrm{St}(F)$; an annulus is the topological product of an interval I and a circle S^1. As a consequence D is not ~ 0 on Q.

We now propose to construct a polygonal simple closed curve D' on Q such that

(1) $D' \sim D$ on Q

(2) $T(D') \cap D' = 0$ (0 = empty set)

(3) D' and F bound an annulus A in St(F) for which $A \cap T(A) = F$ and $A \cap Q = D'$. For the construction of D' we note that for each of the 3-cells σ_3, dual to the vertices of F, the boundary 2-sphere Bd σ_3 intersects Q in an annulus. From the known behavior of periodic transformations of 2-cell and 2-sphere (see **(2)** and the references therein to Brouwer and Kérekjárto) it follows that T on such an annulus Bd $\sigma_3 \cap Q$ is equivalent to the standard transformation of $I \times S^1$, obtained by rotating S^1 through 180°. It is now easy to construct D' as union of polygonal arcs, one arc in each annulus Bd $\sigma_3 \cap Q$, connecting the two components of the boundary of the annulus. The existence of A follows then from the fact that each σ_3 is the join of Bd σ_3 and the vertex of F, to which it is dual. We now apply Theorem 3, p. 180 (see also p. 161), of Schubert **(6)**; according to this there exists a semi-linear homeomorphism ϕ of S^3 which is the identity outside of V, maps Q into itself and sends D into D'.

We now obtain the desired Möbius band M' as the union of A and the image under ϕ of the part of M in $S^3 - \mathrm{St}(F)$. The requirements of Lemma 1 are now satisfied with V' the interior of St(F). This completes the proof.

We now call the M' thus obtained again M and proceed with further modifications leading towards proving Lemma A.

3. Reduction to a finite set of simple closed curves. Suppose that $M \cap T(M)$ contains a 2-cell σ of Σ with vertices a, b, c; note that σ is disjoint from F by Lemma 1. Then σ and $T(\sigma)$ have no point in common since otherwise there would be a fixed point of T in $\sigma \cup T(\sigma)$. Now σ is a face of a 3-cell τ of Σ with vertices a, b, c, d. Let e be a point inside τ and subdivide τ into the join $e \circ \mathrm{Bd}\,\tau$ of e and the boundary Bd τ of τ. Because of the normality of M the intersection $\tau \cap F$ is empty; it follows again that τ and $T(\tau)$ are disjoint since otherwise there would be a fixed point of T in $\tau \cup T(\tau)$. We subdivide $T(\tau)$ by the join $T(e) \circ T(\mathrm{Bd}\,\tau)$; the new subdivision of S^3 is T-invariant.

We now replace the cell σ of M by the join of e and Bd σ; similarly replace

$T(\sigma)$ in $T(M)$ by the join of $T(e)$ and $T(\mathrm{Bd}\,\sigma)$. The band M' so obtained has boundary F and any alteration of C has been in accord with condition (m). However

$$M' \cap T(M') = M \cap T(M) - [\text{interior } \sigma \cup \text{interior } T(\sigma)];$$

i.e., the interiors of the 2-cells σ and $T(\sigma)$ no longer belong to the intersection of M and $T(M)$, whereas no other intersections are introduced. By a repetition of this process *we arrive at a Möbius band* M' *such that* $M' \cap T(M')$ *contains no* 2-*cell.*

Given a 2-cell σ, as in the above, we are free to make the indicated modification on either side of σ. Further if we have in the intersection a union of 2-cells σ_i of Σ which forms a 2-cell K or an annulus R such that

$$T(K) \cap K = 0 \text{ resp. } T(R) \cap R = 0,$$

then K or R has two well-defined "sides" and we may modify each σ_i to either of the two sides.

Next suppose that $M \cap T(M)$ contains a 1-cell $\eta = (a, b)$ of Σ not on F, and that M and $T(M)$ do not cross each other at η in an obvious sense. In M the 1-cell η lies on two 2-cells, say (a, b, c) and (a, b, d). There exists a set of vertices $d_0 = c, d_1, \ldots, d_k = d$, $1 \leqslant k$ such $(abcd_1)$, $(abd_1d_2), \ldots (abd_{k-1}d)$ are 3-cells of σ and such that d_i, $0 \leqslant i \leqslant k$ does not belong to $T(M)$. Using a subdivision of these cells and their images under T we may modify M to become M' in accordance with condition (m) and so that

$$M' \cap T(M') = M \cap T(M) - [\text{interior } \eta \cup \text{interior } T(\eta)].$$

We omit the details of this process of pulling M and $T(M)$ apart. Using this method we obtain a band, again called M, such that M and $T(M)$ cross along any 1-cell in $M \cap T(M) - F$.

Similar methods make it possible to obtain a Möbius band, again called M such that $M \cap T(M)$ contains no isolated points (and of course no 2-cells, and no 1-cells along which M and $T(M)$ do not cross). We note in particular: Any component of the intersection of M and $T(M)$ that is a 2-cell, disjoint from its image, can be completely removed from the intersection (by modifying M in accordance with condition (m)) without introducing new intersections. Any component of $M \cap T(M)$ that is an annulus, disjoint from its image and along which M and $T(M)$ do not cross in the obvious sense, can also be removed in this way. If M and $T(M)$ do cross along such an annulus, then M can be so modified that in the intersection with $T(M)$ there appears instead of the annulus any preassigned simple closed polygon that represents a generator of the fundamental group of the annulus, e.g., either one of the boundary curves; the modified M and $T(M)$ cross each other along this curve. Any component of the intersection that is a simple closed curve along which M and $T(M)$ do not cross can be removed.

Finally let p be a point of M, necessarily a vertex of Σ, such that the order of the graph $M \cap T(M)$ at p is greater than 2, that is, that more than two 1-cells of $M \cap T(M)$ end at p. Let R be the barycentric star of p, and let E_1 and E_2 be the intersections of M and of $T(M)$ with the boundary 2-sphere Bd R of R. Clearly E_1 and E_2 are simple closed polygonal curves which cross each other at their points of intersection.

The curve E_1 separates Bd R into two 2-cells; let K be one of these. If we replace $R \cap M$ by K and make the corresponding replacement at $T(p)$ on $T(M)$, then on the modified M the number of points of $M \cap T(M)$ of order greater than two has been reduced whereas all other properties are retained. We state the main result of this section as a lemma.

LEMMA 2. *The original Möbius band can be modified to a new band, again called M, satisfying condition* (m) *and such that the intersection $M \cap T(M)$ consists of a finite number of simple closed curves, along which M and $T(M)$ cross each other.*

4. Curves on a Möbius band. For further simplifications we shall need information about the simple closed curves on a Möbius band, which we formulate as a lemma.

LEMMA 3. *Let $\tilde{M}$ be any Möbius band. Any simple closed curve $\tilde{C}$ on $\tilde{M}$ belongs to exactly one of the following three, topologically invariant, categories:*

(1) *the curves which are contractible to a point;*

(2) *the curves which represent a generator of the fundamental group $\pi_1(\tilde{M})$ of $\tilde{M}$; we call these middle lines of $\tilde{M}$;*

(3) *the curves which represent the square of a generator of $\pi_1(\tilde{M})$; we call these edge-like* (*the boundary of $\tilde{M}$ is in this category*).

A middle line, disjoint from the boundary of $\tilde{M}$, has arbitrarily small neighborhoods on $\tilde{M}$ that are Möbius bands. An edge-like curve, disjoint from the boundary of $\tilde{M}$, separates $\tilde{M}$ into a Möbius band and an annulus; it has arbitrarily small neighborhoods on $\tilde{M}$ homeomorphic to an annulus.

Any two middle lines intersect. Any two disjoint edge-like curves bound an annulus contained in $\tilde{M}$; a middle line, disjoint from the two edge-like curves, does not meet the annulus bounded by them.

We omit the proofs of these statements and only note the following: The Möbius band has as an orientable double covering an annulus R, represented by the region $1 \leqslant x^2 + y^2 \leqslant 2$ in an (x, y)-plane; the Möbius band is obtained by considering the involutory transformation α of R onto itself, given, in polar coordinates, by

$$\alpha(r, \theta) = (3 - r, \theta + \pi),$$

and identifying pairs of points, which correspond under α, to single points. The identification amounts to a map H from R to the Möbius band, which is the covering map. Proofs for the statements above on curves on $\tilde{M}$ can be made by considering the inverse images under H.

5. Elimination of bounding curves. Using the facts of §4 on curves on a Möbius band we can now make further simplifications.

LEMMA 4. *The band M can be modified* (*consistent with condition* (m)) *so that no component of $M \cap T(M)$ bounds a 2-cell on M or on $T(M)$.*

For the proof we start with the Möbius band M obtained in Lemma 2. Let $C_1, \ldots, C_n$ be those simple closed curves in $M \cap T(M)$, which bound 2-cells in M. We can assume that C_1 is such that the 2-cell K_1, bounded by it on M, contains no point of $M \cap T(M)$ in its interior. We show first that C_1 also bounds a 2-cell on $T(M)$. If not, then it is edge-like or a middle line (in the sense of §4) on $T(M)$.

If it is edge-like on $T(M)$, then it bounds a Möbius band M' on $T(M)$. The set $M' \cup K_1$ would then be a projective plane imbedded in S^3, which is well known to be impossible.

If it is a middle line, then one gets a contradiction by considering the intersection curves of K_1 and $T(M)$ with the boundary torus of the barycentric star of C_1.

We now show how to remove C_1 from $M \cap T(M)$, without introducing new intersections (and of course still satisfying condition (m)). We show first that C_1 cannot meet $T(C_1)$. If it did, then the two would be identical, since T permutes the components of $M \cap T(M)$. Then $K_1 \cup T(K_1)$ would be a 2-sphere, invariant under T. Since this 2-sphere does not meet F, the two 3-cells, into which S^3 is divided by it according to Alexander's theorem **(1)**, would have to be invariant, and would therefore contain fixed points of T in their interiors. But this contradicts the fact that F is connected. We have therefore $C_1 \cap T(C_1) = 0$. As shown before, C_1 bounds a 2-cell K'_1 in $T(M)$. We consider now two cases,

(I): $K_1 \cap T(K'_1) = 0$,

(II): $K_1 \cap T(K'_1) \neq 0$.

We start with case (I).

As a preliminary modification we define a set M^* by

5.1 $$M^* = (M - T(K'_1)) \cup T(K_1).$$

We have of course

5.11 $$T(M^*) = (T(M) - K'_1) \cup K_1.$$

M^* is again a Möbius band, since we have replaced the cell $T(K'_1)$ by the cell $T(K_1)$, which has no point of M in its interior; note that $T(K_1)$ and $T(K'_1)$ have the same boundary curve. Condition (m) concerning the middle line is still satisfied. The intersection of M^* and its transform is given by

5.2 $$\begin{aligned} M^* \cap T(M^*) &= ((M - T(K'_1)) \cap (T(M) - K'_1)) \cup ((M - T(K'_1) \cap K_1) \\ &\qquad \cup ((T(M) - K'_1) \cap T(K_1)) \cup (K_1 \cap T(K_1)). \end{aligned}$$

https://doi.org/10.4153/CJM-1955-027-1 Published online by Cambridge University Press

Since $K_1 \cap T(K'_1) = 0$, the second term on the right is just K_1; similarly the third term is $T(K_1)$. As a consequence, we have

5.3 $$K_1 \cup T(K_1) \subset M^* \cap T(M^*) \subset (M \cap T(M)) \cup K_1 \cup T(K_1).$$

The two 2-cells K_1 and $T(K_1)$ are disjoint; this follows from the fact that C_1 and $T(C_1)$ are disjoint, and that the interior of K_1 does not meet $T(M)$ at all. By the method described in §3, we can therefore remove K_1 and $T(K_1)$ from the intersection of M^* and $T(M^*)$ without introducing new intersections, and arrive at a new Möbius band M_1, such that

$$M_1 \cap T(M_1) \subset M \cap T(M) - (C_1 \cup T(C_1)),$$

i.e., the intersection of M_1 and $T(M_1)$ is contained in that of M and $T(M)$, but does not contain the curves C_1 and $T(C_1)$ any more; this finishes case (I).

In case (II) we define M^*, and $T(M^*)$ as in case (I) by 5.1 and 5.11. For the intersection $M^* \cap T(M^*)$ we have again formula 5.2. It is again true that $K_1 \cap T(K_1) = 0$. Further we now actually have $K_1 \subset T(K'_1)$, since the boundary curve $T(C_1)$ of $T(K'_1)$ is disjoint from C_1, as shown above, and therefore also from K_1. Similarly $T(K_1) \subset K'_1$. It follows then from 5.2 that

5.4 $$M^* \cap T(M^*) = (M - T(K'_1)) \cap (T(M) - K'_1),$$

and so

5.5 $$M^* \cap T(M^*) \subset M \cap T(M) - (C_1 \cup T(C_1))$$

(note that C_1 is the boundary curve of K'_1, and $T(C_1)$ that of $T(K'_1)$), so that C_1 and $T(C_1)$ are no longer in the intersection. This finishes case (II); condition (m) is still satisfied.

Iteration of this process must come to an end, since at each application the number of components of $M \cap T(M)$ decreases, and Lemma 4 follows.

6. Reduction to a middle line. We now take M to satisfy the condition of Lemma 4, and proceed with a last reduction, in order to arrive at Lemma A. The intersection $M \cap T(M)$ now consists of F and a number of curves, say $C_1, \ldots, C_m$, which are either edge-like or middle lines on M, with at most one in the last category (because of Lemma 3). Suppose there are actually edge-like curves in this collection. One concludes, with the help of Lemma 3, that there is one of these, which we can assume to be C_1, such that the annulus R bounded on M by F and C_1 contains no point of $M \cap T(M)$ in its interior. We show that C_1 cannot be invariant under T, or, which amounts to the same, that $C_1 \cap T(C_1) = 0$. If $C_1 = T(C_1)$, then $R \cup T(R)$ would be a T-invariant torus, which separates S^3 into two domains. From the action of T near F it follows that T interchanges the two domains. On the other hand, from the action of T near C_1 we see that T cannot interchange the two domains. This contradiction establishes our assertion.

The curve $T(C_1)$ is therefore one of $C_2, \ldots, C_m$; we claim that it is edge-like on M. If it were a middle line, we would get a contradiction by considering the intersection curves of M and $T(M)$ with the boundary of its barycentric neighborhood, as in the reasoning of §5; recall that $T(C_1)$ is edge-like on $T(M)$. In particular, the number of edge-like curves among $C_1, \ldots, C_m$ is >1.

With the help of Lemma 3 we see that there is a curve in $M \cap T(M)$, edge-like on M and on $T(M)$, which we can call C_2, such that the interior of the annulus R_1 bounded by C_1 and C_2 on M does not meet $T(M)$. Let R'_1 be the annulus bounded by C_1 and C_2 on $T(M)$; the interior of R'_1 may meet M. We have two possibilities:

(A) $$C_2 = T(C_1),$$

(B) $$C_2 \neq T(C_1).$$

In case (A) we have $C_1 = T(C_2), R'_1 = T(R_1), R_1 = T(R'_1)$. We define then

$$M^* = (M - R_1) \cup T(R_1),$$

so that

$$T(M^*) = (T(M) - T(R_1)) \cup R_1.$$

M^* is again a Möbius band. The intersection $M^* \cap T(M^*)$ is identical with $M \cap T(M)$; but M^* and $T(M^*)$ do not cross along C_1 and C_2, so that C_1 and C_2 can be removed from the intersection by the methods of §4; condition (m) is kept intact.

In case (B) we have to distinguish two subcases:

(I) $R_1 \cap T(R'_1) = 0,$

(II) $R_1 \cap T(R'_1) \neq 0.$

We begin with (I). Then R_1 and $T(R'_1)$ are non-intersecting annuli on M. We define new sets by

6.1 $$M^* = (M - T(R'_1) \cup T(R_1),$$

6.11 $$T(M^*) = (T(M) - R'_1) \cup R_1.$$

Then M^* is again a Möbius band; an annulus between two edge-like curves has been replaced by an annulus; this has no effect on the middle lines, so that condition (m) is satisfied. For the intersection with $T(M^*)$ we have the formula

6.2 $$\begin{aligned} M^* &\cap T(M^*) \\ &= (M - T(R'_1)) \cap (T(M) - R'_1) \cup (M - T(R'_1)) \cap R_1 \\ &\qquad \cup (T(M) - R'_1) \cap T(R_1) \cup R_1 \cap T(R_1). \end{aligned}$$

Because of (I) the second and third term are just R_1 and $T(R_1)$. It follows that

6.3 $$M^* \cap T(M^*) = (M - T(R'_1)) \cap (T(M) - R'_1) \cup R_1 \cup T(R_1).$$

As in §5 we see that R_1 and $T(R_1)$ are disjoint: the T-image of the boundary of R_1 does not meet R_1, since we have case (I), and the interior of R_1 does not meet $T(M)$ at all. We are therefore in the situation of §3: R_1 and $T(R_1)$ are disjoint components of $M^* \cap T(M^*)$. We can therefore replace M^* by another permissible Möbius band such that either R_1 and $T(R_1)$ are absent from the intersection with the T-transform (this if M^* and $T(M^*)$ do not cross along R_1) or R_1 is replaced by C_2, and $T(R_1)$ by $T(C_2)$ (this if M^* and $T(M^*)$ do cross along R_1). In either case C_1 and $T(C_1)$ have been removed from the intersection.

We come to case (II). $T(R'_1)$ is an annulus on M, bounded by $T(C_1)$ and $T(C_2)$. Because of our assumptions $T(C_1)$ is different from C_1 and C_2. We show first that we cannot have $C_2 = T(C_2)$. We assume then for the moment that C_2 is invariant. Let R_2 be the annulus bounded by F and C_2 on M; its boundary is then invariant. If the interiors of R_2 and $T(R_2)$ meet, this can happen only along C_1, so that then C_1 would be T-invariant; but this we have shown to be impossible. It follows that $R_2 \cup T(R_2)$ is a T-invariant torus. The reasoning applied to $R \cup T(R)$ in the beginning of this section applies here too, and shows that C_2 is different from $T(C_2)$. It follows that the boundary of $T(R'_1)$ is disjoint from R_1. Since R_1 is connected, it would have to be contained in the interior of $T(R'_1)$. But this is impossible, since no component of $M \cap T(M)$ lies in the annulus R between F and C_1 on M, and so case (II) cannot occur at all.

The processes described in this section can be applied as long as there are edge-like curves, different from F, in the intersection of the band and its transform. By a finite number of steps we arrive therefore at a Möbius band M_1 satisfying Lemma A; as noted earlier, the set $M_1 \cap T(M_1) - F$ cannot be empty, and on the other hand, two middle lines on a Möbius band are never disjoint, by Lemma 3, so that exactly one middle line will be left over in the intersection (besides F).

7. The proof of Theorem I. The knot groups of the original curve C and of the curve C_1 of Lemma A are isomorphic, and the linking numbers of F with C and C_1 are equal, because of condition (m). It is therefore sufficient to prove Theorem I for C_1 instead of C. We return to the notation M for the Möbius band, obtained in Lemma A, and C for the single (invariant) curve constituting $M \cap T(M) - F$. We denote the polyhedron $M \cup T(M)$ by S; it is clear that $T(S) = S$. The second homology group $H_2(S)$ is infinite cyclic and the generating 2-cycle contains all 2-cells, properly oriented, with coefficient $+1$. (In fact S can be described as obtained from a torus $W_2 = S^1 \times S^1$ by identifying pairs of antipodal points on some generating circle $p \times S^1$.)

It follows from Alexander's duality theorem that $S^3 - S$ has two components which we call A_1 and A_2. We have $A_2 = T(A_1)$, $A_1 = T(A_2)$. The reason is that T, at any point of F, at the same time reverses the orientation of S and preserves the orientation of the 3-sphere; it interchanges therefore the two domains into which S separates the 3-sphere locally (and globally). It is also true that $\mathrm{Cl}A_i$, the closure of A_i, is $A_i \cup S$, and that $\mathrm{Cl}A_1 \cap \mathrm{Cl}A_2 = S$.

Let N be the closed barycentric neighborhood of the middle curve C with respect to the triangulation of S^3 reached after the various modifications. Then N is a solid torus and its boundary $\mathrm{Bd}N = B$ is a torus.

One shows with elementary deformations that the three sets $S^3 - N$, the closure of $S^3 - N$, and $S^3 - C$ have isomorphic fundamental groups, and that in fact the inclusion maps induce these isomorphisms. The intersection

$$P = N \cap M$$

is the closed barycentric neighborhood of C on M. It is a Möbius band with C as middle line; similar remarks apply to

$$Q = N \cap T(M).$$

It can be shown by elementary constructions that the following facts hold:

The set $N - (P \cup Q)$ consists of two connected sets N_1, N_2 which are the intersections of N with A_1, A_2.

The boundary curves $\mathrm{Bd}P$, $\mathrm{Bd}Q$ of P and Q, which lie on B, separate B into sets B_1, B_2 each homeomorphic with an open annulus, and $B_i = B \cap A_i$; the closure $\mathrm{Cl}B_i$ of B_i is $B_i \cup \mathrm{Bd}P \cup \mathrm{Bd}Q$. The generators of the fundamental group of $\mathrm{Cl}B_i$ represented by $\mathrm{Bd}P$ or $\mathrm{Bd}Q$ are homotopic in N to the square of the generator of the fundamental group of N which is in turn represented by C. All sets N, $\mathrm{Cl}N_i$, P, Q are deformation retractable onto C; also $\mathrm{Cl}A_i$ is a deformation retract of $\mathrm{Cl}A_i \cup N$. Note that N and B are invariant under T and that P and Q, $\mathrm{Bd}P$ and $\mathrm{Bd}Q$, N_1 and N_2, B_1 and B_2 are interchanged by T.

Finally let

$$X = \mathrm{Cl}(S - (P \cup Q)).$$

This is an annulus for which F, the fixed point curve of T, is a generator of the fundamental group. We shall now prove the first half of Theorem I.

Proposition 1. *The fundamental group of $S^3 - C$ or of $\mathrm{Cl}(S^3 - N)$ is infinite cyclic.*

Proof. We form $X \cup B_1$, which by construction is a torus; it is also the boundary of each of the polyhedra $N \cup \mathrm{Cl}A_2$ and $\mathrm{Cl}(A_1 - N)$. By Alexander's theorem **(1)** one of these two is homeomorphic with a solid torus. We consider two cases.

(a) Suppose that $\mathrm{Cl}(A_1 - N)$ is a solid torus. Then $\mathrm{Cl}(A_2 - N)$ is also a solid torus, as T-image of $\mathrm{Cl}(A_1 - N)$. The union of $\mathrm{Cl}(A_1 - N)$ and $\mathrm{Cl}(A_2 - N)$ is $\mathrm{Cl}(S^3 - N)$; the intersection of $\mathrm{Cl}(A_1 - N)$ and $\mathrm{Cl}(A_2 - N)$ is X. To compute $\pi_1 \mathrm{Cl}(S^3 - N)$, we use the addition theorem **(7**, p. 177**)**, according to which $\pi_1 \mathrm{Cl}(S^3 - N)$ is the free product of $\pi_1 \mathrm{Cl}(A_1 - N)$ and $\pi_1 \mathrm{Cl}(A_2 - N)$ with the additional relations obtained by equating elements which correspond to the same element of $\pi_1(X)$. Let g_1 and g_2 be the generators of the (infinite cyclic) groups $\pi_1 \mathrm{Cl}(A_1 - N)$ and $\pi_1 \mathrm{Cl}(A_2 - N)$; F, considered as a curve in $\mathrm{Cl}(A_1 - N)$, represents some power g_1^m of g_1. Since $T(F) = F$ and $T(g_1) = g_2$, there will be only one new relation, namely, $g_1^m = g_2^m$. The

https://doi.org/10.4153/CJM-1955-027-1 Published online by Cambridge University Press

homology group $H_1\mathrm{Cl}(S^3 - N)$ is then the abelian group with two generators γ_1 and γ_2 and the relation $m\gamma_1 = m\gamma_2$. Because of the Alexander duality theorem this group must be infinite cyclic; it follows that $m = \pm 1$. But this clearly means that $\pi_1\,\mathrm{Cl}(S^3 - N)$ is infinite cyclic, and that F represents a generator.

(b) Suppose that $\mathrm{Cl}A_2 \cup N$ is a solid torus. We show first that the curve C represents a generator of $\pi_1(\mathrm{Cl}A_2 \cup N)$. For the proof we can consider $\mathrm{Cl}A_2$ instead of $\mathrm{Cl}A_2 \cup N$, since it is a deformation retract of the latter. C represents some power β^k of a generator β of $\pi_1\mathrm{Cl}A_2$; we wish to show $k = \pm 1$, and assume temporarily that this is not so. It follows, going to the homology, that C is homologous to 0 mod k in $\mathrm{Cl}A_2$. Applying T, one gets the same behavior in $\mathrm{Cl}A_1$. On the other hand, from the explicit structure of $S = \mathrm{Cl}A_1 \cap \mathrm{Cl}A_2$ we see that C is not homologous to 0 mod k on S. But this leads to a contradiction with the Mayer-Vietoris theorem **(3)** for the decomposition $S^3 = \mathrm{Cl}A_1 \cup \mathrm{Cl}A_2$, since $H_2(S^3) = 0$. It follows that $k = \pm 1$, as we claimed.

We now represent $\mathrm{Cl}A_2 \cup N$ as union of the two polyhedra $\mathrm{Cl}(A_2 - N)$ and N, whose intersection is $\mathrm{Cl}B_2$. Let γ denote the generator of $\pi_1(N)$, represented by C. A generator of $\pi_1(\mathrm{Cl}B_2)$, e.g., $\mathrm{Bd}P$, represents then the element γ^2 in $\pi_1(N)$, and a certain element α in $\pi_1\mathrm{Cl}(A_2 - N)$. The group $\pi_1(\mathrm{Cl}A_2 \cup N)$ is obtained, according to the addition theorem used above, by adding the new generator γ to the generators of $\pi_1\mathrm{Cl}(A_2 - N)$ and adding the relation $\alpha = \gamma^2$ to the relations in $\pi_1\mathrm{Cl}(A_2 - N)$. Since γ is represented by C, it follows that γ is a generator of the group $\pi_1(\mathrm{Cl}A_2 \cup N)$, which is infinite cyclic, since $\mathrm{Cl}A_2 \cup N$ is a solid torus. Consequently the element α of $\pi_1\mathrm{Cl}(A_2 - N)$ is also of infinite order, and we see that $\pi_1(\mathrm{Cl}A_2 \cup N)$ is the free product of $\pi_1\mathrm{Cl}(A_2 - N)$ and $\pi_1(N)$ with amalgamated subgroups $\{\gamma^2\}$ and $\{\alpha\}$ in the sense of Schreier **(5)**. It follows from this theory that $\pi_1\mathrm{Cl}(A_2 - N)$ is isomorphically contained in $\pi_1(\mathrm{Cl}A_2 \cup N)$, and is therefore itself infinite cyclic. But then the argument of (a) can be used to prove Proposition 1.

We now come to the second part of Theorem I:

PROPOSITION 2. *The linking number of F and C is* ± 1.

Proof. The reasoning of (a) and (b) above showed that the groups $\pi_1\mathrm{Cl}(A_i - N)$ and $\pi_1\mathrm{Cl}(S^3 - N)$ are infinite cyclic, with F representing a generator of $\pi_1\mathrm{Cl}(S^3 - N)$ and so also of $\pi_1(S^3 - C)$. Since obviously there are elements in $\pi_1(S^3 - C)$, whose linking number with C is ± 1, it follows that F must also have this property, and Theorem I is proved.

8. The proof of Theorem II. Again T is an involution of S^3 which preserves orientation with fixed point curve F; now F is unknotted. This hypothesis means that F bounds a cell N where N is a polyhedron of an invariant subdivision of S^3. Earlier arguments show:

(1) that N may be modified (§3) so that $N \cap TN$ consists of a finite set of simple closed curves;

(2) that by a further reduction (§5) it can be arranged that

$$N \cap TN = F.$$

Then $N \cup TN$ is a 2-sphere (polyhedral) and S^3 is the union of two closed 3-cells, L and $T(L)$, each having for boundary the 2-sphere $N \cup TN$. Let T_0 be the rotation of S^3, of period 2, given by $x_1 \to -x_1$, $x_2 \to -x_2$, $x_3 \to x_3$, $x_4 \to x_4$. The fixed point curve F_0 is given by $x_1 = x_2 = 0$. An invariant sphere is given by $x_1 = 0$; it is divided by F_0 into two 2-cells N_0 and $T_0(N_0)$, given by $x_1 = 0$, $x_2 \geqslant 0$, respectively $x_2 \leqslant 0$. The whole sphere S^3 is the union of the two 3-cells L_0 and $T_0(L_0)$, given by $x_2 \geqslant 0$, respectively $x_2 \leqslant 0$, each having for boundary the sphere $N_0 \cup T_0(N_0)$.

We begin now by setting up a homeomorphism ϕ between N and N_0, arbitrarily chosen. This can be extended to a homeomorphism ϕ between $N \cup T(N)$ and $N_0 \cup T_0(N_0)$ by defining $\phi = T_0 \cdot \phi \cdot T$ on $T(N)$. The homeomorphism ϕ between $N \cup T(N)$ and $N_0 \cup T_0(N_0)$ can be extended to a homeomorphism ϕ between L and L_0. This in turn can be extended to a homeomorphism of S^3 with itself by defining $\phi = T_0 \cdot \phi \cdot T$ on $T(L)$. The ϕ so constructed satisfies $\phi = T_0 \cdot \phi \cdot T$, or $T = \phi^{-1} \cdot T_0 \cdot \phi$, which shows that our involution T is equivalent to the rotation T_0.

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