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A Problem in Conics.

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The locus of the focus of a parabola touching three given fixed straight lines is both exceedingly simple and widely known. On the other hand the analogous locus of the focus of a parabola passing through three given fixed points is excessively complicated, and its investigation has, so far as the present writer knows, never appeared in any text-book.

The analysis for the general case is so cumbrous that it would appear the easiest course to deal first with the simplest case and then gradually build up and generalise.

Let Ox, Oy be rectangular axes upon which fixed points A, B are respectively taken ($OA = OB = 2a$). If p, q, r are perpendiculars from O, A, B (Fig. 3) upon the directrix of a parabola passing through OAB, then*

$$\frac{(q-p)^2}{4a^2} + \frac{(r-p)^2}{4a^2} = 1$$

or

$$(q-p)^2 + (r-p)^2 = 4a^2.$$

Hence if S is the focus, since $SP = PM$ in the parabola,

$$(SA - SO)^2 + (SB - SO)^2 = 4a^2,$$

which may be regarded as a sort of "tripolar" equation to the locus of S.

* This relation is of course true for any straight line.

For want of courage I should never have proceeded to reduce this equation to Cartesian coordinates. It has, however, been done very kindly for me by Professor W. H. H. Hudson, of King's College, London: and I have verified the result, which is as follows:

$$(2s-3a)r^4-2s(s-a)^2r^2+a(s-a)^4=0$$

where $s=x+y$, $r^2=x^2+y^2$. The result is therefore a quintic having the lines $x=\frac{a}{2}$, $y=\frac{a}{2}$, $x+y-\frac{3a}{2}=0$ as asymptotes.

Mr C. E. Youngman has investigated, by trilinears, the case where the three given fixed points form an equilateral triangle in the present (November) number of the *Educational Times*. He also finds the locus to be a quintic having three real asymptotes parallel to the sides of the original triangle.

The question I should like to put is—Why does the locus come out of the fifth degree? Is there any geometrical view of the question which could account for this?

The foregoing result has been amply verified. It may be derived

(i) By reducing

$$(\mathrm{SA}-\mathrm{SO})^2+(\mathrm{SB}-\mathrm{SO})^2=4a^2$$

to Cartesians.

(ii) By eliminating λ from

$$2x/a=(\lambda^4+2\lambda-1)/\ (\lambda-1)(\lambda^2+1)$$
$$2y/a=(\lambda^4-2\lambda^3-1)/\lambda(\lambda-1)(\lambda^2+1).$$

(iii) By identifying

$$(x+\lambda y)^2-2ax-2\lambda^2ay=0$$

with

$$(x-\xi)^2+(y-\eta)^2=(\lambda x-y-\kappa)^2/(\lambda^2+1)$$

and then eliminating λ, κ from

$$(\lambda^2+1)\,\xi-\lambda\kappa=a$$
$$(\lambda^2+1)\,\eta+\kappa=\lambda^2a$$
$$(\lambda^2+1)(\xi^2+\eta^2)=\kappa^2.$$

These eliminations are none too easy: and analysis seems powerless to throw any light upon the cause for this particular locus being of such a high degree.

In studying this locus, I was led by analysis to the following theorem, which I afterwards verified geometrically:—

Let a variable parabola be described through three given fixed points E_1, E_2, E_3, and let a tangent triangle $t_1t_2t_3$ be drawn whose sides are parallel to, and one fourth the length of, the sides of $E_1E_2E_3$. Then the locus of t_1 is a cubic hyperbola (locus of a point the product of whose perpendiculars upon three fixed straight lines is constant).

The triangle $t_1t_2t_3$ being of fixed size and shape and moving without rotation, any of its special points (such as its circum-centre) will describe the same curve as t_1 shifted through a constant space in a certain direction.

Thus the circum-circle of $t_1t_2t_3$ has a constant radius and its centre moves on a cubic hyperbola. All that can be said at present is that the focus S lies *somewhere* on this circle. The fact, however, that the quintic curve, which is the locus of S, has three asymptotes coincident with the three asymptotes of the cubic hyperbola, which is the locus of the circum-centre of $t_1t_2t_3$, would seem to show that a geometrical solution (if it is ever to be found) will in some way proceed from the association of these two loci.