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INTERSECTION OF CONTINUA AND RECTIFIABLE CURVES

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Abstract We prove that for any non-degenerate continuum $K \subseteq \mathbb{R}^d$ there exists a rectifiable curve such that its intersection with K has Hausdorff dimension 1. This answers a question of Kirchheim.

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1. Introduction

A topological space K is called a *continuum* if it is compact and connected. The following question was asked by B. Kirchheim (personal communication, 2011).

Question 1.1. Does there exist a non-degenerate curve (or, more generally, a continuum) $K \subseteq \mathbb{R}^d$ such that every rectifiable curve intersects K in a set of Hausdorff dimension less than 1?

The motivation behind this question was [**3**, Example (b), p. 208], where Gromov seems to suggest that such non-degenerate curves exist. In this paper we answer Question 1.1 in the negative.

Theorem 1.2 (main theorem). *For any non-degenerate continuum $K \subseteq \mathbb{R}^d$ there exists a rectifiable curve such that its intersection with K has Hausdorff dimension* 1.

Remark 1.3. Finding a one-dimensional intersection is the best we can hope for, since any purely unrectifiable curve K in the plane (e.g. the Koch snowflake curve) has the property that the intersection of K and a rectifiable curve has zero $\mathcal{H}^1$ measure.

2. Preliminaries

The diameter and the boundary of a set A are denoted by $\operatorname{diam} A$ and ∂A, respectively. For $A \subseteq \mathbb{R}^d$ and $s \geqslant 0$ the *s-dimensional Hausdorff measure* is defined as

$$\mathcal{H}^s(A) = \lim_{\delta \to 0+} \mathcal{H}^s_\delta(A),$$

where

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} A_i)^s \colon A \subseteq \bigcup_{i=1}^{\infty} A_i \ \forall i \operatorname{diam} A_i \leqslant \delta\right\}.$$

Then, the *Hausdorff dimension* of A is

$$\dim_H A = \sup\{s \geqslant 0 \colon \mathcal{H}^s(A) > 0\}.$$

Let $A \subseteq \mathbb{R}^d$ be non-empty and bounded, and let $\delta > 0$. Set

$$N(A,\delta) = \min\left\{k \colon A \subseteq \bigcup_{i=1}^{k} A_i \ \forall i \operatorname{diam} A_i \leqslant \delta\right\}.$$

The *upper Minkowski dimension* of A is defined as

$$\overline{\dim}_M(A) = \limsup_{\delta\to 0+} \frac{\log N(A,\delta)}{-\log \delta}.$$

If $A \subseteq \mathbb{R}^d$ is non-empty and bounded, then it follows easily from the above definitions that

$$\dim_H A \leqslant \overline{\dim}_M(A).$$

For more information on these concepts see [**2**, **4**].

A continuous map $f \colon [a,b] \to \mathbb{R}^d$ is called a *curve*. Its *length* is defined as

$$\operatorname{length}(f) = \sup\left\{\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \colon n \in \mathbb{N}^+,\ a = x_0 < \cdots < x_n = b\right\}.$$

If $\operatorname{length}(f) < \infty$, then f is said to be *rectifiable*. We say that f is *naturally parametrized* if for all $x, y \in [a,b]$, $x \leqslant y$, we have that

$$\operatorname{length}(f|_{[x,y]}) = |x - y|.$$

We simply write $\Gamma = f([a,b])$ instead of f if the parametrization is obvious or not important for us. For every non-degenerate rectifiable curve Γ we have $0 < \mathcal{H}^1(\Gamma) < \infty$, so $\dim_H \Gamma = 1$. If $|f(x) - f(y)| \leqslant |x - y|$ for all $x, y \in [a,b]$, then f is called *1-Lipschitz*. Every naturally parametrized curve is clearly 1-Lipschitz.

3. The proof

First we need some lemmas. The following lemma is probably known, but we could not find a reference, so we outline its proof.

Lemma 3.1. *If a non-empty bounded set $A \subseteq \mathbb{R}^d$ has upper Minkowski dimension less than 1, then a rectifiable curve covers A.*

Proof. We can assume that A is compact and that $A \subseteq [0,1]^d$, since we can take its closure and transform it into the unit cube with a similarity; this does not change the upper Minkowski dimension of the set or whether it can be covered by a rectifiable curve.

For every $n \in \mathbb{N}$ we divide $[0,1]^d$ into non-overlapping cubes with edge length 2^{-n} in the natural way, and we denote the cubes that intersect A by

$$Q_{n,1}, Q_{n,2}, \dots, Q_{n,r_n},$$

where r_n is the number of such cubes. As every set with diameter at most 2^{-n} can intersect at most 3^d of the above cubes, we obtain that $r_n \leqslant 3^d N(A, 2^{-n})$. We fix s such that $\overline{\dim}_M(A) < s < 1$. By the definition of the upper Minkowski dimension, there exists a constant $c_1 \in \mathbb{R}$ such that, for all $n \in \mathbb{N}$,

$$r_n \leqslant c_1 2^{sn}. \tag{3.1}$$

Let $n \in \mathbb{N}$ and $i \in \{1, \dots, r_n\}$ be arbitrarily fixed. Let $P_{n,i}$ be the vertex of $Q_{n,i}$ that is the closest to the origin. If $Q_{n+1,j_1}, \dots, Q_{n+1,j_m}$ are the next level cubes contained by $Q_{n,i}$, then consider the broken line

$$\Gamma_{n,i} = P_{n,i} P_{n+1,j_1} P_{n+1,j_2} \cdots P_{n+1,j_m} P_{n,i}.$$

Thus,

$$\operatorname{length}(\Gamma_{n,i}) \leqslant (m+1) \operatorname{diam} Q_{n,i} \leqslant 2m\sqrt{d}\, 2^{-n}. \tag{3.2}$$

Let l_n be the sum of these lengths for all $i \in \{1, \dots, r_n\}$. Then, (3.2) and (3.1) imply that

$$l_n \leqslant 2r_{n+1}\sqrt{d}\, 2^{-n} \leqslant 2c_1 2^{s(n+1)}\sqrt{d}\, 2^{-n} = c_2 2^{(s-1)n}, \tag{3.3}$$

where $c_2 = c_1\sqrt{d}\, 2^{s+1}$. We set

$$L_n = \sum_{k=0}^{n} l_k \quad \text{and} \quad L = \sum_{k=0}^{\infty} l_k.$$

Since $s < 1$, (3.3) implies that $L < \infty$.

We now define the rectifiable curve covering A. First, we take the broken line $\Gamma_0 = \Gamma_{0,1}$ with its natural parametrization $g_0\colon [0, L_0] \to \Gamma_0$. Assume that the curves $g_k\colon [0, l_k] \to \Gamma_k$ are already defined for all $k < n$. At every point $P_{n,i}$, $i \in \{1, \dots, r_n\}$, we insert the broken line $\Gamma_{n,i}$ in Γ_{n-1}, so we obtain a naturally parametrized curve $g_n\colon [0, L_n] \to \Gamma_n$.

For every $n \in \mathbb{N}$ we define $f_n\colon [0, L] \to \Gamma_n$ such that

$$f_n(x) = \begin{cases} g_n(x) & \text{if } x \in [0, L_n], \\ g_n(L_n) & \text{if } x \in [L_n, L]. \end{cases}$$

We now prove that the sequence $\langle f_n \rangle$ uniformly converges. We fix $n \in \mathbb{N}$ and $x \in [0, L]$ arbitrarily. As

$$\sum_{n=0}^{\infty} l_n < \infty,$$

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it is enough to prove that $|f_{n+1}(x) - f_n(x)| \leqslant l_{n+1}$. By construction, there exists $y \in [0, L]$ such that $f_n(x) = f_{n+1}(y)$ and $|x - y| \leqslant l_{n+1}$. Since g_{n+1} is naturally parametrized, we obtain that

$$|f_{n+1}(x) - f_n(x)| = |f_{n+1}(x) - f_{n+1}(y)| \leqslant |x - y| \leqslant l_{n+1}.$$

Therefore, $\langle f_n \rangle$ uniformly converges to some $f\colon [0, L] \to \mathbb{R}^d$. As a uniform limit of 1-Lipschitz functions, f is also 1-Lipschitz, thus rectifiable.

It remains to prove that $A \subseteq f([0, L])$. Let $\boldsymbol{z} \in A$. We need to show that there exists $x \in [0, L]$ such that $f(x) = \boldsymbol{z}$. For every $n \in \mathbb{N}$ there exists $i_n \in \{1, \ldots, r_n\}$ such that $\boldsymbol{z} \in Q_{n,i_n}$. Let $x_n \in [0, L]$ such that $f_n(x_n) = P_{n,i_n}$ for all $n \in \mathbb{N}$. By choosing a subsequence, we may assume that x_n converges to some $x \in [0, L]$. Therefore,

$$f(x) = \lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} P_{n,i_n} = \boldsymbol{z}.$$

The proof is complete. □

The next lemma is [**1**, Lemma 6.1.25].

Lemma 3.2. *If A is a closed subspace of a continuum X such that $\emptyset \neq A \neq X$, then for every connected component C of A we have that $C \cap \partial A \neq \emptyset$.*

We also need the following technical lemma.

Lemma 3.3. *Suppose that $K \subseteq \mathbb{R}^d$ is a continuum contained by a unit cube Q and that K has a point on each of two opposite sides of Q. Then, for any positive integer N we can find N pairwise non-overlapping cubes $Q_1, \ldots, Q_N$, with edge length $1/N$ such that for each $i \in \{1, \ldots, N\}$ there exists a continuum $K_i \subseteq K \cap Q_i$, with the property that K_i has a point on each of two opposite sides of Q_i.*

Proof. Let $N \in \mathbb{N}^+$ be fixed. Set $S_0 = \{0\} \times [0, 1]^{d-1}$ and for all $i \in \{1, \ldots, N\}$ consider

$$S_i = \{i/N\} \times [0, 1]^{d-1} \quad \text{and} \quad T_i = [(i - 1)/N, i/N] \times [0, 1]^{d-1}.$$

We may assume that $Q = [0, 1]^d$ and that the two opposite sides intersecting K are S_0 and S_N. Let $\boldsymbol{x} \in K \cap S_0$ and $\boldsymbol{y} \in K \cap S_N$.

We now prove that for each $i \in \{1, \ldots, N\}$ there exists a continuum $C_i \subseteq K \cap T_i$ such that $C_i \cap S_{i-1} \neq \emptyset$ and $C_i \cap S_i \neq \emptyset$. Let C_1 be the component of $K \cap T_1$ containing $\boldsymbol{x}$. Applying Lemma 3.2 for $X = K$, $A = K \cap T_1$ and $C = C_1$ yields that $C_1 \cap S_1 \neq \emptyset$ (observe that if we consider $A = K \cap T_1$ as a subspace of K, then its boundary is contained in $K \cap S_1$). Let C_2' be the component of $K \cap (T_2 \cup \cdots \cup T_N)$ containing $\boldsymbol{y}$. Similarly as above, we obtain that $C_2' \cap S_1 \neq \emptyset$. If we continue this process, we get the required continua $C_2, \ldots, C_N$.

Finally, for each $i \in \{1, \ldots, N\}$ we construct a cube $Q_i \subseteq T_i$ with edge length $1/N$ and a continuum $K_i \subseteq Q_i$ such that K_i has a point on each of two opposite sides of Q_i. Clearly, the cubes Q_i will be pairwise non-overlapping, and it is enough to construct

Q_1 and K_1 (one can get Q_i, K_i similarly). We consider the standard basis of $\mathbb{R}^d$, $\boldsymbol{e}_1 = (1, 0, \ldots, 0), \ldots, \boldsymbol{e}_d = (0, 0, \ldots, 1)$. Set $A_1 = C_1$, $V_1 = \{0\} \times \mathbb{R}^{d-1}$, $W_1 = \{1/N\} \times \mathbb{R}^{d-1}$, $Z_1 = [0, 1/N] \times \mathbb{R}^{d-1}$ and $m(1) = 1$. Then, the definitions yield that A_1 has a point on both $V_{m(1)}$ and $W_{m(1)}$. Let $j \in \{2, \ldots, d\}$ and assume that A_k, V_k, W_k, Z_k and $m(k)$ are already defined for all $k < j$ such that A_k has a point on both $V_{m(k)}$ and $W_{m(k)}$. Let $\boldsymbol{x}_j \in A_{j-1}$ be a point that has minimal jth coordinate, and let V_j be the affine hyperplane that is orthogonal to $\boldsymbol{e}_j$ and contains $\boldsymbol{x}_j$. Set $W_j = V_j + (1/N)\boldsymbol{e}_j$, and let Z_j be the closed strip between V_j and W_j. If $A_{j-1} \subseteq Z_j$, then let $A_j = A_{j-1}$ and $m(j) = m(j-1)$. If $A_{j-1} \nsubseteq Z_j$, then let A_j be the component of $\boldsymbol{x}_j$ in $A_{j-1} \cap Z_j$ and $m(j) = j$; in this case, Lemma 3.2 yields $A_j \cap W_j \neq \emptyset$. Thus, A_j has a point on both $V_{m(j)}$ and $W_{m(j)}$. Let

$$Q_1 = \bigcap_{j=1}^{d} Z_j$$

and $K_1 = A_d$. Then, $Q_1 \subseteq S_1$ is a cube with edge length $1/N$ and $K_1 \subseteq Q_1$ is a continuum. As K_1 has a point on both $V_{m(d)}$ and $W_{m(d)}$, we obtain that K_1 has a point on each of two opposite sides of Q_1. The proof is complete. □

Now we are ready to prove Theorem 1.2.

Proof. By considering a similar copy of K, we may assume that K is contained by a unit cube Q and that K has a point on each of two opposite sides of Q.

Let $\varepsilon > 0$ be arbitrary. First, we prove the weaker result that there exists $A \subseteq K$ such that $1 - \varepsilon \leqslant \dim_H A = \overline{\dim}_M(A) < 1$. By Lemma 3.1, A is covered by a rectifiable curve. We fix an integer $N \geqslant 2$, for which

$$s := \frac{\log(N-1)}{\log N} \geqslant 1 - \varepsilon.$$

We construct $A \subseteq K$ such that $\dim_H A = \overline{\dim}_M(A) = s$. Set $\mathcal{I}_n = \{1, \ldots, N-1\}^n$ for every $n \in \mathbb{N}^+$. Iterating Lemma 3.3 implies that for all $n \in \mathbb{N}^+$ and $(i_1, \ldots, i_n) \in \mathcal{I}_n$ there exist cubes $Q_{i_1 \cdots i_n}$ in Q with edge length $1/N^n$ such that $Q_{i_1 \cdots i_n} \subseteq Q_{i_1 \cdots i_{n-1}}$, and there exist continua $K_{i_1 \cdots i_n} \subseteq K$ such that $K_{i_1 \cdots i_n} \subseteq Q_{i_1 \cdots i_n} \cap K_{i_1 \cdots i_{n-1}}$ and $K_{i_1 \cdots i_n}$ has a point on each of two opposite sides of $Q_{i_1 \cdots i_n}$. Set

$$A_n = \bigcup_{i_1=1}^{N-1} \cdots \bigcup_{i_n=1}^{N-1} K_{i_1 \cdots i_n},$$

and let

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Clearly, $A \subseteq K$ is compact.

On the one hand, as $A \subseteq A_n$ and A_n is covered by $(N-1)^n$ cubes of edge length $1/N^n$, we obtain that $N(A_n, \sqrt{d}/N^n) \leqslant (N-1)^n$ for all $n \in \mathbb{N}^+$. Therefore,

$$\overline{\dim}_M(A) \leqslant \frac{\log(N-1)}{\log N} = s.$$

On the other hand, we prove that $\mathcal{H}^s(A) > 0$. Assume that

$$A \subseteq \bigcup_{j=1}^{\infty} U_j;$$

it is enough to prove that

$$\sum_{j=1}^{\infty} (\operatorname{diam} U_j)^s \geqslant \frac{1}{2^d(N-1)}.$$

Clearly, we may assume that U_j is a non-empty open set with $\operatorname{diam} U_j < 1$ for each j, and the compactness of A implies that there is a finite subcover

$$A \subseteq \bigcup_{j=1}^{k} U_j.$$

We fix $n_0 \in \mathbb{N}^+$ such that $1/N^{n_0} < \min_{1 \leqslant j \leqslant k} \operatorname{diam} U_j$. For $j \in \{1, \ldots, k\}$ let

$$t_j = \#\{(i_1, \ldots, i_{n_0}) \in \mathcal{I}_{n_0} \colon U_j \cap K_{i_1 \cdots i_{n_0}} \neq \emptyset\}.$$

Since

$$A \subseteq \bigcup_{j=1}^{k} U_j,$$

we have that

$$\sum_{j=1}^{k} t_j \geqslant (N-1)^{n_0}. \tag{3.4}$$

We now show that, for all $j \in \{1, \ldots, k\}$,

$$(\operatorname{diam} U_j)^s \geqslant \frac{t_j}{2^d (N-1)^{n_0+1}}. \tag{3.5}$$

We fix $j \in \{1, \ldots, k\}$. There exists $0 \leqslant m < n_0$ such that $1/N^{m+1} \leqslant \operatorname{diam} U_j < 1/N^m$. Clearly, the number of cubes $Q_{i_1 \cdots i_m}$ at level m that intersect U_j is at most 2^d. Therefore, $t_j \leqslant 2^d (N-1)^{n_0 - m}$. On the other hand, $\operatorname{diam} U_j \geqslant 1/N^{m+1}$ implies that $(\operatorname{diam} U_j)^s \geqslant 1/(N-1)^{m+1}$, and (3.5) follows. By (3.4) and (3.5), we obtain that

$$\sum_{j=1}^{k} (\operatorname{diam} U_j)^s \geqslant \sum_{j=1}^{k} \frac{t_j}{2^d (N-1)^{n_0+1}} \geqslant \frac{1}{2^d (N-1)}.$$

Hence, $\mathcal{H}^s(A) > 0$. Therefore, $\dim_H A \geqslant s$, so $s \leqslant \dim_H A \leqslant \overline{\dim}_M(A) \leqslant s$. Thus, $1 - \varepsilon \leqslant \dim_H A = \overline{\dim}_M(A) < 1$.

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We are now in a position to prove that there exists a rectifiable curve $\varGamma$ with $\dim_H(\varGamma \cap K) = 1$. Pick an arbitrary point $\boldsymbol{x} \in K$ and let K_n be the intersection of K and the closed ball of radius $1/2^n$ centred at $\boldsymbol{x}$. Let C_n denote the component of K_n containing $\boldsymbol{x}$. Since C_n is a non-degenerate continuum by Lemma 3.2, we know that there exists $B_n \subseteq C_n$ such that $1 - (1/n) \leqslant \dim_H B_n = \overline{\dim}_M(B_n) < 1$. Therefore, Lemma 3.1 implies that there exist rectifiable curves $\varGamma_n$ covering B_n. We may assume that the end points of $\varGamma_n$ are in B_n. We can also assume that the length of $\varGamma_n$ is at most $1/2^n$. (Otherwise we split $\varGamma_n$ up into finitely many parts, each having length at most $1/2^n$; then, one of these parts intersects B_n in a set of Hausdorff dimension at least $1 - (1/n)$.) We concatenate the curves $\varGamma_n$ with line segments. Then, the full length of the line segments is at most

$$2\sum_{n=1}^{\infty} \frac{1}{2^n} = 2,$$

the full length of the curves $\varGamma_n$ is at most

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

so we get a rectifiable curve $\varGamma$ that covers

$$\bigcup_{n=1}^{\infty} B_n.$$

As

$$\dim_H\biggl(\bigcup_{n=1}^{\infty} B_n\biggr) = 1,$$

the intersection $\varGamma \cap K$ has Hausdorff dimension 1. The proof is complete. $\square$

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