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A THEORY OF UNIFORMITIES FOR GENERALIZED ORDERED SPACES

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1. Introduction. Let $(X,\mathscr{T})$ be a topological space equipped with a partial order $\leqq$ and let $C(\leqq)$ denote the continuous increasing functions mapping X into $\mathbf{R}$ (a function f: $X \rightarrow \mathbf{R}$ is increasing provided $f(x) \leqq f(y)$ whenever $x \leqq y$). Then $(X,\mathscr{T},\leqq)$ is an *N-space* (in the terminology of [**16**], a completely regular order space) provided $\mathscr{T}$ is the weak topology of $C(\leqq)$ and if $x \leqq y$ is false, then there is an $f \in C(\leqq)$ such that $f(y) < f(x)$. L. Nachbin's introduction of N-spaces was perspicacious, for these spaces now find application in a wide spectrum of mathematics. They are used in such diverse subjects as topological vector spaces [**17**; **18**; **23**], topological lattices, semi-lattices and semigroups [**10**; **14**] and topological dynamics [**7**]; papers of J. Blatter [**1**], J. Blatter and G. Seever [**2**] and R. Redfield [**19**; **20**] have established that these spaces play an important role in general topology as well. One attribute of N-spaces that makes them important in many areas of mathematics is that these spaces admit a full theory of uniform completion and compactification.

While we are particularly interested in N-spaces whose partial orders are linear orders, we have taken care to present our results in the more general setting of partial orders primarily for two reasons. First, as Nachbin observed, one can consider any completely regular space to be an N-space by taking equality as the partial order. Second, we believe that our use of [**11**] in the study of N-spaces unifies and clarifies many of the results concerning arbitrary N-spaces that have been obtained previously. For example, if $\mathscr{U}$ is a convex uniformity compatible with an N-space $(X,\mathscr{T},\leqq)$ and $(\tilde{X},\tilde{\mathscr{U}})$ is the uniform completion of $(X,\mathscr{U})$, then Redfield has constructed a partial order $\lesssim$ on $\tilde{X}$ that extends $\leqq$ so that $(\tilde{X},\mathscr{T}(\tilde{\mathscr{U}}),\lesssim)$ is an N-space. The methods of Section 2 of our paper characterize such partial orders and show that they can be obtained in a natural way.

In Section 3, we consider only those N-spaces whose partial order is a linear order. We show that these spaces are the generalized order spaces (GO spaces). It follows from this result and [**19**, Proposition 3.3] that a space $(X,\mathscr{T})$ is a GO space if, and only if, there is a linear order $\leqq$ on X and a compatible uniformity $\mathscr{U}$ on X that is order convex with respect to $\leqq$. Thus for GO spaces the appropriate theory of uniform spaces is the theory of order convex uniform spaces; and, as might be expected, some insight into both order convex uniform spaces and generalized ordered spaces is provided by the interplay that exists between these two classes of spaces.

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We adopt the terminology and notation of [**16**] except that, as is now customary, we use the term "quasi-uniformity" in lieu of Nachbin's original "semi-uniformity". In particular, as in [**16**], if $\mathscr{V}$ is a quasi-uniformity then $\mathscr{V}^*$ denotes the coarsest uniformity that contains $\mathscr{V}$. All N-spaces are assumed to be Hausdorff N-spaces.

2. N-spaces. We begin by recalling a fundamental result of [**16**].

THEOREM 2.1. *If* $(X, \mathscr{T})$ *is a topological space and* $\leqq$ *is a partial order on* X, *then* $(X, \mathscr{T}, \leqq)$ *is an N-space if, and only if, there is a quasi-uniformity* $\mathscr{V}$ *on* X *such that*

(1) $\mathscr{T} = \mathscr{T}(\mathscr{V}^*)$ and

(2) $\bigcap \mathscr{V} = G(\leqq)$, *where* $G(\leqq) = \{(x, y) | x \leqq y\}$.

We say that $\mathscr{V}$ *determines* $(X, \mathscr{T}, \leqq)$. If $(X, \mathscr{T}, \leqq)$ is an N-space and $\mathscr{U}$ is a uniformity on X compatible with $(X, \mathscr{T})$, then a quasi-uniformity $\mathscr{V}$ *determines* $\mathscr{U}$ provided $\mathscr{V}^* = \mathscr{U}$ and $\mathscr{V}$ determines $(X, \mathscr{T}, \leqq)$. Our assumption that $(X, \mathscr{T})$ is a Hausdorff space is equivalent to the assumption that $\bigcap \mathscr{V}$ is a partial order on X [**15**, Theorem 3.1].

Although Theorem 2.1 provides a characterization of N-spaces, these spaces may also be characterized as the pairwise completely regular bitopological spaces. Indeed the equivalence of pairwise completely regular spaces and pairwise uniform spaces observed in [**5**] may already be found expressed in terms of partial orders in [**16**].

We have need of the following reformulation of the principal results of [**11**].

THEOREM 2.2. *Let* $(X, \mathscr{U})$ *be a Hausdorff uniform space. If* $\mathscr{V}$ *is a quasi-uniformity on* X *such that* $\mathscr{V}^* = \mathscr{U}$, *then there is exactly one quasi-uniform space* $(\tilde{X}, \tilde{\mathscr{V}})$ *such that* $\tilde{\mathscr{V}} | X \times X = \mathscr{V}$ and $(\tilde{X}, \tilde{\mathscr{V}}^*)$ *is the completion of the uniform space* $(X, \mathscr{V}^*)$.

Throughout this paper we let $\sim$ denote the reflection defined in Theorem 2.2 and $\tilde{\leq}$ denote $\bigcap \tilde{\mathscr{V}}$. Note that the classical result that every uniform space has a unique completion is a special case of Theorem 2.2. The following theorem is also a direct consequence of Theorem 2.2.

THEOREM 2.3. *Let* $(X, \mathscr{T}, \leqq)$ *be an N-space determined by* $\mathscr{V}$. *Then* $\tilde{\mathscr{V}}$ *determines the N-space* $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{V}}^*), \tilde{\leq})$.

Let $(X, \mathscr{T}, \leqq)$ be an N-space. Then a compact N-space $(X', \mathscr{T}', \leqq')$ is an *N-compactification* of $(X, \mathscr{T}, \leqq)$ provided there is a mapping $k : X \rightarrow X'$ such that both k and k^{-1} are continuous and order preserving and $k(X)$ is a dense subset of X'.

LEMMA 2.4. *Let* $(X', \mathscr{T}', \leqq')$ *be an N-space determined by a quasi-uniformity* $\mathscr{V}'$ *and let* $(X, \mathscr{T})$ *be a subspace of* $(X', \mathscr{T}')$. *Then* $(X, \mathscr{T}, \leqq' | X \times X)$ *is an N-space, which is determined by* $\mathscr{V}' | X \times X$.

THEOREM 2.5. *Let* $(X, \mathscr{T}, \leqq)$ *be an N-space determined by a totally bounded quasi-uniformity* $\mathscr{V}$. *Then the N-space* $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{V}}^*), \tilde{\leqq})$ *determined by* $\tilde{\mathscr{V}}$ *is an N-compactification of* $(X, \mathscr{T}, \leqq)$. *Furthermore if* $(X', \mathscr{T}')$ *is a compactification of* $(X, \mathscr{T})$, $\leqq'$ *is a closed partial order on* X' *and* $k : X \to X'$ *is a dense topological and order embedding, then there is a totally bounded quasi-uniformity* $\mathscr{V}$ *that determines* $(X, \mathscr{T}, \leqq)$ *such that* $\tilde{\mathscr{V}}$ *determines* $(X', \mathscr{T}', \leqq')$.

Let $\mathscr{V}$ be a quasi-uniformity. It follows from [**8**, Theorem 1] that there is a largest totally bounded quasi-uniformity, denoted $\mathscr{V}_\omega$, that is contained in $\mathscr{V}$. Moreover $\mathscr{T}(\mathscr{V}) = \mathscr{T}(\mathscr{V}_\omega)$.

THEOREM 2.6. *Let* $(X, \mathscr{T}, \leqq)$ *be an N-space and let* $\mathscr{V}$ *be a quasi-uniformity that determines* $(X, \mathscr{T}, \leqq)$. *Then* $\mathscr{V}_\omega$ *also determines* $(X, \mathscr{T}, \leqq)$.

Proof. To show that $\mathscr{V}_\omega$ determines $(X, \mathscr{T}, \leqq)$ it suffices to show that $\mathscr{T}((\mathscr{V}_\omega)^*) = \mathscr{T}(\mathscr{V}^*)$ and that $\bigcap \mathscr{V}_\omega = \bigcap \mathscr{V}$. Since $\mathscr{T}((\mathscr{V}^{-1})_\omega) = \mathscr{T}(\mathscr{V}^{-1}) = \mathscr{T}((\mathscr{V}_\omega)^{-1})$, we have that $\mathscr{T}((\mathscr{V}_\omega)^*) = \mathscr{T}(\mathscr{V}_\omega \vee (\mathscr{V}_\omega)^{-1}) = \mathscr{T}(\mathscr{V}_\omega) \vee \mathscr{T}((\mathscr{V}_\omega)^{-1}) = \mathscr{T}(\mathscr{V}_\omega) \vee \mathscr{T}((\mathscr{V}^{-1})_\omega) = \mathscr{T}(\mathscr{V}) \vee \mathscr{T}(\mathscr{V}^{-1}) = \mathscr{T}(\mathscr{V}^*)$.

We now show that $\bigcap \mathscr{V}_\omega = \bigcap \mathscr{V}$. Since $\mathscr{V}_\omega \subset \mathscr{V}$, $\bigcap \mathscr{V} \subset \bigcap \mathscr{V}_\omega$. Suppose that $(x, y) \notin \bigcap \mathscr{V}$. Then there is a $V \in \mathscr{V}$ such that $V \subset X \times X - \{(x, y)\}$. Thus $X \times X - \{(x, y)\} \in \mathscr{V}_\omega$ and $(x, y) \notin \bigcap \mathscr{V}_\omega$.

In Theorem 2.6 we established that $\mathscr{T}((\mathscr{V}_\omega)^*) = \mathscr{T}(\mathscr{V}^*)$ and since $(\mathscr{V}^*)_\omega$ is compatible with $\mathscr{T}(\mathscr{V}^*)$, it follows that $\mathscr{T}((\mathscr{V}_\omega)^*) = \mathscr{T}((\mathscr{V}^*)_\omega)$. We note, however, that Example 4.1 shows that in general $(\mathscr{V}_\omega)^* \neq (\mathscr{V}^*)_\omega$.

THEOREM 2.7 [**16**, § 2 of Appendix]. *Let* $(X, \mathscr{T}, \leqq)$ *be an N-space. Then* $(X, \mathscr{T}, \leqq)$ *has an N-compactification.*

Proof. Let $\mathscr{V}$ be a quasi-uniformity that determines $(X, \mathscr{T}, \leqq)$. By Theorem 2.6, $\mathscr{V}_\omega$ is a totally bounded quasi-uniformity that determines $(X, \mathscr{T}, \leqq)$, and the result follows from Theorem 2.5.

A natural analogue in the theory of N-spaces of the well-known result that a compact space admits a unique uniformity has been obtained by J. Blatter and G. L. Seever as a consequence of their "Translation Lemma" [**2**, Lemma 5.8].

TRANSLATION LEMMA. *Let* $(X, \mathscr{T})$ *be a compact space, let* $\mathscr{V}$ *be a quasi-uniformity on* X *such that* $\mathscr{T}(\mathscr{V}^*) \subset \mathscr{T}$ *and let* $\leqq$ *denote* $\bigcap \mathscr{V}$. *Let* A *and* B *be subsets of* X. *Then there is an* $x \in \mathrm{cl}_{\mathscr{T}}(A)$ *and a* $y \in \mathrm{cl}_{\mathscr{T}}(B)$ *such that* $x \leqq y$ *if, and only if, for each* $V \in \mathscr{V}$, $V \cap A \times B \neq \emptyset$.

The equivalence of conditions (3) and (4) of the following theorem were obtained as a consequence of the Translation Lemma in [**2**], while the equivalence of conditions (1) and (4) were obtained by S. Salbany in [**21**, Theorem 4.5].

THEOREM 2.8. *Let* X *be a set, let* $\mathscr{W}$ *be a quasi-uniformity on* X *and let* $\mathscr{V}$ *be a quasi-uniformity on* X *such that* $\mathscr{T}(\mathscr{V}^*)$ *is compact. Then each two of the following statements are equivalent.*

(1) $\mathscr{T}(\mathscr{V}) = \mathscr{T}(\mathscr{W})$ *and* $\mathscr{T}(\mathscr{V}^{-1}) = \mathscr{T}(\mathscr{W}^{-1})$.
(2) $\mathscr{T}(\mathscr{V}^*) = \mathscr{T}(\mathscr{W}^*)$ *(equivalently* $\mathscr{V}^* = \mathscr{W}^*$*) and* $\mathscr{T}(\mathscr{V}^{-1}) = \mathscr{T}(\mathscr{W}^{-1})$.
(3) $\mathscr{T}(\mathscr{V}^*) = \mathscr{T}(\mathscr{W}^*)$ *(equivalently* $\mathscr{V}^* = \mathscr{W}^*$*) and* $\bigcap\mathscr{V} = \bigcap\mathscr{W}$.
(4) $\mathscr{V} = \mathscr{W}$.

Proof. (1) ⇒ (2): $\mathscr{T}(\mathscr{V}^*) = \mathscr{T}(\mathscr{V}) \vee \mathscr{T}(\mathscr{V}^{-1}) = \mathscr{T}(\mathscr{W}) \vee \mathscr{T}(\mathscr{W}^{-1}) = \mathscr{T}(\mathscr{W}^*)$.

(2) ⇒ (3): Let $p \in X$. Then $(\bigcap\mathscr{V})(p) = \mathrm{cl}_{\mathscr{T}(\mathscr{V}^{-1})}(p) = \mathrm{cl}_{\mathscr{T}(\mathscr{W}^{-1})}(p) = (\bigcap\mathscr{W})(p)$. Thus $\bigcap\mathscr{V} = \bigcap\mathscr{W}$.

(3) ⇒ (4): Since $\mathscr{T}(\mathscr{V}^*) = \mathscr{T}(\mathscr{W}^*)$ is compact, $\mathscr{V}^* = \mathscr{W}^*$; and so $\mathscr{V}$ and $\mathscr{W}$ are totally bounded. By [**2**, Corollary 5.10] and [**8**, Theorem 1] $\mathscr{V} = \mathscr{W}$.

(4) ⇒ (1): This implication is evident.

COROLLARY 2.9. *Let* $(X', \mathscr{T}', \leqq')$ *be an N-compactification of an N-space* $(X, \mathscr{T}, \leqq)$. *Then there is only one quasi-uniformity* $\mathscr{V}$, *which is totally bounded, that determines* $(X, \mathscr{T}, \leqq)$ *such that* $\tilde{\mathscr{V}}$ *determines* $(X', \mathscr{T}', \leqq')$.

COROLLARY 2.10 [**2**, Theorem 5.16] *Let* $(X, \mathscr{T}, \leqq)$ *be an N-space. Then there is a one-to-one correspondence between the N-compactifications of* $(X, \mathscr{T}, \leqq)$ *and the totally bounded quasi-uniformities (equivalently quasi-proximities) that determine* $(X, \mathscr{T}, \leqq)$.

3. Generalized ordered spaces. A *linearly ordered space* (abbreviated LOTS) is a triple $(X, \lambda, \leqq)$ where $\leqq$ is a linear order on X and λ is the usual open interval toplogy of the order $\leqq$. A *generalized ordered space* (abbreviated GO space) [**3**] is a triple $(X, \mathscr{T}, \leqq)$ where $\leqq$ is a linear order on X and $\mathscr{T}$ is a topology on X such that (1) the open interval topology of $\leqq$ is coarser than $\mathscr{T}$ and (2) every point of X has a local $\mathscr{T}$-base consisting of (possibly degenerate) intervals of X. In [**3**, 17 A 23], it is established that a topological space $(X, \mathscr{T})$ is a subspace of a LOTS if, and only if, there is a linear order on X such that $(X, \mathscr{T}, \leqq)$ is a GO-space. We begin this section by characterizing GO spaces in terms of N-spaces. If $(X, \mathscr{T}, \leqq)$ is an N-space and $\leqq$ is a linear order on X, then $(X, \mathscr{T}, \leqq)$ is called a *linear N-space*.

LEMMA 3.1. *Let* $(X, \mathscr{T}, \leqq)$ *be a linear N-space determined by a quasi-uniformity* $\mathscr{V}$. *Then* $\lesssim$ *is a linear order on X so that* $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{V}}), \lesssim)$ *is a linear N-space.*

Proof. The proof is by contradiction. By [**11**, Theorem 15],

$$\tilde{X} = \{\mathscr{F} \mid \mathscr{F} \text{ is a minimal } \mathscr{V}^* - \text{Cauchy filter on } X\}$$

and for each $V \in \mathscr{V}$,

$$\tilde{V} = \{(\mathscr{F}, \mathscr{G}) \in \tilde{X} \times \tilde{X} \mid \text{there is an } F \in \mathscr{F} \text{ and a } G \in \mathscr{G} \text{ with } F \times G \subset V\}.$$

Suppose that $\lesssim$ is not a linear order on $\tilde{X}$. Then there are $\mathscr{F}, \mathscr{G} \in \tilde{X}$ and

$V \in \mathscr{V}$ such that $(\mathscr{F}, \mathscr{G}) \notin \tilde{V}$ and $(\mathscr{G}, \mathscr{F}) \notin \tilde{V}$. Let $W \in \mathscr{V}$ such that $W^3 \subset V$. There exists an $F \in \mathscr{F}$ and a $G \in \mathscr{G}$ such that $F \times F \subset W$ and $G \times G \subset W$. Since $(\mathscr{F}, \mathscr{G}) \notin \tilde{W}$, there exists $(x, y) \in F \times G - W$ and since $\leqq$ is a linear order on X and $G(\leqq) = \bigcap \mathscr{V}$, $(y, x) \in W$. Since $(y, x) \in W$, $G \times F \subset W^{-1}(y) \times W(x) \subset W^3 \subset V$. It follows that $(\mathscr{G}, \mathscr{F}) \in \tilde{V} - a$, a contradiction.

THEOREM 3.2. *$(X, \mathscr{T}, \leqq)$ is a linear N-space if, and only if, it is a* GO *space.*

Proof. Suppose that $(X, \mathscr{T}, \leqq)$ is a GO space. Then $(X, \mathscr{T}, \leqq)$ can be topologically and order embedded in a compact ordered space $(X', \mathscr{T}', \leqq')$ [**3**, Theorem 17 A 23] and [**12**, Theorem 2.9]. By [**16**, Proposition 13], $(X', \mathscr{T}', \leqq')$ is a linear N-space so that $(X, \mathscr{T}, \leqq)$ as a subspace is a linear N-space.

Now suppose that $(X, \mathscr{T}, \leqq)$ is a linear N-space and let $\mathscr{V}$ be a totally bounded quasi-uniformity that determines $(X, \mathscr{T}, \leqq)$. By Theorem 2.5, $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{V}}^*), \lesssim)$ is an N-compactification of $(X, \mathscr{T}, \leqq)$ and by the preceding lemma $\lesssim$ is a linear order on $\tilde{X}$. As $\lesssim$ is closed in $\tilde{X} \times \tilde{X}$, the open interval topology is coarser than $\mathscr{T}(\tilde{\mathscr{V}}^*)$ and as the open interval topology is Hausdorff and $\mathscr{T}(\tilde{\mathscr{V}}^*)$ is compact, it follows that $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{V}}^*), \lesssim)$ is a LOTS. As in the proof of [**3**, Theorem 17 A 22] and [**12**, Theorem 2.9], it follows that $(X, \mathscr{T}, \leqq)$ is a GO space.

If $\leqq$ is a linear order on a (quasi-) uniform space $(X, \mathscr{U})$, then $\mathscr{U}$ is *convex* with respect to $\leqq$ provided that for each $U \in \mathscr{U}$ there is a $V \in \mathscr{U}$ such that for each $x \in X$, $V(x)$ is convex and $V \subset U$.

THEOREM 3.3 [**19**, Proposition 3.1]. *Let $(X, \mathscr{U})$ be a uniform space. If there is a quasi-uniformity $\mathscr{V}$ on X such that $\mathscr{V}^* = \mathscr{U}$ then $\mathscr{U}$ is convex with respect to $\bigcap \mathscr{V}$.*

THEOREM 3.4 [**19**, Proposition 3.3] *Let $(X, \mathscr{U})$ be a uniform space and let $\leqq$ be a linear order on X. Then $\mathscr{U}$ is convex with respect to $\leqq$ if, and only if, there is a quasi-uniformity that determines both $\mathscr{U}$ and $(X, \mathscr{T}(\mathscr{U}), \leqq)$.*

Proof. If $\mathscr{V}$ is a quasi-uniformity that determines both $\mathscr{U}$ and $(X, \mathscr{T}(\mathscr{U}), \leqq)$, then by Theorem 3.3, $\mathscr{U}$ is convex with respect to $\bigcap \mathscr{V} = G(\leqq)$.

Now suppose that $\mathscr{U}$ is a convex uniformity with respect to $\leqq$. Then, as is easily verified, $\mathscr{U}$ satisfies the conditions of [**16**, Theorem 10] so that $(X, \mathscr{T}(\mathscr{U}), \leqq)$ is determined by the quasi-uniformity generated by $\{G(\leqq) \circ W | W \in \mathscr{U}\}$.

THEOREM 3.5. *Let $(X, \mathscr{V})$ be a quasi-uniform space and suppose that $\bigcap \mathscr{V}$ is the graph of a linear order. Then $\mathscr{V}$ is convex with respect to this linear order.*

Proof. By Theorem 3.3, $\mathscr{V}^*$ is convex. Let $U \in \mathscr{V}$. Then there is an entourage $V \in \mathscr{V}$ such that $V \cap V^{-1} \subset U$ and for each $x \in X$, $(V \cap V^{-1})(x)$ is convex. We assert that for each $x \in X$, $V(x)$ is convex. For suppose that $\{a, c\} \subset V(x)$ and that $b \in X - V(x)$ such that $a \leqq b \leqq c$. Since $[x, \infty) \subset$

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$V(x)$, $b < x$; and as $a \leqq b < x$, $a \in V^{-1}(x)$. Thus $\{a, x\} \subset (V^{-1} \cap V)(x)$ and $b \in [a, x] \subset (V^{-1} \cap V)(x) \subset V(x)$, a contradiction.

To complete the proof it suffices to show that $V \subset U$. Let $(x, y) \in V$. If $x < y$, then $y \in (\bigcap \mathscr{V})(x) \subset U(x)$. If $y < x$, then $y \in V^{-1}(x)$ so that $y \in (V \cap V^{-1})(x) \subset U(x)$.

One consequence of Corollary 2.9 is that if $(X, \mathscr{T}, \leqq)$ is an N-space and $(X, \mathscr{T})$ is compact, then there is exactly one quasi-uniformity that determines $(X, \mathscr{T}, \leqq)$. This result is comparable to the following theorem.

THEOREM 3.6. *Let $(X, \mathscr{U})$ be a uniform space that is convex with respect to a linear order $\leqq$. Then there is exactly one quasi-uniformity $\mathscr{V}$ such that $\bigcap \mathscr{V} = G(\leqq)$ and $\mathscr{V}^* = \mathscr{U}$.*

Proof. In Theorem 3.4 it was established that there is a quasi-uniformity $\mathscr{V}$ such that $\bigcap \mathscr{V} = G(\leqq)$ and $\mathscr{V}^* = \mathscr{U}$. Suppose that $\mathscr{W}$ is another such quasi-uniformity and assume, without loss of generality, that there is a $V \in \mathscr{V} - \mathscr{W}$. Since $V \in \mathscr{V}^* = \mathscr{W}^*$, there is a $W \in \mathscr{W}$ such that $W \cap W^{-1} \subset V$. Moreover there exists $(x, y) \in W - V$. Since $(x, y) \notin \bigcap \mathscr{V} = G(\leqq)$ and $\leqq$ is a linear order, $(y, x) \in G(\leqq) = \bigcap \mathscr{W}$ so that in particular $(x, y) \in W^{-1}$. Then $(x, y) \in W \cap W^{-1} \subset V$, a contradiction.

THEOREM 3.7. *Let $(X, \mathscr{T}, \leqq)$ be a* GO *space and let $\mathscr{U}$ be a compatible convex uniformity. Then there is exactly one linear order $\lesssim$ on the completion $(\tilde{X}, \tilde{\mathscr{U}})$ of $(X, \mathscr{U})$ extending $\leqq$ such that $\tilde{\mathscr{U}}$ is convex with respect to $\lesssim$.*

Proof. Since $\mathscr{U}$ is convex, by Theorem 3.4 there is a quasi-uniformity $\mathscr{V}$ on X that determines $\mathscr{U}$. By Theorem 2.3 and Lemma 3.1, $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}), \lesssim)$ is a linear N-space. Suppose that $\tilde{\mathscr{W}}$ is a quasi-uniformity on $\tilde{X}$ such that $\tilde{\mathscr{W}}^* = \tilde{\mathscr{U}}$ and $\bigcap \tilde{\mathscr{W}}$ is the graph of some order on $\tilde{X}$ that extends $\leqq$. By Theorem 3.6, $\tilde{\mathscr{W}}|X \times X = \mathscr{V}$. By Theorem 2.2, $\tilde{\mathscr{V}} = \tilde{\mathscr{W}}$ and so $\bigcap \tilde{\mathscr{W}} = \bigcap \tilde{\mathscr{V}} = G(\lesssim)$.

Let $(X, \mathscr{T}, \leqq)$ be a GO space and let $(X', \mathscr{T}', \leqq')$ be a compact LOTS. Then $(X', \mathscr{T}', \leqq')$ is an *ordered compactification of* $(X, \mathscr{T}, \leqq)$ provided there is a mapping $k : X \to X'$ that is both a topological and an order embedding and $k(X)$ is a dense subset of X'.

THEOREM 3.8. *Let $(X', \mathscr{T}', \leqq')$ be any ordered compactification of a* GO *space $(X, \mathscr{T}, \leqq)$. Then there is exactly one quasi-uniformity $\mathscr{V}$ determining $(X, \mathscr{T}, \leqq)$ such that $\mathscr{V}$ determines $(X', \mathscr{T}', \leqq')$ and this quasi-uniformity is totally bounded.*

Proof. Since every ordered compactification is an N-compactification, the result is an immediate consequence of Corollary 2.9.

THEOREM 3.9. *Let $(X, \mathscr{T}, \leqq)$ be a* GO *space. Then the fine uniformity is convex.*

Proof. As every GO space can be embedded as a closed subspace of a LOTS,

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it follows from [**13**, Theorems 3.4 and 4.1 and Corollary 2.7] that the fine uniformity for X consists of all $\mathscr{T} \times \mathscr{T}$-neighborhood of the diagonal. Let U be a symmetric neighborhood of Δ and for each $x \in X$ let W_x be a convex open set about x such that $W_x \times W_x \subset U$. Then $W = \cup \{W_x \times W_x | x \in X\}$ is a member of the fine uniformity such that $W \subset U$ and for each $x \in X$, $W(x)$ is convex.

COROLLARY 3.10. *Let* $(X, \mathscr{T})$ *be a* GO *space and let* $\mathscr{U}$ *be a compatible uniformity of weight* $w(\mathscr{U})$. *Then there is a convex uniformity compatible with* $\mathscr{T}(\mathscr{U})$ *of weight* $w(\mathscr{U})$.

Proof. Let $\mathscr{F}$ denote the fine uniformity, which is convex. Let $\mathscr{B}$ be a base for $\mathscr{U}$ of minimal cardinality. For each $B \in \mathscr{B}$ there is a $V_{B,1} \in \mathscr{F}$ such that $V_{B,1} \subset B$ and for each $x \in X$, $V_{B,1}(x)$ is convex. Inductively we define for each $B \in \mathscr{B}$ a sequence $\langle V_{B,i} \rangle$ so that for each positive integer i, $V_{B,i+1} \circ V_{B,i+1} \subset V_{B,i}$, $V_{B,i} \in \mathscr{F}$ and for each $x \in X$, $V_{B,i}(x)$ is convex. Evidently

$$\{V_{B,i} | B \in \mathscr{B}, i \in N\}$$

is a subbase of cardinality $w(\mathscr{U})$ for a compatible convex uniformity.

COROLLARY 3.11. *Every metrizable* GO *space can be embedded as a dense subspace of a completely metrizable* GO *space.*

Proof. Let $(X, \mathscr{T}, \leqq)$ be a metrizable GO space. By the previous corollary there is a compatible convex uniformity $\mathscr{U}$ with a countable base. By the proof of Theorem 3.4, there is a quasi-uniformity $\mathscr{V}$ that has a countable base and determines both $(X, \mathscr{T}, \leqq)$ and $\mathscr{U}$. Consequently, $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{V}}), \lesssim)$ is a completely metrizable GO space in which $(X, \mathscr{T}, \leqq)$ is embedded as a dense subspace.

4. Examples and questions.

The theory of convex uniformities for GO spaces provides problems that arise naturally from the theory and so require no insight to pose. For example if $(X, \mathscr{T}, \leqq)$ is a GO space and $(X, \mathscr{T})$ admits a uniformity with a certain uniform property, it is of evident concern whether one or all the convex uniformities for $(X, \mathscr{T}, \leqq)$ also possess the given uniform property. The following problem is of equal interest but in general of greater difficulty. Suppose we assume that all the uniformities on a topological space X that are convex with respect to at least one of the linear orders on X that renders X a GO space have a certain uniform property. What topological properties are consequences of our assumption? This section does not consider all problems such as those suggested above, but it does point out some questions of a different vein and provides a number of examples in which the theory of quasi-uniformities prevails.

Example 4.1. Convex uniformities on $[0, 1)$.

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Let $X = \{x|0 \leqq x < 1\}$ with the topology and order inherited from $\mathbf{R}$, as usual let I denote the one point compactification of X and let $\mathscr{U}$ denote the only compatible uniformity on I. Since I is an ordered compactification, by Theorem 3.9, $\mathscr{U}$ is convex so that $\mathscr{U}|X \times X$ is a non-complete totally bounded convex uniformity on X. Suppose that $\mathscr{W}$ is a quasi-uniformity on X that determines a non-complete convex uniformity on X. By Theorem 2.2 and Lemma 3.1, $(\tilde{X}, \mathscr{T}(\widetilde{\mathscr{W}}^*), \lesssim)$ is a GO space in which X is order embedded as a dense subspace. Since I is the only GO space in which X can be so embedded, it follows that $(\tilde{X}, \mathscr{T}(\widetilde{\mathscr{W}}^*), \lesssim)$ is the closed unit interval with its usual topology and order. Hence $\mathscr{U} = \tilde{\mathscr{W}}^*$ and $\mathscr{U}|X \times X = \mathscr{W}^*$. We have established that $\mathscr{W}^*$ is the only compatible non-complete convex uniformity on X and that $\mathscr{W}^*$ is totally bounded.

Let $\mathscr{V}$ be a quasi-uniformity on X that determines a convex uniformity. By Theorem 3.3, $(\mathscr{V}_\omega)^* = \mathscr{W}^*$ and by Theorem 3.6, $\mathscr{V}_\omega = \mathscr{W}$. In particular, let $\mathscr{V}$ determine the fine uniformity. Then $(\mathscr{V}^*)_\omega$ is $\mathscr{C}^*(X)$ so that by [**9**, Corollary 10], $(\mathscr{V}^*)_\omega$ is not convex; thus $(\mathscr{V}^*)_\omega \neq (\mathscr{V}_\omega)^*$.

In [**22**, § 2], W. J. Thron and S. J. Zimmerman associate with each topological space $(X, \mathscr{T})$ a quasi-order

$$R_{\mathscr{T}} = \{(x, y) \in X \times X| \text{ every open set containing } y \text{ contains } x\}.$$

It is easily verified that if $\mathscr{V}$ is a compatible quasi-uniformity for $(X, \mathscr{T})$, then $\bigcap \mathscr{V}^{-1} = R_{\mathscr{T}}$.

THEOREM 4.2. *Let $(X, \mathscr{T}, \leqq)$ be a* LOTS *determined by a quasi-uniformity $\mathscr{V}$. Then $\mathscr{T}(\mathscr{V})$ is a minimal T_0 topology.*

Proof. Since $R_{\mathscr{T}(\mathscr{V})} = \bigcap \mathscr{V}^{-1}$, $R_{\mathscr{T}(\mathscr{V})}^{-1} = G(\leqq)$ and $R_{\mathscr{T}(\mathscr{V})}$ is a linear order. Thus for each $x \in X$, $\{y|x \leqq y\} = (\bigcap \mathscr{V})(x)$ and $\{y|y \leqq x\} = (\bigcap \mathscr{V}^{-1})(x)$. By [**22**, Corollary 1], it suffices to establish that

$$\mathscr{B} = \{\{X\} \cup \{\{y|x < y\} : x \in X\}$$

is a base for $\mathscr{T}(\mathscr{V})$. Let $G \in \mathscr{T}(\mathscr{V})$ and let $x \in G$. If there is a $y \in G$ such that $y < x$, then there is a $V \in \mathscr{V}$ such that $x \in (\bigcap \mathscr{V})(y) \subset V(y) \subset G$. Now suppose that x is the least member of G. There is a $V \in \mathscr{V}$ such that $V(x) \subset G$. Then $V(x) = (\bigcap \mathscr{V})(x)$ so that $\{y|x \leqq y\} = V(x) = G \in \mathscr{T}(\mathscr{V}) \subset \mathscr{T}(\mathscr{V}^*) = \mathscr{T}$. It follows that either $\{y|x \leqq y\} = G = X$ or x has immediate predecessor x^- and $x \in \{y|x^- < y\} = \{y|x \leqq y\} = G$.

In order to establish that $\mathscr{B}$ is a base for $\mathscr{T}(\mathscr{V})$ it remains to show that for each $x \in X$, $\{z|x < z\} = [(\bigcap \mathscr{V})(x) - \{x\}] \in \mathscr{T}(\mathscr{V})$. Let $x \in X$ and suppose that $x < y$. Then there is an entourage $V \in \mathscr{V}$ such that $x \notin V(y)$. By Theorem 3.5, $\mathscr{V}$ is convex. Hence there is a $W \in \mathscr{V}$ such that $W \subset V$ and $W(y)$ is convex. Thus $W(y) \subset \{z|x < z\}$.

COROLLARY 4.3. *Let $(X, \mathscr{T}, \leqq)$ be a* LOTS *determined by a quasi-uniformity $\mathscr{V}$. Then $(\mathscr{T})\mathscr{V}$ and $\mathscr{T}(\mathscr{V}^{-1})$ are minimal T_0 topologies.*

Example 4.4. A GO space that is determined by a quasi-uniformity $\mathscr{V}$ such that $\mathscr{T}(\mathscr{V})$ is minimal T_0 but $\mathscr{T}(\mathscr{V}^{-1})$ is not.

Let $(X, \mathscr{T}, \leqq)$ be the Sorgenfrey line. For each $p \in X$ and $\epsilon > 0$ let $V(\epsilon, p) = \{(x, y) | x \geqq y - \epsilon$ and if $p \leqq x < p + \epsilon$ then $p \leqq y\}$. Then $\{V(\epsilon, p) | p \in X, \epsilon > 0\}$ is a base for the desired quasi-uniformity.

Our first question is kin to the problem, solved by Thron and Zimmerman, of giving necessary and sufficient conditions that a topological space be a LOTS.

Question 4.5. Let $(X, \mathscr{T}, \leqq)$ be a GO space determined by a quasi-uniformity $\mathscr{V}$. Under what conditions on $\mathscr{V}$ is $(\tilde{X}, \mathscr{T}(\mathscr{V}^*), \lesssim)$ necessarily a LOTS?

If $(X, \mathscr{T}, \leqq)$ is a LOTS for which the cardinal of every Q-sequence whose limit is a gap is nonmeasurable, then νX is a GO space [**6**, Theorem 10.7]. It follows from Theorem 3.9 that if $(X, \mathscr{T}, \leqq)$ is a GO space of non-measurable cardinality, then νX is a GO space. This result and the result of [**9**, Observation following Corollary 11] that $\mathscr{C}^*(X)$ is convex if, and only if, X is pseudocompact motivate the following question.

Question 4.6. Let $(X, \mathscr{T}, \leqq)$ be a GO space. Is $\mathscr{C}(X)$ necessarily convex with respect to $\leqq$? If not, what are necessary and sufficient conditions that $\mathscr{C}(X)$ be convex?

Example 4.7 A LOTS $(X, \mathscr{T}, \leqq)$ determined by a quasi-uniformity $\mathscr{V}$ such that $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{V}}^*), \lesssim)$ is a GO space that is not a LOTS.

In [**6**, Example 10.11], L. Gillman and M. Henriksen give an example of a LOTS X for which νX is not a LOTS. By Theorem 3.9 the fine uniformity $\mathscr{U}$ on X is convex. Thus $\mathscr{U}$ is determined by a quasi-uniformity $\mathscr{V}$, and $\tilde{\mathscr{V}}$ determines νX. Consequently $(X, \mathscr{T}(\mathscr{V}^*), \cap \mathscr{V})$ is a LOTS and $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{V}}^*), \lesssim) = \nu X$ is a GO space that is not a LOTS.

We began this section by pointing out that to each problem in the study of uniform spaces and completely regular topological spaces there is a corresponding problem in the study of convex uniform spaces and linear N-spaces. We end the section by selecting three archetypal problems that illustrate this correspondence.

Question 4.8. For which topological spaces $(X, \mathscr{T})$ is it true that for each compatible uniformity $\mathscr{U}$ there is a linear order $\leqq$ on X such that $(X, \mathscr{T}, \leqq)$ is a GO space and $\mathscr{U}$ is convex with respect to $\leqq$? For which GO spaces $(X, \mathscr{T}, \leqq)$ is every compatible uniformity convex? We conjecture that every compatible uniformity for $(X, \mathscr{T}, \leqq)$ is convex if, and only if, $(X, \mathscr{T})$ is pseudocompact and admits an orderable one point compactification. The conjecture obtains if $(X, \mathscr{T}, \leqq)$ is a LOTS.

Question 4.9. Suppose $(X, \mathscr{T})$ is an N-space that admits a convex uniformity. If every compatible convex uniformity is totally bounded is $(X, \mathscr{T})$ pseudocompact?

Question 4.13. Let $(X, \mathcal{T}, \leqq)$ be a GO space. Under what conditions is the collection of all neighborhoods of $G(\leqq)$ a base for a quasi-uniformity on X that determines $(X, \mathcal{T}, \leqq)$?

References

1. J. Blatter, *Order compactifications of totally ordered topological spaces*, J. Approx. Theory *13* (1975), 56–65.
2. J. Blatter and G. L. Seever, *Interposition and lattice cones of functions*, Trans. Amer. Math Soc. *222* (1976), 65–96.
3. E. Čech, *Topological spaces*, Academia (Czechoslovak Acad. Sci.) Prague 1966.
4. R. Engelking and D. Lutzer, *Paracompactness in ordered spaces*, Fund. Math. *94* (1977), 49–58.
5. P. Fletcher, *Pairwise uniform spaces*, Notices Amer. Math. Soc. *83* (1965), 612.
6. L. Gillman and M. Henriksen, *Concerning rings of continuous functions*, Trans. Amer. Math. Soc. *77* (1954), 340–362.
7. O. Hajek, *Absolute stability of noncompact sets*, J. Differential Equations *9* (1971), 496–508.
8. W. Hunsaker and W. F. Lindgren, *Construction of quasi-uniformities*, Math. Ann. *188* (1970), 39–42.
9. R. P. Kaufman, *Ordered sets and compact spaces*. Colloq. Math. *17* (1967), 35–39.
10. J. D. Lawson, *Intrinsic topologies in topological lattices and semilattices*, Pacific J. Math. *44* (1973), 593–602.
11. W. F. Lindgren and P. Fletcher, *A construction of the pair completion of a quasi-uniform space*, Can. Math. Bull., to appear.
12. D. J. Lutzer, *On generalized ordered spaces*, Dissertationes Math. Rozprawy Mat. *89* (1971).
13. M. J. Mansfield, *Some generalizations of full normality*, Trans. Amer. Math. Soc. *86* (1957), 489–505.
14. M. W. Mislove, *Semigroups over trees*, Trans. Amer. Math. Soc. *195* (1974), 383–400.
15. M. G. Murdeshwar and S. A. Naimpally, *Trennungsaxioms in quasi-uniform spaces*, Nieuw Arch. Wisk. *14* (1966), 97–101.
16. L. Nachbin, *Topology and order*, Van Nostrand Mathematical Studies, No. 4 (Van Nostrand, N.Y., 1965).
17. Kung-Fu Ng, *On order and topological completeness*, Math. Ann. *196* (1972), 171–176.
18. A. L. Peressini, *Ordered topological vector spaces* (Harper and Row N.Y., 1967).
19. R. H. Redfield, *Uniformly convex totally ordered sets*, Proc. Amer. Math. Soc. *51* (1975), 289–294.
20. ——— *Ordering uniform completions of partially ordered sets*, Can. J. Math. *26* (1974), 644–664.
21. S. Salbany, *Bitopological spaces, compactifications and completions*, Mathematical monographs of the University of Cape Town, No. 1. Cape Town (1974).
22. W. J. Thron and S. J. Zimmerman, *A characterization of order topologies by means of minimal T_0-topologies*, Proc. Amer. Math. Soc. *27* (1971), 161–167.
23. Yau-chuen Wong and Kung-Fu Ng, *Partially ordered topological vector spaces* (Clarendon Press, Oxford, 1973).

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