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# LOCALIZATION, ALGEBRAIC LOOPS AND $H$-SPACES I

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If $(Y, \mu)$ is an $H$-Space (here all our spaces are assumed to be finitely generated) with homotopy associative multiplication $\mu$ and $X$ is a finite CW complex then $[X, Y]$ has the structure of a nilpotent group. Using this and the relationship between the localizations of nilpotent groups and topological spaces one can demonstrate various properties of $[X, Y]$ (see [**1**], [**2**], [**6**] for example). If $\mu$ is not homotopy associative then $[X, Y]$ has the structure of a nilpotent loop [**7**], [**9**]. However this algebraic structure is not rich enough to reflect certain significant properties of $[X, Y]$. Indeed, we will show that there is no theory of localization for nilpotent loops which will correspond to topological localization or will restrict to the localization of nilpotent groups.

In this paper we will describe a class of loops which we call *h-loops*. We will show that every loop which arises as $[X, Y]$ is an $h$-loop. As a consequence of this we will prove that every finite subloop of $[X, Y]$ is a product of loops of prime power order. We will define a type of localization for $h$-loops which is universal when the loop is finite and also corresponds to the topological localization. When the $h$-loop is not finite localization yields a set of loops, one of which corresponds to topological localization.

In a future paper [**9**] we define a category of loops which does admit a functorial localization and which satisfies the above restrictions.

The organization of the paper is as follows: Section one contains preliminary definitions and general properties for loops and $P$-local loops. Section two is composed of the definition of $h$-loop and extension properties for $h$-loops. The third section deals with the localization of $h$-loops. The final section concerns itself with the relationship between the topological and algebraic situations and with examples.

## Section 1. Preliminaries on loops and local loops.

*Definitions.* An *algebraic loop*, $G$, is a set with a multiplication and an identity $e$ such that the equations $xa = b$, $ay = b$ have unique solutions for $a, b$ in $G$.

A subloop $N$ of $G$ is *normal* if $aN = Na$, $(ab)N = a(bN)$ and $N(ab) = (Na)b$ for $a, b$ in $G$. Normal subloops correspond to kernels of loop homomorphisms.

The *center* of a loop $G$, $Z(G)$, is the collection,

$$\{x \in G|\quad ax = xa, a(bx) = (ab)x, a(xb) = (ax)b, x(ab) = (xa)b, a, b \in G\}.$$

For details on the above see [**3**].

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A short exact sequence $K \overset{i}{\rightarrowtail} G \overset{p}{\twoheadrightarrow} M$ is a *central extension* of $K$ by $M$ if $i(K)$ is contained in the center of $G$.

The following may be taken as the definition of a nilpotent loop. A loop $G$ is *nilpotent* of class $\leqq n$ if there exists a finite collection $\{K_r \rightarrow G_r \rightarrow M_r\}$, $r = 0, 1, \ldots, n$ of central extensions such that $M_0 = e$, $M_r \subset G_{r-1}$ and $M_n = G$.

In essentially the same manner as one classifies central extensions of groups, central extensions, $G$, of $K$ by $M$ are classified by set maps (cocycles), $f : M \times M \rightarrow K$ with $f(a, b) = 0$ if either $a$ or $b$ is $e$ with $G$ taken to be the product $M \times {}_f K$ and multiplication defined by $(m, k)(n, l) = (mn, k + l + f(m, n))$. Two cocycles $f$ and $g$ represent equivalent (in the standard way) extensions if and only if there is a map $h : M \rightarrow K$, $h(e) = 0$ such that $f = f' + \delta h$ where $\delta h(a, b) = h(a) - h(ab) + h(b)$.

If we denote by $Z(M, K)$ the abelian group of cocycles and by $B(M, K)$ the image of $\delta$ we get that central extensions of $K$ by $M$ correspond to $H(M, K) = Z(M, K)/B(M, K)$. It is straightforward to show that if $\alpha : K \rightarrow L$, $\beta : M \rightarrow N$ are homomorphisms then the induced maps $\alpha_* : H(M, K) \rightarrow H(M, L)$ and $\beta^* : H(N, K) \rightarrow H(M, K)$ correspond to push-outs and pullbacks.

Let $P$ be a set of primes.

A loop $G$ is *P-local* if the map defined by $x \rightarrow x^n$ is a bijection for every association if $(n, P) = 1$.

A homomorphism $g : G \rightarrow G'$ is a *P-equivalence* if

i) $x \in \ker g$ implies that there is an $n$, $(n, P) = 1$ and some association such that $x^n = e$ and

ii) there is an association and an $m$, $(m, P) = 1$ such that $x^m \in \operatorname{im} g$ if and only if $x \in G'$.

PROPOSITION 1.1. *Let $K \overset{i}{\rightarrow} G \overset{p}{\rightarrow} M$ be a central extension. If, for some association the map $x \rightarrow x^n$ is a bijection for any two of $K$, $G$, $M$, then it is a bijection for the third.*

*Proof.* Assume that $K$ and $M$ have the above property and let $g_1, g_2 \in G$ with $g_1{}^n = g_2{}^n$. Then $p(g_1{}^n) = p(g_2{}^n)$ so that $p(g_1) = p(g_2)$ since $M$ has the property. If we let $k'$ be such that $g_1 k' = g_2$ we have $k' = i(k)$. Since $k' \in Z(G)$ we get $g_2{}^n = g_1{}^n (k')^n$ or that $(k')^n = e$. But $K$ has the property so that $k' = e$ and $g_1 = g_2$. On the other hand if $g \in G$ let $m \in M$ be such that $m^n = p(g)$ and let $p(g') = m$. If we take $k \in K$ such that $(g')^n i(k^n) = g$ we get $(g' i(k))^n = g$.

The other cases proceed similarly.

COROLLARY 1.2. *If $M$ is a loop with the property that for some association $x \rightarrow x^n$ is a bijection for $n$, $(n, P) = 1$ implies that $M$ is P-local then any central extension of a P-local $K$ by $M$ is P-local.*

PROPOSITION 1.3. *Let*

$$\begin{array}{ccccc} K & \longrightarrow & G & \longrightarrow & M \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle r} \\ K' & \longrightarrow & G' & \longrightarrow & M' \end{array}$$

*be a commuting diagram of central extensions. If $k$ and $r$ are $P$-equivalences, so is $l$.*

*Proof.* The proof is the same as for groups ([**6**], 3.2).

PROPOSITION 1.4. *Let* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *be an exact sequence of abelian groups. Then the sequence*

$$H(M, A) \xrightarrow{\alpha_*} H(M, B) \xrightarrow{\beta_*} H(M, C)$$

*is exact.*

*Proof.* Trivially $\beta_*\alpha_* = 0$ so let $f : M \times M \to B$ be such that $\beta f = \delta h$ for some $h : M \to c$. Let $\lambda : C \to B$ be a section (i.e. $\lambda(0) = 0$, $\beta\lambda = 1$) and define $h' : M \to B$ by $h'(x) = \lambda h(x)$. Then $\beta(f - \delta h') = 0$ and hence $f - \delta h' : M \times M \to A$.

COROLLARY 1.5. $H(M, \prod_1^n A_i) = \prod_1^n (M, A_i)$

THEOREM 1.6. *Let $g : M \to M'$ be an onto $P$-equivalence for some set of primes $P$ and let $K$ be $P$-local. Then $g^* : H(M', K) \to H(M, K)$ is one to one.*

*Proof.* Let $f : M' \times M' \to K$ be such that $f(g \times g) = \delta h$ for $h : M \to K$, $h(e) = 0$.

If $x \in \ker g$, let $c' = g(c)$ and note that

$$0 = f(e, c') = f(g \times g)(x, c) = h(x) - h(xc) + h(c).$$

In a similar manner we can get $h(ab) = h(a) + h(b)$ if either $a$ or $b$ is in $\ker g$. Furthermore if $x \in \ker g$ then there is an $n$, $(n, P) = 1$ such that $x^n = e$ so that we get $0 = h(x^n) = nh(x)$. But $K$ is $P$-local so that $h(x) = 0$ if $x \in \ker g$.

Now define $h' : M' \to K$ by $h'(a') = h(a)$ if $g(a) = a'$. If $g(b) = a'$ then $a = bx$ with $x \in \ker g$. By the above $h(a) = h(bx) = h(b) + h(x) = h(b)$ so that $h'$ is well defined.

It then follows that $f(a', b') = f(g \times g)(a, b) = h(a) - h(ab) + h(b) = h'(a') + h'(b') = \delta h'(a', b')$ where $g(a) = a'$, $g(b) = b'$. Thus $f = \delta h'$ and $g^*$ is one to one.

## Section 2. $h$-loops.

*Definition* 2.1. A nilpotent loop $G$ is an *h-loop* if for any set of primes $Q$

i) $T_Q(G) = \{x \in G | x^n = e,$ some association, $n \in \langle Q \rangle\}$ is a finite subloop of $G$ ($\langle Q \rangle$ is the multiplicative set generated by $Q$);

ii) There is an $n \in \langle Q \rangle$ and a fixed association, $\mu^n$, such that $x \in T_Q(G)$ implies $x^n = e$; (call the pair $(n, \mu^n)$ a *power* of $T_Q(G)$.)

iii) Given $a$, $b$, $c$ in $G$ let $a'$ (resp. $c'$) be defined by $a'(bc) = (ab)c$, (resp. $a(bc) = (ab)c'$). If $a \in T_Q(G)$ (resp. $c \in T_Q(G)$) then $a' \in T_Q(G)$ (resp. $c' \in T_Q(G)$).

(It can be shown that i) implies ii).)

The following properties of $h$-loops are easily verified:

a) Subloops of $h$-loops are $h$-loops.

b) A finite product of $h$-loops is an $h$-loop, and hence

c) Pullbacks of $h$-loops are $h$-loops.

Let $f : M \times M \to K$ be a cocycle and let $E = M \times {}_f K$. Note that if $(a, k) \in E$, $(a, k)^n = (a^n, nk + F(n, a))$, where $F(n, a) = \sum f(a^i, a^j)$ with the summation and the associations for $a^i$, $a^j$ determined by the association for $n$. Thus $(a, k)^n = (e, 0)$ if and only if $a^n = e$ and $nk = -F(n, a)$.

LEMMA 2.2. *Let $M$ be an h-loop and let the central extension $E = M \times {}_f K$ with $K$ having finite torsion be such that $T_Q(E)$ is a finite subloop of order $m$ and $T_Q(E)$ has a power $(n, \mu^n)$. Then $m \in \langle Q \rangle$.*

*Proof.* Since $T_Q(E)$ has power $n$, $T_Q(E) = \{(a, k) | a^n = e, -nk = F(a, n)\}$. For a fixed $a \in M$ with $a^n = e$ let $S_a = \{k \in K | -nk = F(a, n)\}$. Since $S_a = \{r + k' | -nr = F(a, n), nk' = 0\}$ for some fixed $r \in K$ we see that $S_a$ is either empty or has order equal to the order of $\ker(n : K \to K)$ and hence is in $\langle Q \rangle$.

Consider $U = \{a \in M | S_a$ is not empty$\}$. If $a$, $a'$ are in $U$ then $S_a$ is their product $aa'$. But $U$ is a finite subset of $T_Q(M)$ and hence is a subloop. Since $T_Q(M)$ is nilpotent the order of $U$ is in $\langle Q \rangle$. Thus since $m$ is the product of the orders of $S_a$ and $U$ which implies the required result.

Since $h$-loops are nilpotent and since the order of $T_Q(M)$ is in $\langle Q \rangle$ if $M$ is a finitely generated abelian group, straightforward induction yields:

THEOREM 2.3. *Let $G$ be an h-loop. The order of $T_Q(G)$ is in $\langle Q \rangle$.*

Let $\alpha = (a, k_1)$, $\beta = (b, k_2)$ and $\gamma = (c, k_3)$ be in $E$ and let $\alpha' = (a', k_1')$ (resp. $\gamma' = (c', k_3')$) be such that $\alpha'(\beta\gamma) = (\alpha\beta)\gamma$ (resp. $\alpha(\beta\gamma) = (\alpha\beta)\gamma'$).

LEMMA 2.4. *Let $M$ be an h-loop and let $f : M \times M \to K$ be a co-cycle such that $E = M \times {}_f K$ is an h-loop. Let $Q$ be a set of primes with $(n, \mu^n)$ a power of $T_Q(E)$. Then*

i) $\alpha \in T_Q(E)$ *implies*

$$nf(a, b) + nf(ab, c) + F(n, a') = nf(b, c) + nf(a', bc) + F(n, a)$$

ii) $\gamma \in T_Q(E)$ *implies*

$$nf(b, c) + nf(a, bc) + F(n, c') = nf(a, b) + nf(ab, c') + F(n, c).$$

*Proof.* Since $\alpha'(\beta\gamma) = (\alpha\beta)\gamma$ we can solve for $\alpha' = (a', k_1')$ and get that $a'$ is such that $a'(bc) = (ab)c$ and

$$k_1' = k_1 + f(a, b) + f(ab, c) - f(b, c) - f(a', bc),$$
$$(\alpha')^n = ((a')^n, nk_1 + nf(a, b) + nf(ab, c) - nf(b, c) - nf(a', bc) + F(n, a'))$$

But $\alpha^n = (e, 0)$ implies $nk_1 = -F(n, a)$. Since $E$ is an $h$-loop and $(n, \mu^n)$ is a power of $T_Q(E)$ we see that $(\alpha')^n = (e, 0)$ which implies i).

The proof of ii) is entirely analogous.

PROPOSITION 2.5. *Let $K$ be $P$-local and $E = M \times_f K$ an $h$-loop. Then $f = f' + \delta h$, where $f'(a, b) = 0$ if either $a$ or $b$ are in the image of $T_{P'}(E)$. ($P'$ is the compliment of $P$).*

*Proof.* Define $h(b) = \sum_a f(a, b)$ where the sum is taken over all $a$ in the image of $T_{P'}(E)$.

Let $(n, \mu^n)$ be a power for $T_{P'}(E)$ and let $\alpha = (a, k)$, $\alpha' = (a', k')$ be as above. By 2.4 we get

$$n\textstyle\sum_a f(a, b) + n\sum_a f(ab, c) + \sum_a F(n, a') = n\sum_a f(b, c) + n\sum_a f(a', bc) + \sum_a F(n, a).$$

Note that the mapping $a \to a'$ is a bijection of the image of $T_{P'}(E)$ as is $a \to ab$ if $b$ is also in the image.

Thus if $b$ is in the image of $T_{P'}(E)$ the above reduces to

$$nh(b) + nh(c) = nmf(b, c) + nh(bc)$$

where $m$ is the order of the image of $T_{P'}(E)$. But both $n$ and $m$ are prime to $P$ and $K$ is $P$-local. Thus $f(b, c) = \delta h'(b, c)$ if $b$ is in the image of $T_{P'}(E)$. Using ii) of 2.4 we can complete the proof.

COROLLARY 2.6. *Let $K$ be $P$-local and $E = M \times_f K$ an $h$-loop. Then $f = f' + \delta h$ where $f'(a, b) = 0$ if either $a$ or $b$ are in $T_{P'}(M)$.*

*Proof.* By 2.5 it suffices to show that $T_{P'}(M)$ is the image of $T_{P'}(E)$.

Let $a \in T_{P'}(E)$ and let $\alpha \in E$ be such that $p(\alpha) = a$. Let $a^m = e$, $(m, P) = 1$ and let $\alpha = (a, k)$ where $-mk = F(m, a)$. Then $\alpha^m = (e, 0)$.

Let $M$ be an $h$-loop and let $H_h(M, K)$ denote the subset of $H(M, K)$ consisting of extensions which are $h$-loops. It is easy to see that $H_h(M, K)$ is a subgroup and that 1.6 holds for $H_h(M, K)$. 1.5 holds if the $A_i$ are finite groups of relatively prime orders.

THEOREM 2.7. *Let $g : M \to M'$ be an onto $P$-equivalence between $h$-loops and let $K$ be $P$-local. Then $g^* : H_h(M', K) \to H_h(M, K)$ is an isomorphism.*

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*Proof.* Since the pullback of $h$-loops is an $h$-loop $g^* : H_h(M', K) \to H_h(M, K)$. By 1.6 $g^*$ is one to one so let $f : M \times M \to K$ represent an element of $H_h(M, K)$. By 2.4 we can assume $f(a, b) = 0$ if $a \in T_{P'}(M)$ or $b \in T_{P'}(M)$.

Define $f : M' \times M' \to K$ by $f'(a', b') = f(a, b)$ where $g(a) = a'$, $g(b) = b'$. We show that $f'$ is well defined. If $g(a_1) = a'$ let $xa = a_1$ so that $x \in \ker g \subset T_{P'}(M)$. Let $(n, \mu^n)$ be a power for $T_{P'}(M)$. By 2.4 and 2.6

$$nf(x, a) + nf(xa, b) + F(n, x') = nf(c, b) + nf(x', ab) + F(n, x)$$

with $x' \in T_{P'}(M)$. But $f(x, a)$, $f(x', ab)$, $F(n, x)$ and $F(n, x')$ are all zero whence $nf(xa, b) = nf(a, b)$. Since $K$ is $P$ local $f(xa, b) = f(a, b)$. The same holds for $b$ and hence $f'$ is well defined.

It only remains to show that $f'$ represents an $h$-loop. To do this we need the following three results.

PROPOSITION 2.8. *Let $f : M \times M \to K$ represent an $h$-extension where $K$ is finite. If $M = \prod_{p_i} T_{p_i}(M)$ is finite with the $p_i$ distinct primes then $M \times {}_f K$ is also such a product.*

*Proof.* By 1.5 we need only consider the case where $K$ is an abelian $p$ group. Then the projection $\pi : M \to T_P(M)$ is an onto $p$-equivalence and $K$ is $p$-local. If $f : M \times M \to K$ is an $h$-extension the extension $f'$ of 2.7 is trivially an $h$-extension with $\pi^*[f'] = [f]$. But $\pi^* f'$ is a product of the required type.

COROLLARY 2.9. *If $G$ is a finite $h$-loop then $G$ is a product of loops of prime power order ($p$-loops).*

*Proof.* Since $G$ is nilpotent and since any finite abelian group has this property the result follows from 2.8 by induction on the class of $G$.

COROLLARY 2.10. *The homomorphic image of an $h$-loop with finite kernel is an $h$-loop.*

*Proof.* Let $g : G \to G'$ be onto with finite kernel and let $x' \in G'$ with $(x')^m = e$. Denote by $(m)$ the set of primes in $m$. By 2.9 $\ker g = T(m) \times T(m)'$ where $T(m)$ and $T(m)'$ are composed of $(m)$ torsion and $(m)'$ torsion respectively. Let $g(x) = x'$. Then $x^m \in \ker g$. If $x^m \in T(m)$ then $(x^m)^n = e$ for some $n \in \langle (m) \rangle$ and hence $x' \in g(T_{(m)}(G))$. If $x^m \in T(m)'$ then the map $(ax)^m \to ((ax)^m)^n$ for $n \in \langle (m) \rangle$, $a \in T(m)'$ is a bijection of $T(m)'$ and hence there is an $ax \in G$ such that $(ax)^m = e$ again yielding $x' \in g(T_{(m)}(G))$. If $x^m$ is arbitrary let $(n, \mu^n)$ be a power for $T_{(m)}(G)$. Then the map $a \to a^n$ maps ker $g$ into $T(m)'$ and we are reduced to the second case.

Thus we get $T_Q(G') = g(T_Q(G))$ whence the result follows.

We may now complete the proof of 2.7 by noting that the extension $M' \times {}_{f'} K$ is the image of $M \times {}_f K$ with finite kernel.

Note that if $g : M \to M'$ is a $P$-equivalence between $h$-loops such that every element of $M'$ not in the image of $g$ has no torsion then we may extend the $h$-

extension $f'$ from the image of $g$ by defining $f'(a, b) = 0$ if $a$ or $b$ is not in the image and still have an $h$-extension. If an element $x$ of $M'$ with $x \notin \operatorname{im} g$ is torsion it must be of the form $x = ab$ with $a \in T_P(M')$, $b \in T_{P'}(M')$. Furthermore, since $g$ is a $P$-equivalence $a \in \operatorname{im} g$. Thus by the above we may define $f'(x, g) = f'(a, y)$ and $f'(b, y) = 0$ and get an $h$-extension. Thus we still get $g^* : H_h(M', K) \to H_h(M, K)$ is onto if $g$ is a $P$-equivalence and $K$ is $P$-local. Similarly if all $x \notin \operatorname{im} g$ is torsion we see by 2.6 that $g^*$ is one to one.

Summing this up we get

THEOREM 2.11. *Let $g : M \to M'$ be a $P$-equivalence between $h$-loops and let $K$ be $P$-local. Then $g^* : H_h(M', K) \to H_h(M, K)$ is onto if $K$ is $P$-local. Furthermore, if every element of $M'$ not in the image of $g$ is torsion then $g^*$ is an isomorphism.*

## Section 3. Localizing $h$-loops.

*Definition* 3.1. Let $P$ be a set of primes. A *$P$-localization* of a loop $G$ is a homomorphism $l : G \to G'$ which is a $P$-equivalence with $G'$ a $P$-local $h$-hoop.

PROPOSITION 3.2. *$P$-localizations exist for $h$-loops.*

*Proof.* The proof is the standard induction on the class of the loop. Since there are localizations for abelian groups we may localize $h$-loops of class $\leqq 1$.

Assume we can localize all $h$-loops of class $< n$ and let $l_K : K \to K'$, $l_M : M \to M'$ be $P$-localizations. If $G$ is represented by an element of $H_h(M, K)$ define $l_G : G \to G'$ where $G'$ is any element of

$$(l_M^*)^{-1}(l_{K*}) : H_h(M, K) \to H_h(M, K') \leftarrow H_h(M', K').$$

This set is not empty by 2.11 and the induced map $l_G$ is a $P$-equivalence by 1.3. By 1.1 $G'$ is $P$ local.

In general it will be seen that $P$-localization is not unique and we denote the collection of $P$-localizations of $G$ by $\mathfrak{G}_P$.

However if $G$ is finite we do get uniqueness of $P$-localization.

PROPOSITION 3.3. *Let $G$ be a finite $h$-loop and let $l_P : G \to G'$ be a $P$-localization. Then for any homomorphism $g : G \to H$ with $H$ $P$-local there is a unique $\hat{g} : G' \to H$ with $g = \hat{g} l_P$.*

*Proof.* Since $G_P$ has no $P'$-torsion $l_P : G \to G'$ is an onto $P$-equivalence. Since $H$ is $P$-local we get ker $g$ contains ker $l_P$, whence the result follows.

Note that 3.3 implies that $P$-localization is unique for finite $h$-loops. Since finite $h$-loops are products of $p$-loops we get:

PROPOSITION 3.4. *Let $G$ be a finite $h$-loop. Then the $P$-localization of $G$ is the projection $\pi : G \to T_P(G)$.*

From the above it can be seen that many of the basic properties of nilpotent groups and their localizations ([**6**], Chapter 1) hold for finite $h$-loops and those that do not use uniqueness hold for all $h$-loops. We list two for reference:

PROPOSITION 3.5. ([**6**], 2.1) *Let $G$ be an $h$-loop. Then for any set of primes $Q$, $T_Q(G)$ is a normal subloop of $G$.*

PROPOSITION 3.6. ([**6**], 3.9) *If $G$ is a finite $h$-loop then $G$ is the pullback of $G_P$ and $G_{P'}$ over $G_0$.*

## Section 4. Topological results and examples.

PROPOSITION 4.1. *Let $X$ be a finite* CW *complex and $Y$ an $H$-space. Then $[X, Y]$ is an $h$-loop.*

*Proof.* We first show that if $l : Y \to T_P$ is the (topological) localization of $Y$ that $\ker(l_x : [X, Y] \to [X, Y_P])$ is $T_{P'}([X, Y])$.

To do this note that $l$ may be defined by $l : Y \to \lim_{\to} Y = Y_P$ where the limit is taken over an ordering of $n \in \langle P' \rangle$ with the maps raising elements to their $n^{\text{th}}$ power with any association [**2**]. Thus if $\alpha \in [X, Y]$ with $\alpha^n = e$ for some $n \in \langle P' \rangle$, $\alpha \in \ker l_*$. Conversely if $\alpha \in \ker l_*$ then $\alpha^n = e$ for some $n \in \langle P' \rangle$ and some association. Thus $T_{P'}([X, Y]) = \ker l_*$, and is finite by [**6**] or [**5**]. Thus condition i) of the definition of $h$-loop is satisfied. Condition ii) is shown similarly. For condition iii) note that if $\alpha \in \ker l_*$ $(\gamma \in \ker l_*)$ then

$$l_*((\alpha\beta)\gamma) = l_*(\beta\gamma) = l_*(\alpha'(\beta\gamma)) = l_*(\alpha')l_*(\beta\gamma)$$

so that $\alpha' \in \ker l_*$ $(\gamma' \in \ker l_*)$.

PROPOSITION 4.2. *If $X$ is finite* CW, *$Y$ is an $H$ space and $[X, Y]$ is finite then $[X, Y_P] \cong [X, Y]_P$. If $[X, Y]$ is not finite there is some $P$-localization of $[X, Y]$ isomorphic to $[X, Y_P]$.*

*Proof.* We may localize the space $Y$ by localizing the $k$ invariants of the Postnikov system for $Y$. Applying the functor $[X, -]$ to the Postnikov system yields a system of central extensions with localizations as in 3.2.

PROPOSITION 4.3. *Let $X$ be finite* CW, *$Y$ a finitely generated $H$-space (or the localization of such a space) and let $l : Y \to Y_P$ be a localization. Then* $\ker(l_* : [X, Y] \to [X, Y_P])$ *is prime to $P$.*

*Proof.* By 4.2 there is a localization of the loop $[X, Y]$ isomorphic to $[X, Y_P]$. By 4.1 $\ker l_* = T_{P'}([X, Y])$ and by 2.3 we get the required result.

Note that 4.3 is really the only nontrivial part in the proof of $A$ of [**8**]. So we get a purely algebraic proof of

THEOREM 4.4. ($A$ of [**8**]) *Let $X$, $Y$ and $l$ be as in* 4.3 *and let $\alpha \in [X, Y_P]$, then the order of $l_*^{-1}(\alpha)$ is prime to $P$ or is empty. Furthermore there is always a localization map $L : Y \to Y_P$ such that $L_*^{-1}(\alpha)$ is not empty.*

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PROPOSITION 4.5. *Let $X$ be finite* CW *and $Y$ an $H$-space. Any finite subloop of $[X, Y]$ is a product of $p$-loops.*

*Proof.* This is now an immediate corollary of 2.7.

Note that this result is more general than $B$ of [**8**].

At this point we show that the above program is not vacuous and raises some interesting questions.

PROPOSITION 4.6. *Not all nilpotent loops are h-loops.*

*Proof.* Define the extension $G$ of $Z/2$ by $Z/3$ by $f : Z/3 \times Z/3 \rightarrow Z/2$ with $f(1, 2) = 1$, $f = 0$ otherwise. Then the element $\alpha = (1, 0)$ of $G$ has the property that $\alpha\alpha^2 = e$ while repeated right multiplication of $\alpha$ generates $G$. This is essentially due to Bruck [**3**].

PROPOSITION 4.7. *If localization is defined by the universal property* 3.3 *then localization of h-loops can not restrict to either localization of nilpotent groups or the induced localization of homotopy classes of maps into an H-space.*

*Proof.* There are an infinite number of elements of $H_h(Z_2, Z/2)$ which restrict to the product in $H_h(Z, Z/2)$.

Finally we note that if $G$ is power associative (i.e. the subloop generated by a single element is a group) then properties i) and ii) in the definition of $H$-loop are automatically satisfied. It is not known if the loop $[X, Y]$ is power associative in general.

PROPOSITION 4.8. *Not all h-loops are power associative.*

*Proof.* Let $f : Z/3 \times Z/3 \rightarrow Z/3$ by $f(1, 2) = 1$, $f = 0$ otherwise. Then $Z/3 \times {}_f Z/3$ is an $h$-loop and not power associative.

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