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10.1017/mag.2024.83 © The Authors, 2024
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University Press on behalf of
The Mathematical Association

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108.33 Some inequalities for a triangle

In a recent Article [1] an upper bound was derived for $h_a + h_b + h_c$, the sum of the (lengths of the) altitudes of a triangle. In this Note we find a different upper bound in terms of R, the radius of the circumcircle. We also derive several other inequalities for a triangle which we have been unable to find in the literature, despite the fact that they follow quickly from known results.

Our notation is standard – for a triangle ABC, a, b and c are the side-lengths, $2s = a + b + c$ and r is the radius of the incircle. R is the radius of the circumcircle and r_a, r_b and r_c are the radii of the excircles, while h_a, h_b and h_c are the altitudes. The shorthand [WEIFFTTIE]. will indicate the phrase, "With equality if and only if the triangle is equilateral", throughout.

We need these known preliminary results, all easily proved and widely available in [2] and [3], for example.

Lemma 1: We have $h_a + h_b + h_c \leqslant \frac{\sqrt{3}}{2}(a + b + c)$. [WEIFFTTIE]. See [3, p. 274].

Lemma 2: We have $a = 2R \sin A$; $b = 2R \sin B$; $c = 2R \sin C$. See [2, p. 200].

Lemma 3: We have $\sin A + \sin B + \sin C \leqslant \frac{3}{2}\sqrt{3}$. [WEIFFTTIE]. See [2, p. 315}.

Lemma 4: We have $r_a + r_b + r_c - r = 4R$. See [2, p. 207].

Lemma 5 (Euler 1767): We have $R \geqslant 2r$. [WEIFFTTIE]. See [2, p. 216].

Euler's proof of this result was very beautiful. He showed that the distance d between the incentre and the circumcentre is given by $d^2 = R(R - 2r)$ and since $d^2 \geqslant 0$, we have $R \geqslant 2r$.

https://doi.org/10.1017/mag.2024.84 Published online by Cambridge University Press

Lemma 6: We have $r_a = s\tan\frac{1}{2}A$, $r_b = s\tan\frac{1}{2}B$ and $r_c = s\tan\frac{1}{2}C$. See [2, p. 205].

Lemma 7: We have $a\cot A + b\cot B + c\cot C = 2(R + r)$. See [2, p. 207].

Lemma 8: We have $r_a + r_b + r_c = \frac{1}{2}\left[a\cot\frac{1}{2}A + b\cot\frac{1}{2}B + c\cot\frac{1}{2}C\right]$. See [2, p. 206].

Lemma 9: We have $\dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c} = \dfrac{1}{r}$. See [2, p. 207].

Lemma 10: We have $R = \dfrac{abc}{4\Delta}$ and $r = \dfrac{\Delta}{s}$. See [2, p. 207].

Lemma 11: We have $r_ar_br_c = \dfrac{\Delta^2}{r}$. See [2, p. 207].

Main results

Theorem 1: We have $\frac{9}{2}R \geqslant h_a + h_b + h_c \geqslant 9r$ [WEIFFTTIE].

Proof: We already know that $h_a + h_b + h_c \geqslant 9r$ [1]. Now, by Lemma 1,

$$\begin{aligned} h_a + h_b + h_c &\leqslant \frac{\sqrt{3}}{2}(a + b + c) \\ &= \tfrac{1}{2}\sqrt{3}\left(2R\sin A + 2R\sin B + 2R\sin C\right) && \text{(by Lemma 2)} \\ &= \sqrt{3}\left(R\sin A + R\sin B + R\sin C\right) \\ &\leqslant \tfrac{1}{2}R\left(\sqrt{3} \times 3\sqrt{3}\right) && \text{(by Lemma 3)} \\ &= \tfrac{9}{2}R. \end{aligned}$$

Theorem 2: We have $\frac{9}{2}R \geqslant r_a + r_b + r_c \geqslant 9r$ [WEIFFTTIE].

Proof: By Lemma 4, $r_a + r_b + r_c = 4R + r \geqslant 8r + r$, by Lemma 5, so $r_a + r_b + r_c \geqslant 9r$. Also, by Lemma 5, $r_a + r_b + r_c = 4R + r \leqslant 4R + \frac{1}{2}R = \frac{9}{2}R$, so $\frac{9}{2}R \geqslant r_a + r_b + r_c$. [WEIFFTTIE] applies in both cases.

Corollary 1: In any triangle, at least one of r_a, r_b or r_c is less than or equal to $\frac{3}{2}R$,

Corollary 2: In any triangle, at least one of r_a, r_b or r_c is greater than or equal to $3r$.

Theorem 3: We have $\frac{9}{2}R \geqslant s\left[\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right] \geqslant 9r$. [WEIFFTTIE].

Proof: This follows at once from Lemma 6 and Theorem 2.

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Theorem 4: We have $3R \geqslant a\cot A + b\cot B + c\cot C \geqslant 6r$. [WEIFFTTIE]. This follows at once from Lemma 7.

Theorem 5: We have $2\left(a\cot A + b\cot B + c\cot C\right) - \left(r_a + r_b + r_c\right) = 3r$ and $2\left(r_a + r_b + r_c\right) - \left(a\cot A + b\cot B + c\cot C\right) = 6R$.

This follows at once from solving the equations in Lemmas 4 and 7 for r and R.

Theorem 6: $9R \geqslant a\cot\frac{1}{2}A + b\cot\frac{1}{2}B + c\cot\frac{1}{2}C \geqslant 18r$ [WEIFFTTIE].

This follows at once from Theorem 2 and Lemma 8.

Theorem 7: We have $3\sqrt{3}R \geqslant a + b + c \geqslant 6\sqrt{3}r$ [WEIFFTTIE].

Proof: The left-hand-side inequality is known (see [3]) but for completeness here is a quick proof:

By Lemma 2, $a + b + c = 2R\left(\sin A + \sin B + \sin C\right) \leqslant 2R\cdot\frac{3}{2}\sqrt{3} = 3\sqrt{3}R$, by Lemma 3. To show $a + b + c \geqslant 6\sqrt{3}r$, we proceed as follows:

Apply the AM-GM inequality to $s - a$, $s - b$, $s - c$ to get

$$(s - a) + (s - b) + (s - c) \geqslant 3\sqrt[3]{(s - a)(s - b)(s - c)}$$

$$\text{or} \qquad 3s - 2s = s \geqslant 3\sqrt[3]{(s - a)(s - b)(s - c)}$$

$$\text{or} \qquad s^3 \geqslant 27(s - a)(s - b)(s - c).$$

Next,

$$s^4 \geqslant 27s(s - a)(s - b)(s - c) = 27\Delta^2,$$

so $s^2 \geqslant 3\sqrt{3}\Delta$ and $s \geqslant 3\sqrt{3}\dfrac{\Delta}{s} = 3\sqrt{3}r$.

Finally, $2s = a + b + c \geqslant 6\sqrt{3}r$. [WEIFFTTIE].

Theorem 8: Let $t = \sqrt[4]{27} = 2.279507\ldots$. Then $\left(\frac{1}{2}t\right)R \geqslant \sqrt{\Delta} \geqslant tr$. [WEIFFTTIE].

Proof: We have $r_a + r_b + r_c = 4R + r$ (Lemma 4) and $r_ar_br_c = \frac{1}{r}\Delta^2$ (Lemma 11). Applying the AM-GM inequality, we get

$$r_a + r_b + r_c \geqslant 3\sqrt[3]{r_ar_br_c}$$

$$\text{or} \qquad 4R + r \geqslant 3\sqrt[3]{\frac{\Delta^2}{r}}$$

$$\text{or} \qquad r(4R + r)^3 \geqslant 27\Delta^2.$$

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Using $\frac{1}{2}R \geqslant r$ this becomes $27R^4 \geqslant 16\Delta^2$ or $\frac{1}{2}tR \geqslant \sqrt{\Delta}$, as claimed. Also, applying the AM-GM inequality to $\frac{1}{r_a}$, $\frac{1}{r_b}$ and $\frac{1}{r_c}$ we get

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} \geqslant 3\sqrt[3]{\frac{1}{r_a} \cdot \frac{1}{r_b} \cdot \frac{1}{r_c}}$$

which by Lemmas 9 and 11 gives $\Delta^2 \geqslant 27r^4$ or $\sqrt{\Delta} \geqslant tr$. So $\frac{t}{2}R \geqslant \sqrt{\Delta} \geqslant tr$. [WEIFFTTIE].

Theorem 9: We have $4s^3 \geqslant 27\Delta R$. [WEIFFTTIE].

Proof: Since $2s = a + b + c \geqslant 3\sqrt[3]{abc}$, the result follows at once by Lemma 10.

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108.34 One sharpening of the Garfunkel-Bankoff inequality and some applications

Garfunkel-Bankoff inequality

For a triangle *ABC* we use the notation $\sum \tan^2 \frac{A}{2}$ and $\prod \sin \frac{A}{2}$ for the cyclic sum and the cyclic product respectively. Then we have

Theorem 1: In any triangle *ABC* holds

$$\sum \tan^2 \tfrac{A}{2} \geqslant 2 - 8 \prod \sin \tfrac{A}{2} + \left(1 - 8 \prod \sin \tfrac{A}{2}\right) \prod \tan^2 \tfrac{A}{2}. \qquad (1)$$

Proof: By the well-known identities

$$\sum \tan^2 \frac{A}{2} = \frac{(4R + r)^2}{s^2} - 2, \quad \prod \sin \frac{A}{2} = \frac{r}{4R}, \quad \prod \tan \frac{A}{2} = \frac{r}{s}$$

where R, r and s are the circumradius, inradius and semiperimeter of the triangle, inequality (1) is transformed to

$$\frac{(4R + r)^2}{s^2} - 2 \geqslant 4 - \frac{2r}{R} + \frac{r^2}{s^2}\left(1 - \frac{2r}{R}\right)$$