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A GENERAL CONSTRUCTION OF SPACES OF THE TYPE OF R. C. JAMES

ROBERT H. LOHMAN AND PETER G. CASAZZA

In 1950, R. C. James [**7**] exhibited a nonreflexive Banach space with a basis that is of finite codimension in its second dual. This space is the first example of a quasi-reflexive space. General results on quasi-reflexive spaces have been obtained by P. Civin and B. Yood [**3**], and quasi-reflexive spaces with bases have been studied by D. Dean, B. L. Lin, and I. Singer [**4**; **12**]. More recently, L. Sternbach [**14**] has shown that if a Banach space X is quasi-reflexive of order n, there exist subspaces X_k of X, $1 \leqq k \leqq n$, such that $X_1 \subset \ldots \subset X_n = X$ and X_k is quasi-reflexive of order k. It follows that X_{k+1}/X_k is quasi-reflexive of order 1. Thus quasi-reflexive spaces of order 1 are rather fundamental and are important from the structural point of view just described. Moreover, they are generally useful in exhibiting pathological properties not usually found in the standard classical spaces.

J. Lindenstrauss [**9**] generalized James' example in one direction by showing that, given any separable Banach space X, there is a Banach space Y with a basis such that Y^{**}/Y is isomorphic to X. The purpose of this paper is somewhat different. Our aim is to examine James' original example and show that there is a general procedure underlying his construction, thus permitting a general construction of quasi-reflexive spaces of order 1 relative to a whole family of Banach spaces.

James' original example is obtained by introducing an interesting norm on a space of scalar sequences. One can, however, think of this norm as a norm that is constructed, using a certain procedure, with respect to l_2 with its usual unit vector basis. In this paper, we generalize James' construction by considering norms determined by Banach spaces with bases. We define norms of this type with respect to a Banach space X with a basis and discuss several properties of the generalized James' space of scalar sequences. It is shown, for instance, that this construction yields a Banach space [Theorem 1] which has a basis when the basis for X is symmetric and boundedly complete [Theorem 3]. If X is a reflexive Banach space with a symmetric, block p-Hilbertian basis, then the generalized James' space has a shrinking basis [Theorem 9] and is of codimension one in its second dual [Theorem 10], thus giving a general method for manufacturing quasi-reflexive spaces of order one. Under the same hypotheses, it is shown that the generalized James' space is topologically isomorphic to its second dual [Theorem 12].

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1. Preliminaries. Throughout the paper, X will denote a Banach space, over the reals or complexes, with a normalized, monotone Schauder basis (x_i). For basic definitions concerning Schauder bases, we refer the reader to [**13**]. If (x_i) is a symmetric basis, we assume that X is equipped with a symmetric norm $||\cdot||$; that is, $||\cdot||$ is symmetric in the sense that if $\sum_{i=1}^{\infty}\alpha_i x_i$ converges and γ, δ are permutations of the set of positive integers, then $\sum_{i=1}^{\infty}\alpha_{\gamma(i)}x_{\delta(i)}$ converges and

$$\left|\left|\sum_{i=1}^{\infty}\alpha_i x_i\right|\right| = \left|\left|\sum_{i=1}^{\infty}\alpha_{\gamma(i)}x_{\delta(i)}\right|\right|.$$

We let S denote the space of all scalar sequences and $P_n : S \to S$ denote the nth section mapping; that is, if $\alpha = (\alpha_i) \in S$, then $P_n(\alpha) = \sum_{i=1}^{n}\alpha_i e_i$, where e_i is the sequence with 1 in the ith place and 0 elsewhere. For any Banach space X with norm $||\cdot||$, X^* denotes the dual of X and $||\cdot||^*$ denotes the dual norm. The mapping $j : X \to X^{**}$ is defined by $j(x)(f) = f(x)$ for all $x \in X$ and $f \in X^*$. X is quasi-reflexive of order n(n finite) if $X^{**}/j(X)$ is of dimension n.

2. A family of James type spaces. Let $\mathscr{P}$ denote the family of all finite, increasing sequences $P = \{p_1, p_2, \ldots, p_{2n+1}\}$ of positive integers, where n is a positive integer. Let X be a Banach space with a (normalized, monotone) basis (x_i). For $\alpha = (\alpha_i) \in S$ and $P \in \mathscr{P}$ define

$$||\alpha||_P = \left|\left|\sum_{i=1}^{n}(\alpha_{p_{2i-1}} - \alpha_{p_{2i}})x_i + \alpha_{p_{2n+1}}x_{n+1}\right|\right|.$$

The generalized James space $J(x_i)$ is defined by

$$J(x_i) = \left\{\alpha = (\alpha_i) : \sup_{P\in\mathscr{P}} ||\alpha||_P < \infty \text{ and } \lim_i \alpha_i = 0\right\}.$$

Define the function N by

(1) $\quad N(\alpha) = \sup_{P\in\mathscr{P}} ||\alpha||_P, \alpha \in S.$

THEOREM 1. *The function N satisfies*

(i) $N(e_i) = 1$ *for all* i,

(ii) $\sup_n |\alpha_n| \leqq N(\alpha)$ *for all* $\alpha = (\alpha_n) \in J(x_i)$,

(iii) $N(\alpha) = \sup_k N(P_k(\alpha))$ *for all* $\alpha \in J(x_i)$.

Furthermore, $(J(x_i), N)$ *is a Banach space.*

Proof. Properties (i)–(iii) follow easily from (1) and the fact that (x_i) is a normalized, monotone basis.

In order to prove the last assertion, note that each of the functionals $||\cdot||_P$ is a semi-norm on S. Thus $||\alpha + \beta||_P \leqq ||\alpha||_P + ||\beta||_P \leqq N(\alpha) + N(\beta)$ for all $\alpha, \beta \in S$ and $P \in \mathscr{P}$. Taking the supremum over all $P \in \mathscr{P}$, we have $N(\alpha + \beta) \leqq N(\alpha) + N(\beta)$. From this it follows that $J(x_i)$ is a linear space and that N is semi-norm on $J(x_i)$. Since (ii) holds, N is a norm on $J(x_i)$.

Now let $(\alpha^{(r)})_{r=1}^{\infty}$ be a Cauchy sequence in $J(x_i)$, where $\alpha^{(r)} = (\alpha_i^{(r)})_{i=1}^{\infty}$. It follows from (ii) that there exists $\alpha = (\alpha_i) \in c_0$ such that $\alpha^{(r)} \to \alpha$ coordinatewise. Given $P = \{p_1, p_2, \ldots, p_{2n+1}\} \in \mathscr{P}$ and any r, we have

$$(2) \quad \left\| \sum_{i=1}^{n} (\alpha_{p_{2i-1}}{}^{(r)} - \alpha_{p_{2i}}{}^{(r)})x_i + \alpha_{p_{2n+1}}{}^{(r)}x_{n+1} \right\| \leqq N(\alpha^{(r)}) \leqq \sup_r N(\alpha^{(r)}).$$

Letting $r \to \infty$ in (2) shows that $||\alpha||_P \leqq \sup_r N(\alpha^{(r)})$ so that $\alpha \in J(x_i)$. Given $\epsilon > 0$, there exists an integer r_0 such that for all $r, s \geqq r_0$ and $P = \{p_1, p_2, \ldots, p_{2n+1}\} \in \mathscr{P}$, we have

$$(3) \quad \Bigg\| \sum_{i=1}^{n} [(\alpha_{p_{2i-1}}{}^{(r)} - \alpha_{p_{2i-1}}{}^{(s)}) - (\alpha_{p_{2i}}{}^{(r)} - \alpha_{p_{2i}}{}^{(s)})]x_i + (\alpha_{p_{2n+1}}{}^{(r)} - \alpha_{p_{2n+1}}{}^{(s)})x_{n+1} \Bigg\| \leqq \epsilon.$$

Fixing $s \geqq r_0$ and P and letting $r \to \infty$ in (3) shows that $||\alpha - \alpha^{(s)}||_P \leqq \epsilon$. Consequently, $N(\alpha - \alpha^{(s)}) \leqq \epsilon$ for all $s \geqq r_0$, completing the proof.

Remark 2. Throughout the rest of the paper, it is assumed that N is the norm on $J(x_i)$. The function N almost satisfies the properties of a proper sequential norm as defined by W. Ruckle [**11**]. However, if $\alpha = (\alpha_n) \in S$, property (iii) of Theorem 1 may fail, unless we require $\lim_n \alpha_n = 0$. For example, if $X = l_2$ and (x_i) is the usual unit vector basis of l_2, then for $\alpha = (2, 1, 1, \ldots)$, we have $N(\alpha) = \sqrt{2}$ and $N(P_k\alpha) = 2$ for all k.

THEOREM 3. *The sequence (e_i) is a monotone basic sequence in $J(x_i)$. If (x_i) is a boundedly complete, symmetric basis, then (e_i) is a basis for $J(x_i)$.*

Proof. Let n, m be positive integers and let $\alpha_1, \ldots, \alpha_{n+m}$ be scalars. It follows from Theorem 1 that

$$N\left(\sum_{i=1}^{n} \alpha_i e_i\right) = N\left(P_n\left(\sum_{i=1}^{n+m} \alpha_i e_i\right)\right) \leqq N\left(\sum_{i=1}^{n+m} \alpha_i e_i\right).$$

By Nikol'skiĭ's theorem [**13**, p. 58], (e_i) is a monotone basis for its closed linear span. Assume that (x_i) is a boundedly complete, symmetric basis. Then (e_i) will be a basis for $J(x_i)$, provided the linear span of the e_i's is dense in $J(x_i)$. Suppose this is not the case. Then there exists $\alpha \in J(x_i)$ and $\epsilon > 0$ such that $N(\alpha - P_k\alpha) > \epsilon$ for all k. Choose k_1 such that $|a_k| < \epsilon/12$ for all $k \geqq k_1$. Let $\alpha^{(1)} = \alpha - P_{k_1}\alpha = (\alpha_i^{(1)})$. A simple inductive argument shows that we obtain a sequence

$$k_1 \leqq p_{1,1} < \ldots < p_{1,2n_1+1} \leqq p_{2,1} < \ldots < p_{2,2n_2+1} \leqq p_{3,1} < \ldots < p_{3,n_3+1} \leqq \ldots$$

with the property that for each integer r,

$$\alpha^{(r)} = (\alpha_i^{(r)}), \alpha^{(r+1)} = \alpha^{(r)} - P_{p_{r,2n_r+1}}\alpha^{(r)}$$

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and

$$(4) \quad \left|\left| \sum_{i=1}^{n_r} (\alpha_{p_{r,2i-1}}{}^{(r)} - \alpha_{p_{r,2i}}{}^{(r)})x_i + \alpha_{p_{r,2n_r+1}}{}^{(r)}x_{n_r+1} \right|\right| > \epsilon.$$

For notational convenience we denote the sequence

$$p_{1,1}, \ldots, p_{1,2n_1}, p_{2,1}, \ldots, p_{2,2n_2}, p_{3,1}, \ldots, p_{3,2n_3}, \ldots$$

by $q_1, q_2, \ldots$. For any positive integer n, we have

$$(5) \quad \left|\left| \sum_{i=1}^{n} (\alpha_{q_{2i-1}} - \alpha_{q_{2i}})x_i \right|\right| \leqq N(\alpha).$$

It follows from (5) and the bounded completeness of (x_i) that the series $\sum_{i=1}^{\infty} (\alpha_{q_{2i-1}} - \alpha_{q_{2i}})x_i$ converges. Choose an integer i_0 such that for all j

$$\left|\left| \sum_{i=i_0}^{i_0+j} (\alpha_{q_{2i-1}} - \alpha_{q_{2i}})x_i \right|\right| < \epsilon/3.$$

We may assume that q_{2i_0-1} is the first term in a block of $p_{r,k}$'s; that is, $q_{2i_0-1} = p_{r,1}$. Choosing j_0 so that $q_{2(i_0+j_0)} = p_{r,2n_r}$, we see that

$$\left|\left| \sum_{i=1}^{n_r} (\alpha_{p_{r,2i-1}}{}^{(r)} - \alpha_{p_{r,2i}}{}^{(r)})x_i \right|\right| = \left|\left| (\alpha_{p_{r,1}}{}^{(r)} - \alpha_{p_{r,2}}{}^{(r)})x_1 \right.\right.$$

$$\left.\left. + \sum_{i=2}^{n_r} (\alpha_{p_{r,2i-1}} - \alpha_{p_{r,2i}})x_i \right|\right| \leqq \epsilon/3 + \left|\left| \sum_{i=i_0}^{i_0+j_0} (\alpha_{q_{2i-1}} - \alpha_{q_{2i}})x_i \right|\right| < 2\epsilon/3.$$

It now follows from (4) that

$$|\alpha_{p_{r,2n_r+1}}{}^{(r)}| = ||\alpha_{p_{r,2n_r+1}}{}^{(r)}x_{n_r+1}|| > \epsilon/3$$

which contradicts the fact that $|\alpha_k| < \epsilon/12$ for all $k \geqq k_1$.

In James' space there are two copies of l_2 whose algebraic direct sum is dense. The next result is the analogue of this property for the spaces $J(x_i)$ we have constructed. The straightforward proof is omitted.

THEOREM 4. *Let (x_i) be a symmetric basis for X. Then*
(i) *(x_i) dominates (e_i),*
(ii) *(x_i) is equivalent to (e_{2i}) and (e_{2i-1}).*

Remark 5. Since $(\sum_{i=1}^{n} e_i)$ is a bounded, non-convergent sequence in $J(x_i)$, (e_i) is never a boundedly complete basic sequence. This means that $J(x_i)$ is never reflexive. When X is reflexive and (x_i) is a symmetric basis for X, Theorem 4 shows that $J(x_i)$ contains two isomorphic copies of X whose algebraic direct sum is dense in $J(x_i)$. Since $J(x_i)$ is not reflexive, this sum is not a topological direct sum.

We close this section with a characterization of equivalence of unit vector bases in the sequence spaces we have introduced. The proof is a simple application of Theorem 4 and is left to the reader.

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COROLLARY 6. *Let (x_i), (y_i) be symmetric bases for the Banach spaces X and Y, respectively, and let N, M be the norms defined by* (1) *for (x_i) and (y_i), respectively. Then (e_i) in $(J(x_i), N)$ is equivalent to (e_i) in $(J(y_i), M)$ if and only if (x_i) is equivalent to (y_i).*

3. Quasi-reflexive spaces generated by symmetric spaces. In this section we consider quasi-reflexive properties of the spaces $J(x_i)$ generated by certain spaces X. Before proceeding to these results, the following definition will prove useful.

Definition 7. Let $1 \leqq p < \infty$. The basis (x_i) for X is block p-Hilbertian if there exists a constant K such that for each norm-bounded block basic sequence (z_k) with respect to (x_i) and each sequence (α_k) of scalars

$$\left\|\sum_{k=1}^{m} \alpha_k z_k\right\| \leqq (K \sup_k \|z_k\|)\left[\sum_{k=1}^{m} |\alpha_k|^p\right]^{1/p} m = 1, 2, \ldots .$$

Remark 8. The family of Banach spaces with block p-Hilbertian bases includes many of the well-known spaces with symmetric bases. For instance, the unit vector basis of the Lorentz sequence spaces [**1**; **2**] is a block p-Hilbertian basis. Also, let $M(x)$ be an Orlicz function satisfying the Δ_2 condition (cf. [**10**]). Then (e_i) is a symmetric basis for the Orlicz sequence space l_M. If $1 \leqq p < \infty$ and $M(x) \leqq x^p$ for all $x \geqq 0$, then (e_i) is a block p-Hilbertian basis for l_M. If X is a super-reflexive Banach space, then by the results of [**8**, Theorem 2], there is a number p for which $1 < p < \infty$ and (x_i) is block p-Hilbertian. Each block Hilbertian basis, as introduced in [**5**], is block 2-Hilbertian. Finally, note that if $1 < p < \infty$ and (x_i) is a block p-Hilbertian basis for X, then (x_i) is a shrinking basis.

THEOREM 9. *If (x_i) is a symmetric, block p-Hilbertian basis for X for some $p (1 \leqq p < \infty)$, then (e_i) is a block p-Hilbertian basic sequence. In particular, if $1 < p < \infty$, (e_i) is a shrinking basic sequence. Thus if X is reflexive and $1 < p < \infty$, (e_i) is a shrinking basis for $J(x_i)$.*

Proof. Let $0 = n_1 < n_2 < \ldots$ be an increasing sequence of integers, let

$$z_k = \sum_{i=n_k+1}^{n_{k+1}} \alpha_i e_i, \quad k = 1, 2, \ldots,$$

be a norm-bounded sequence, and let (t_k) be a sequence of scalars. K denotes the constant corresponding to (x_i) as guaranteed by Definition 7. Let $P = \{p_1, p_2, \ldots, p_{2n+1}\} \in \mathscr{P}$ and let m be a positive integer. Then

$$\left\|\sum_{k=1}^{m} t_k z_k\right\|_P = \left\|\sum_{i=1}^{n} (\beta_{p_{2i-1}} - \beta_{p_{2i}}) x_i + \beta_{p_{2n+1}} x_{n+1}\right\|,$$

where $\sum_{k=1}^{m} t_k z_k = (\beta_i)$. We can write

$$\sum_{i=1}^{n} (\beta_{p_{2i-1}} - \beta_{p_{2i}}) x_i + \beta_{p_{2n+1}} x_{n+1} = \sum_{k=1}^{m} t_k w_k + \sum_{k=1}^{m} t_k w_k',$$

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where each w_k is either 0 or a sum of terms of the form

$$\sum (\alpha_{p_{2i-1}} - \alpha_{p_{2i}})x_i$$

with each of the α_j's coming from the same z_k and each w_k' is either 0, of the form $\alpha_j x_i$, or $(\alpha_{p_{2i-1}} - \alpha_{p_{2i}})x_i$ with the α_j's coming from different z_k's. Using the facts that (x_i) is block p-Hilbertian, that $|\alpha_i| \leqq N(z_k)$ for each $k = 1, \ldots, m$ and $i = n_k + 1, \ldots, n_{k+1}$, and that $||w_k|| \leqq N(z_k)$, we have

$$\left|\left| \sum_{i=1}^{n} (\beta_{p_{2i-1}} - \beta_{p_{2i}})x_i + \beta_{p_{2n+1}}x_{n+1} \right|\right|$$
$$\leqq \left|\left| \sum_{k=1}^{m} t_k w_k \right|\right| + \left|\left| \sum_{k=1}^{m} t_k w_k' \right|\right| \leqq (K \sup_{1 \leqslant k \leqslant m} ||w_k||)$$
$$\times \left[\sum_{k=1}^{m} |t_k|^p \right]^{1/p} + \left(2K \sup_{1 \leqslant i \leqslant n_{m+1}} |\alpha_i|\right) \left[\sum_{k=1}^{m} |t_k|^p \right]^{1/p}$$
$$\leqq \left(3K \sup_{1 \leqslant k \leqslant m} N(z_k)\right) \left[\sum_{k=1}^{m} |t_k|^p \right]^{1/p}.$$

It follows that

$$N\left(\sum_{k=1}^{m} t_k z_k \right) \leqq \left(3K \sup_{1 \leqslant k \leqslant m} N(z_k)\right) \left[\sum_{k=1}^{m} |t_k|^p \right]^{1/p}.$$

Hence (e_i) is a block p-Hilbertian basic sequence.

THEOREM 10. *If X is a reflexive Banach space and (x_i) is a symmetric, block p-Hilbertian basis for $X (1 < p < \infty)$, then $J(x_i)$ is of codimension one in $J(x_i)^{**}$.*

Proof. The arguments are similar to those of R. C. James as presented in [**7**], but we present a proof for the sake of completeness. Let (e_i^*) denote the biorthogonal sequence associated with (e_i). By Theorem 9, (e_i^*) is a basis for $J(x_i)^*$. Let $F \in J(x_i)^{**}$. Since each $f \in J(x_i)^*$ can be written as $f = \sum_{i=1}^{\infty} f(e_i)e_i^*$, a simple computation shows that

$$N^{**}(F) \leqq \sup_n N\left(\sum_{i=1}^{n} F(e_i^*)e_i \right).$$

On the other hand, for a fixed n, let $\alpha = \sum_{i=1}^{n} F(e_i^*)e_i$. The linear functional f defined on the linear span of $\{\alpha, e_{n+1}, e_{n+2}, \ldots\}$ by $f(\alpha) = N(\alpha)$ and $f(e_i) = 0$ for $i > n$ has norm one. Thus f can be extended to $\bar{f} \in J(x_i)^*$ with $N^*(\bar{f}) = 1$. Moreover, it is easily seen that $F(\bar{f}) = N(\alpha) = N(\sum_{i=1}^{n} F(e_i^*)e_i)$. Therefore,

$$N^{**}(F) \geqq \sup_n N\left(\sum_{i=1}^{n} F(e_i^*)e_i \right),$$

so that

$$(7) \quad N^{**}(F) = \sup_n N\left(\sum_{i=1}^{n} F(e_i^*)e_i \right).$$

We now claim that $\lim_i F(e_i^*)$ exists. If not, there exists $\epsilon > 0$ and an increasing sequence (p_k) of positive integers such that $|F(e_{p_{2i-1}}^*) - F(e_{p_{2i}}^*)| > \epsilon$ for all i. Considering sets of the form $P_n = \{p_1, p_2, \ldots, p_{2n+1}\}$, we have for all n

$$\left|\left|\sum_{i=1}^{n} (F(e_{p_{2i-1}}^*) - F(e_{p_{2i}}^*))x_i\right|\right| \leqq \left|\left|\sum_{i=1}^{p_{2n+1}} F(e_i^*)e_i\right|\right|_{P_n} \leqq N^{**}(F)$$

by (7). Since X is reflexive, (x_i) is a boundedly complete basis, whence

$$\sum_{i=1}^{\infty} (F(e_{p_{2i-1}}^*) - F(e_{p_{2i}}^*))x_i$$

converges. Since $||x_i|| = 1$ for all i, $\lim_i |F(e_{p_{2i-1}}^*) - F(e_{p_{2i}}^*)| = 0$. The contradiction proves that $\lim_i F(e_i^*)$ exists.

Write $t = \lim_i F(e_i^*)$ and define F_0 on the span of $\{e_1^*, e_2^*, \ldots\}$ by $F_0(e_i^*) = 1$ for all i. Then F_0 can be extended to a linear functional $\bar{F}_0 \in J(x_i)^{**}$ with $N^{**}(\bar{F}_0) = 1$. Let α_t denote the constant sequence of t's and let $\alpha = (F(e_i^*))$. By (7), $\alpha - \alpha_t \in J(x_i)$. Since $(t\bar{F}_0 + j(\alpha - \alpha_t))(e_i^*) = F(e_i^*)$ for all i, $F = t\bar{F}_0 + j(\alpha - \alpha_t)$, completing the proof.

LEMMA 11. *Let (x_i) be a symmetric basis for X. Then the basic sequences $(e_i)_{i=1}^{\infty}$ and $(e_i)_{i=2}^{\infty}$ are equivalent.*

Proof. Let m be a positive integer and let $\alpha_1, \ldots, \alpha_m$ be scalars. Write

$$\alpha = \sum_{i=1}^{m} \alpha_i e_i \quad \text{and} \quad \beta = \sum_{i=1}^{m} \alpha_i e_{i+1}.$$

If $P = \{p_1, p_2, \ldots, p_{2n+1}\} \in \mathscr{P}$, then $Q = \{p_1 + 1, p_2 + 1, \ldots, p_{2n+1} + 1\} \in \mathscr{P}$ and $||\alpha||_P = ||\beta||_Q$. Consequently, $N(\alpha) \leqq N(\beta)$. On the other hand, let $Q = \{q_1, q_2, \ldots, q_{2n+1}\} \in \mathscr{P}$. If $q_1 \geqq 2$, then

$$P = \{q_1 - 1, q_2 - 1, \ldots, q_{2n+1} - 1\} \in \mathscr{P}$$

and $||\beta||_Q = ||\alpha||_P$. If $q_1 = 1$, then

$$||\beta||_Q \leqq ||\beta_{q_2}x_1|| + \left|\left|\sum_{i=2}^{n} (\beta_{q_{2i-1}} - \beta_{q_{2i}})x_i + \beta_{q_{2n+1}}x_{n+1}\right|\right| \leqq 2N(\alpha).$$

It follows that $N(\alpha) \leqq N(\beta) \leqq 2N(\alpha)$, completing the proof.

As an immediate consequence of Theorem 10 and Lemma 11 we have the following.

THEOREM 12. *If X is a reflexive Banach space and (x_i) is a symmetric, block p-Hilbertian basis for $X (1 < p < \infty)$, then $J(x_i)$ is topologically isomorphic to $J(x_i)^{**}$.*

We conclude with some open questions.

(1) Is there a condition, weaker than block p-Hilbertian, that can be placed

on (x_i) in order to guarantee that (e_i) is a shrinking basis for $J(x_i)$? If so, the proofs of Theorem 10 and Lemma 11 show that $J(x_i)$ will then be quasi-reflexive order one and topologically isomorphic to its second dual.

(2) Under what conditions does every subspace of $J(x_i)$ contain a subspace isomorphic to a subspace of X? This would lead to a generalization of a result R. Herman and R. Whitley [**6**] for James' space.

(3) Are there conditions under which $J(x_i)$ and $J(x_i)^*$ are isomorphic? It is not even known if James' space satisfies this property.

References

1. Z. Altschuler, P. G. Casazza and B. L. Lin, *On symmetric basic sequences in Lorentz sequence spaces*, Israel J. Math. *15* (1973), 140–155.

2. P. G. Casazza and B. L. Lin, *On symmetric basic sequences in Lorentz sequence spaces II*, Israel J. Math. *17* (1974), 191–218.

3. P. Civin and B. Yood, *Quasi-reflexive spaces*, Proc. Amer. Math. Soc. *9* (1957), 906–911.

4. D. W. Dean, B. L. Lin and I. Singer, *On k-shrinking and k-boundedly complete bases in Banach spaces*, Pacific J. Math. *32* (1970), 323–331.

5. E. Dubinsky, A. Pelczynski and H. P. Rosenthal, *On Banach spaces X for which* $\pi_2(\mathscr{L}_\infty, X) = B(\mathscr{L}_\infty, X)$, Studia Math. *44* (1972), 617–648.

6. R. Herman and R. Whitley, *An example concerning reflexivity*, Studia Math. *28* (1967), 289–294.

7. R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. *52* (1950), 518–527.

8. ——— *Super-reflexive spaces with bases*, Pacific J. Math. *41* (1972), 409–419.

9. J. Lindenstrauss, *On James' paper "Separable conjugate spaces,"* Israel J. Math. *9* (1971), 279–284.

10. J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math. *10* (1971), 379–390.

11. W. Ruckle, *On the construction of sequence spaces that have Schauder bases*, Can. J. Math. *18* (1966), 1281–1293.

12. I. Singer, *Bases and quasi-reflexivity of Banach spaces*, Math. Annalen *153* (1964), 199–209.

13. ——— *Bases in Banach spaces I.* (New York, Springer-Verlag, 1970).

14. L. Sternbach, *On k-shrinking and k-boundedly complete basic sequences and quasi-reflexive spaces*, Pacific J. Math. *37* (1971), 817–824.

Kent State University
Kent, Ohio;
University of Alabama
Huntsville, Alabama