Bull. Aust. Math. Soc. **87** (2013), 68–81
doi:10.1017/S0004972712000226

SHARP ESTIMATES FOR FUNCTIONS OF BOUNDED LOWER OSCILLATION

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(Received 30 January 2012)

Abstract

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally integrable function of bounded lower oscillation. The paper contains the proofs of sharp strong-type, weak-type and exponential estimates for the mean oscillation of f. In particular, this yields the precise value of the norm of the embedding $\mathrm{BLO} \subset \mathrm{BMO}_p$, $1 \le p < \infty$. Higher-dimensional analogues for anisotropic BLO spaces are also established.

2010 *Mathematics subject classification*: primary 42B37; secondary 49L20.

Keywords and phrases: BMO, BLO, Bellman function, best constants.

1. Introduction

A real-valued locally integrable function f defined on $\mathbb{R}^n$ is said to be in BMO, the space of functions of bounded mean oscillation, if

$$\sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|\,dx < \infty, \tag{1.1}$$

where the supremum is over all cubes Q in $\mathbb{R}^n$ with edges parallel to the coordinate axes, $|Q|$ denotes the volume of Q and

$$f_Q = \frac{1}{|Q|} \int_Q f(x)\,dx$$

is the mean of f over Q. A function f is said to have bounded lower oscillation if the term f_Q in (1.1) can be replaced by $\operatorname{ess\,inf}_Q f$, the essential infimum of f over Q. That is, $f \in \mathrm{BLO}$ if

$$\sup_Q \Big(f_Q - \operatorname*{ess\,inf}_Q f \Big) < \infty. \tag{1.2}$$

The suprema in (1.1) and (1.2) are denoted by $\|f\|_{\mathrm{BMO}}$ and $\|f\|_{\mathrm{BLO}}$. We consider a slightly less restrictive setting in which only the cubes Q within a given Q^0 are considered; to stress the dependence on Q^0, we use the notation $\mathrm{BMO}(Q^0)$ and $\mathrm{BLO}(Q^0)$.

Partially supported by MNiSW Grant N N201 397437.

The BMO class was introduced by John and Nirenberg in [6] and has played an important role in analysis and probability, since many classical operators (maximal, singular integral, and so on) map L^∞ into BMO. Another remarkable result, due to Fefferman [3], identifies BMO as a dual to the Hardy space H^1. Functions of bounded mean oscillation have very strong integrability properties (see for example [6]). In particular, the p-oscillation

$$\|f\|_{\mathrm{BMO}_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \right)^{1/p}, \quad 1 < p < \infty,$$

is finite for any $f \in \mathrm{BMO}$; it turns out to define the equivalent norm on $\mathrm{BMO}(\mathbb{R}^n)$.

The BLO class first appeared in the paper of Coifman and Rochberg [1], who used it to prove a decomposition property of BMO (see below). It is easy to see that BLO is contained in BMO; more precisely, the bound $\|f\|_{\mathrm{BMO}} \le 2\|f\|_{\mathrm{BLO}}$ holds true. Unlike BMO, the class BLO is not a linear space, as it is not even stable under multiplication by negative numbers ($\log |x|$ is in BLO, but $-\log |x|$ is not). Furthermore, we have $\mathrm{BLO} \cap (-\mathrm{BLO}) = L^\infty$, which follows from the very definition. On the other hand, the aforementioned result of Coifman and Rochberg states that each function from BMO can be written as a difference of two BLO functions. This decomposition has an interesting counterpart in the theory of Muckenhoupt weights: since

$$\mathrm{BMO} = \{\alpha\omega : \alpha \ge 0, \omega \in A_2\} \quad \text{and} \quad \mathrm{BLO} = \{\alpha\omega : \alpha \ge 0, \omega \in A_1\}$$

(see [1]), the statement $\mathrm{BMO} = \mathrm{BLO} - \mathrm{BLO}$ can be regarded as the logarithm of the factorisation $A_2 = A_1/A_1$ of Jones [7]; see also [4, 5].

The purpose of this paper is to provide some sharp upper bounds for the size of functions belonging to BLO. This type of problems, concerning exact information on the size of various classes of functions, has gathered a lot of interest in the literature recently: see for example [2, 8, 11–14] and references therein. We start with the one-dimensional setting and develop a Bellman-type approach which will enable the study of such problems in the BLO class. More precisely, we will show that the validity of a given estimate for BLO functions is equivalent to the existence of a corresponding function which satisfies appropriate concavity and majorisation properties. Then the approach is illustrated with a number of examples. In particular, we identify the embedding constant of BLO into the space $(\mathrm{BMO}, \|\cdot\|_{\mathrm{BMO}_p})$. We now give the precise formulation.

THEOREM 1.1. *For any interval $I^0 \subset \mathbb{R}$ and any function $f : I^0 \to \mathbb{R}$,*

$$\|f\|_{\mathrm{BMO}_p(I^0)} \le C_p \|f\|_{\mathrm{BLO}(I^0)}, \quad 1 \le p < \infty, \tag{1.3}$$

where

$$C_p = \left(\int_{-1}^{\infty} |s|^p e^{-s-1} \, ds \right)^{1/p}.$$

The constant C_p is the best possible.

https://doi.org/10.1017/S0004972712000226 Published online by Cambridge University Press

This leads to the following sharp exponential estimate (which can be viewed as an integral John–Nirenberg inequality for BLO spaces).

THEOREM 1.2. *For any interval $I^0 \subset \mathbb{R}$, any $f : I^0 \to \mathbb{R}$ and any $a > 0$,*

$$\sup_{I \subseteq I^0} \frac{1}{|I|} \int_I \exp(a|f(x) - f_I|)\, dx \le K(a\|f\|_{\mathrm{BLO}(I^0)}), \tag{1.4}$$

where $K(u) = \infty$ if $u \ge 1$ and

$$K(u) = \frac{1}{e}\Big(\frac{1}{1-u} + \frac{e^{u+1} - 1}{1+u}\Big)$$

for $u \in (0, 1)$. For each a, the bound on the right of (1.4) is the best possible.

We also provide sharp estimates for the distribution of BLO functions (which can be regarded as weak John–Nirenberg inequalities for BLO spaces).

THEOREM 1.3. *For any interval $I^0 \subset \mathbb{R}$, any $f : I^0 \to \mathbb{R}$ and any $\lambda > 0$,*

$$\sup_{I \subseteq I^0} \frac{1}{|I|} |\{x \in I : |f(x) - f_I| \ge \lambda\}| \le P\Big(\frac{\lambda}{\|f\|_{\mathrm{BLO}(I^0)}}\Big), \tag{1.5}$$

where

$$P(\lambda) = \begin{cases} 1 & \text{if } \lambda \le \frac{1}{2}, \\ 1 - \lambda(1 - e^{1-2\lambda}) & \text{if } \frac{1}{2} < \lambda \le 1, \\ e^{-\lambda} & \text{if } \lambda > 1. \end{cases}$$

For each $\lambda > 0$, the bound on the right-hand side of (1.5) is the best possible.

The Bellman function method for BLO functions is described in detail in the next section. In Section 3 we show how this approach can be used to obtain the results stated above. The final part of the paper contains higher-dimensional versions of the above theorems for anisotropic BLO spaces.

2. On the method of proof

The Bellman function method is a powerful technique which allows the study of various interesting estimates in probability and analysis. There are several papers which contain a detailed description of the general methodology as well as many examples and applications, see for example [9–11, 13–15]. We present the appropriate modification of the technique so that it works for BLO functions. We would like to stress here that the reasoning we present is not just the mere repetition of the arguments used in the above papers; the passage to the BLO setting will require some additional effort.

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We start with an auxiliary technical fact.

Lemma 2.1. *Let $I \subset \mathbb{R}$ be a bounded interval. Suppose that $\varepsilon \in (0, \frac{1}{2})$ is a fixed number and let $f : I \to \mathbb{R}$ be an arbitrary function satisfying $\|f\|_{\mathrm{BLO}(I)} < 1 - \varepsilon$. Then there exists a splitting of I into two intervals I_- and I_+ for which*

$$\begin{aligned} x_- &:= \frac{1}{|I_-|} \int_{I_-} f(x)\,dx - \operatorname*{ess\,inf}_I f < 1, \\ x_+ &:= \frac{1}{|I_+|} \int_{I_+} f(x)\,dx - \operatorname*{ess\,inf}_I f < 1 \end{aligned} \tag{2.1}$$

and such that the splitting parameters $\alpha_\pm = |I_\pm|/|I|$ belong to $[\varepsilon, 1 - \varepsilon]$.

Proof. First note that for any splitting at least one estimate in (2.1) is valid: in fact, $\min\{x_-, x_+\} < 1 - \varepsilon$, which is due to

$$\alpha_- x_- + \alpha_+ x_+ = \frac{1}{|I|} \int_I f(x)\,dx - \operatorname*{ess\,inf}_I f < 1 - \varepsilon. \tag{2.2}$$

We begin with splitting I into two halves (that is, we put $\alpha_- = \alpha_+ = \frac{1}{2}$). If both estimates in (2.1) are valid, we take this splitting. If this is not the case, assume, with no loss of generality, that $x_- < x_+$ and start decreasing α_- by shrinking the interval I_-. Since x_- and x_+ are continuous functions of α_-, we have two possibilities: either there is $\alpha_- \in [\varepsilon, \frac{1}{2})$ for which $x_- \le x_+ < 1$ (and then we are done), or for all $\alpha_- \in [\varepsilon, \frac{1}{2})$ we have $x_+ \ge 1$. Suppose that the second possibility occurs and take the splitting corresponding to $\alpha_- = \varepsilon$. Since $x_- \ge 0$, dividing both sides of (2.2) by $\alpha_+ = 1 - \varepsilon$ yields $x_+ < 1$, a contradiction. □

We turn to the description of the Bellman method in the BLO setting. Let V be a nonnegative, continuous function defined on $[-1, \infty)$, let $c \ge 0$ and suppose that we are interested in proving that

$$\frac{1}{|I|} \int_I V(f(x))\,dx \le c \tag{2.3}$$

for any bounded interval I and any $f : I \to \mathbb{R}$ satisfying $f_I = 0$ and $\|f\|_{\mathrm{BLO}(I)} \le 1$. (Note that the assumptions on f imply that $\operatorname{ess\,inf}_I f \ge -1$, so the integral in (2.3) is well-defined.) To handle this problem, distinguish the set

$$\mathcal{D} = \{(x, y) : -1 \le y \le x < y + 1\}$$

and suppose that $U : \mathcal{D} \to \mathbb{R}$ is a function which enjoys the following four properties:

$$\text{for any } x \ge -1,\ U(x, x) \ge V(x), \tag{2.4}$$

$$\text{for any fixed } y \ge -1, \text{ the function } U(\cdot, y) \text{ is concave on } [y, y + 1), \tag{2.5}$$

$$\text{for any fixed } x \ge 0, \text{ the function } U(x, \cdot) \text{ is nonincreasing on } (x - 1, x]. \tag{2.6}$$

$$\lim_{y \downarrow -1} U(0, y) \le c. \tag{2.7}$$

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Here, as usual, $U(\cdot, y)$ and $U(x, \cdot)$ denote the functions $x \mapsto U(x, y)$ and $y \mapsto U(x, y)$, respectively. The connection between the inequality (2.3) and the special function U satisfying the above conditions is described in the following statement.

Theorem 2.2. *Suppose that U satisfies the conditions (2.4)–(2.7). Then for any bounded interval $I \subset \mathbb{R}$ and any function $f : I \to \mathbb{R}$ satisfying $f_I = 0$ and $\|f\|_{\mathrm{BLO}(I)} \leq 1$ the inequality (2.3) holds true.*

Proof. Fix I and f as in the statement; with no loss of generality, we may assume that $\|f\|_{\mathrm{BLO}(I)} < 1$. Indeed, if $\|f\|_{\mathrm{BLO}(I)} = 1$, we pick $\kappa \in (0, 1)$ and work with the function κf which has the BLO norm smaller than 1; having shown the estimate

$$\frac{1}{|I|} \int_I V(\kappa f(x))\, dx \leq c,$$

we let $\kappa \to 1$ and get the claim by virtue of Fatou's lemma (since V is continuous and nonnegative).

So, let $\|f\|_{\mathrm{BLO}(I)} < 1$ and pick $\varepsilon \in (0, \frac{1}{2})$ such that $\|f\|_{\mathrm{BLO}(I)} < 1 - \varepsilon$. It is convenient to split the reasoning into three intermediate parts.

Step 1. Consider the following family $\{\mathcal{I}^n\}_{n\geq 0}$ of partitions of I, generated by the inductive use of Lemma 2.1. We start with $\mathcal{I}^0 = \{I\}$; then, given $\mathcal{I}^n = \{I^{n,1}, I^{n,2}, \ldots, I^{n,2^n}\}$, we split each $I^{n,k}$ according to Lemma 2.1, applied to the function f, and put

$$\mathcal{I}^{n+1} = \{I_-^{n,1}, I_+^{n,1}, I_-^{n,2}, I_+^{n,2}, \ldots, I_-^{n,2^n}, I_+^{n,2^n}\}.$$

Next, we define a sequence $(f_n)_{n\geq 0}$ by

$$f_n(x) = \frac{1}{|I^n(x)|} \int_{I^n(x)} f(s)\, ds,$$

where $I^n(x) \in \mathcal{I}^n$ is an interval containing x (if there are two such intervals, we pick the one which has x as its right endpoint). Set $g_n(x) = \operatorname{ess\,inf}_{I^n(x)} f$; then, by the inequality $\|f\|_{\mathrm{BLO}(I)} < 1 - \varepsilon < 1$, for each n and almost all $x \in I$ we have $(f_n(x), g_n(x)) \in \mathcal{D}$. Furthermore, (2.1) guarantees that $(f_{n+1}(x), g_n(x)) \in \mathcal{D}$ for all $n \geq 0$ and almost all $x \in I$.

Step 2. Now we will prove that for any nonnegative integer n and any $I^{n,k} \in \mathcal{I}^n$,

$$\int_{I^{n,k}} U(f_{n+1}(x), g_{n+1}(x))\, dx \leq \int_{I^{n,k}} U(f_n(x), g_n(x))\, dx. \tag{2.8}$$

To do this, note that $(g_n(x))_{n\geq 0}$ is nondecreasing for all $x \in I$, so by (2.6),

$$\int_{I^{n,k}} U(f_{n+1}(x), g_{n+1}(x))\, dx \leq \int_{I^{n,k}} U(f_{n+1}(x), g_n(x))\, dx.$$

Since g_n is constant on $I^{n,k}$, it remains to use (2.5) to get (2.8).

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Step 3. This is the final part. Summing (2.8) over k,

$$\int_I U(f_{n+1}(x), g_{n+1}(x))\,dx \le \int_I U(f_n(x), g_n(x))\,dx$$

and hence for any nonnegative integer n we have

$$\frac{1}{|I|}\int_I U(f_n(x), g_n(x))\,dx \le \frac{1}{|I|}\int_I U(f_0(x), g_0(x))\,dx = U\Big(f_I, \operatorname*{ess\,inf}_I f\Big) \le c,$$

where in the last passage we have used the equality $f_I = 0$ and the conditions (2.6) and (2.7). Again by (2.6), we have $U(f_n, g_n) \ge U(f_n, f_n)$ and hence (2.4) gives

$$\frac{1}{|I|}\int_I V(f_n(x))\,dx \le c. \tag{2.9}$$

Since the splitting ratios $\alpha_\pm$ of Lemma 2.1 are bounded away from 0 and 1, the diameter of $\mathcal{I}^n$ tends to 0 as $n \to \infty$, that is,

$$\lim_{n\to\infty} \sup_{1\le k\le 2^n} \operatorname{diam} I^{n,k} = 0.$$

Consequently, by Lebesgue's differentiation theorem, we have $f_n(x) \to f(x)$ for almost all $x \in I$. Since V is continuous and nonnegative, the application of Fatou's lemma in (2.9) yields the claim. □

An interesting phenomenon is that the implication of the above theorem can be reversed. To prove this fact, we need the following definition. For any $(x, y) \in \mathcal{D}$, let $\mathcal{F}(x, y)$ denote the class of all continuous and nonincreasing functions $f : (0, 1] \to \mathbb{R}$, which satisfy $\|f\|_{\mathrm{BLO}([0,1])} \le 1$, $f_{[0,1]} = x$ and $\operatorname{ess\,inf}_{[0,1]} f = f(1) = y$. We extend each such function to the whole interval $[0, 1]$ by setting $f(0) = \lim_{x\downarrow 0} f(x) \in \mathbb{R} \cup \{\infty\}$. For any (x, y) the class $\mathcal{F}(x, y)$ is nonempty, since it contains the function

$$f(s) = \begin{cases} y + \log \dfrac{x-y}{s} & \text{if } s \le x - y, \\ y & \text{if } s > x - y. \end{cases} \tag{2.10}$$

Indeed, this function is continuous and nonincreasing on $(0, 1]$, and it is easy to compute that $f_{[0,1]} = x$ and $\operatorname{ess\,inf}_{[0,1]} f = y$. To check that $\|f\|_{\mathrm{BLO}([0,1])} \le 1$, observe that in general, for any nonincreasing function f on $[0, 1]$,

$$\|f\|_{\mathrm{BLO}([0,1])} = \sup_{b\in(0,1]} [f_{[0,b]} - f(b)]. \tag{2.11}$$

This identity follows directly from the fact that for any interval $J \subset [0, 1]$ we have $f_J \le f_{[0,\sup J]}$ and $\operatorname{ess\,inf}_J f = \operatorname{ess\,inf}_{[0,\sup J]} f$. Now, for f as in (2.10) and $b \le x - y$,

$$f_{[0,b]} = y + \log\frac{x-y}{b} + 1 = f(b) + 1,$$

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while for $x - y < b \le 1$,

$$f_{[0,b]} = y + \frac{x-y}{b} \le f(b) + 1.$$

This completes the proof of the inclusion $f \in \mathcal{F}(x, y)$.

We are ready to establish the converse to Theorem 2.2.

Theorem 2.3. *Suppose that the inequality (2.3) holds true for $I = [0, 1]$ and all functions $f : [0, 1] \to \mathbb{R}$ such that $f_{[0,1]} = 0$ and $\|f\|_{\mathrm{BLO}([0,1])} \le 1$. Then there is a function $U : \mathcal{D} \to \mathbb{R}$ which satisfies the conditions (2.4)–(2.7).*

Proof. The desired function $U : \mathcal{D} \to \mathbb{R}$ is given by

$$U(x, y) = \sup\left\{\int_0^1 V(f(x))\,dx : f \in \mathcal{F}(x, y)\right\}.$$

Let us verify the required conditions.

The majorisation (2.4). This holds true since $\mathcal{F}(x, x)$ contains only one element, the constant function (so in fact both sides of (2.4) are equal).

The concavity with respect to the first variable. Fix $y \ge 0$, $x_-, x_+ \in [y, y + 1)$ and positive numbers α_-, α_+ satisfying $\alpha_- + \alpha_+ = 1$. Pick two functions f_-, f_+ belonging to $\mathcal{F}(x_-, y)$ and $\mathcal{F}(x_+, y)$, respectively, and splice them together into the function $\overline{f} : [0, 1] \to \mathbb{R} \cup \{\infty\}$ by the formula

$$\overline{f}(x) = \begin{cases} f_-(x/\alpha_-) & \text{if } 0 \le x \le \alpha_-, \\ f_+((x - \alpha_-)/\alpha_+) & \text{if } \alpha_- < x \le 1. \end{cases}$$

Let f be a nonincreasing rearrangement of $\overline{f}$: that is, define

$$f(t) = \inf\{s \in \mathbb{R} : |\{x \in [0, 1] : \overline{f}(x) > s\}| \le t\}, \qquad t \in (0, 1].$$

Then f is nonincreasing and has the same distribution as $\overline{f}$, so $\operatorname{ess\,inf}_{[0,1]} f = y$ and

$$f_{[0,1]} = \int_0^1 \overline{f}(x)\,dx = \alpha_- \int_0^1 f_-(x)\,dx + \alpha_+ \int_0^1 f_+(x)\,dx = \alpha_- x_- + \alpha_+ x_+.$$

Furthermore, since $f_\pm$ are continuous and have the same essential infimum, the function f is also continuous. Finally, we check that $\|f\|_{\mathrm{BLO}([0,1])} \le 1$, which, by (2.11), amounts to saying that $f_{[0,b]} - f(b) \le 1$ for all $b \in (0, 1]$. Fix such a number b and note that we have two possibilities: either one of $\sup f_-$, $\sup f_+$ is smaller than $f(b)$, or both $\sup f_-$, $\sup f_+$ are at least $f(b)$. If the first possibility occurs (say, $\sup f_- < f(b)$), then, on $[f(b), \infty)$, the distribution function of f coincides with the distribution function of f_+ multiplied by α_+. Hence,

$$f_{[0,b]} = (f_+)_{[0,\alpha_+ b]} \le \operatorname*{ess\,inf}_{[0,\alpha_+ b]} f_+ + 1 = f_+(\alpha_+ b) + 1 = f(b) + 1.$$

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Now if both $\sup f_-$, $\sup f_+$ are at least $f(b)$, then there are $b_-, b_+ \in [0,1]$ such that $f_-(b_-) = f_+(b_+) = f(b)$, $\alpha_- b_- + \alpha_+ b_+ = b$ and

$$\begin{aligned}\int_0^b f(x)\,dx &= \alpha_- \int_0^{b_-} f_-(x)\,dx + \alpha_+ \int_0^{b_+} f_+(x)\,dx \\ &\le \alpha_- b_-(f_-(b_-)+1) + \alpha_+ b_+(f_+(b_+)+1) \\ &= bf(b) + b.\end{aligned}$$

Thus, we have $f \in \mathcal{F}(\alpha_- x_- + \alpha_+ x_+, y)$ and hence

$$U(\alpha_- x_- + \alpha_+ x_+, y) \ge \int_0^1 V(f(x))\,dx = \alpha_- \int_0^1 f_-(x)\,dx + \alpha_+ \int_0^1 f_+(x)\,dx.$$

It remains to take the supremum over f_- and f_+ to get (2.5).

The monotonicity with respect to the second variable. Pick x, y, z such that $x - 1 < z < y \le x$ and a function $f \in \mathcal{F}(x, y)$. Fix $\kappa \in (0,1)$ and consider $\overline{f} : [0,1] \to \mathbb{R}$ defined by

$$\overline{f}(x) = \begin{cases} f(x/\kappa) & \text{if } 0 \le x \le \kappa, \\ \dfrac{z-y}{1-\kappa}(x-\kappa) + y & \text{if } \kappa < x \le 1. \end{cases}$$

Then $\overline{f}$ is nonincreasing, continuous and satisfies $\operatorname{ess\,inf}_{[0,1]} \overline{f} = \overline{f}(1) = z$ and

$$\overline{f}_{[0,1]} = \kappa f_{[0,1]} + (1-\kappa)\overline{f}_{[\kappa,1]} = \kappa x + \frac{(1-\kappa)(y+z)}{2}.$$

Furthermore, as we will check now, $\|\overline{f}\|_{\mathrm{BLO}([0,1])} < 1$. According to (2.11), we must verify that for any $b \in (0,1]$ we have $\overline{f}_{[0,b]} - \overline{f}(b) \le 1$. If $b \le \kappa$, then this follows directly from the inequality $\|f\|_{\mathrm{BLO}([0,1])} < 1$; if $b > \kappa$, we derive that

$$\overline{f}_{[0,b]} = \frac{\kappa}{b} f_{[0,1]} + \frac{b-\kappa}{b}\overline{f}_{[\kappa,b]} \le \frac{\kappa}{b}x + \frac{b-\kappa}{b}y < z + 1 \le \operatorname*{ess\,inf}_{[0,b]} f + 1.$$

Thus, the BLO norm of $\overline{f}$ is smaller than 1 and, consequently,

$$U\Big(\kappa x + \frac{(1-\kappa)(y+z)}{2}, z\Big) \ge \int_0^1 V(\overline{f}(x))\,dx \ge \kappa \int_0^1 V(f(x))\,dx.$$

Taking the supremum over f,

$$\frac{1}{\kappa} U\Big(\kappa x + \frac{(1-\kappa)(y+z)}{2}, z\Big) \ge U(x,y).$$

However, as we have shown above, the function $U(\cdot, z)$ is concave on $[z, z+1)$; thus it is continuous in the interior of this interval. Therefore, letting $\kappa \to 1$ yields the monotonicity condition (2.6).

The property (2.7). By the assumption of the theorem, if $y \in (-1, 0]$ and $f \in \mathcal{F}(0, y)$, then (2.3) holds true. Take the supremum over f to get the property. □

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3. Applications

3.1. Proof of Theorem 1.1. Fix $1 \le p < \infty$ and $f \in \mathrm{BLO}(I^0)$. To show (1.3), it suffices to prove that for any $I \subseteq I^0$,

$$\frac{1}{|I|}\int_I |f(x) - f_I|^p\, dx \le C_p^p \|f\|^p_{\mathrm{BLO}(I)},$$

since $\|f\|_{\mathrm{BLO}(I)} \le \|f\|_{\mathrm{BLO}(I^0)}$. By homogeneity, we may and do assume that $\|f\|_{\mathrm{BLO}(I)} \le 1$; furthermore, since the BLO norm is invariant with respect to translations, we may restrict ourselves to the functions satisfying $f_I = 0$. Then the above estimate is precisely of the form (2.3), with $V(x) = |x|^p$ and $c = C_p^p$. The special function $U_p : \mathcal{D} \to \mathbb{R}$, corresponding to this choice of V, is given by

$$U_p(x, y) = (x - y)e^y \int_y^\infty |s|^p s e^{-s}\, ds + (1 - x + y)|y|^p.$$

Indeed, it is evident that (2.4) holds true: in fact, we have equality here. The concavity with respect to the first variable is obvious. The only nontrivial condition is (2.6). We easily check that $U_1(x, y) = x - y + |y|$, so the monotonicity is satisfied for $p = 1$. If p is larger than 1, a straightforward calculation shows that

$$\begin{aligned} U_{py}(x, y) &= (1 - x + y)\Big(|y|^p + p|y|^{p-2}y - e^y \int_y^\infty |s|^p e^{-s}\, ds\Big) \\ &= -p(p-1)e^y \int_y^\infty |s|^{p-2} e^{-s}\, ds \le 0, \end{aligned}$$

and (2.6) follows. Finally, (2.7) is evident, in fact both sides are equal. Therefore, by Theorem 2.2, the inequality (1.3) holds true and it remains to prove that this estimate is sharp. It suffices to consider the interval $I^0 = [0, 1]$ only; the examples for other intervals are obtained by straightforward affine transformations. Consider a function $f : [0, 1] \to \mathbb{R}$, given by $f(x) = -\log x - 1$. Then $\|f\|_{\mathrm{BLO}([0,1])} \le 1$, which can be seen by repeating the analysis of the function (2.10). On the other hand,

$$\|f\|^p_{\mathrm{BMO}_p([0,1])} \ge \int_0^1 \Big| f(x) - \int_0^1 f(y)\, dy\Big|^p dx = \int_0^1 |\log x + 1|^p\, dx = C_p^p,$$

where the last passage can be easily verified using the substitution $s = -\log x - 1$. This shows that the constant C_p is indeed the norm of the inclusion $\mathrm{BLO} \subset \mathrm{BMO}_p$.

3.2. Proof of Theorem 1.2. The bound (1.4) follows at once from (1.3):

$$\begin{aligned} \frac{1}{|I|}\int_I \exp(a|f(x) - f_I|)\, dx &= 1 + \sum_{k=1}^\infty \frac{a^k}{k!}\frac{1}{|I|}\int_I |f(x) - f_I|^k\, dx \\ &\le 1 + \sum_{k=1}^\infty \frac{a^k C_k^k \|f\|^k_{\mathrm{BLO}(I)}}{k!} \le K(a\|f\|_{\mathrm{BLO}(I^0)}). \end{aligned}$$

This estimate is sharp: take $I = I^0 = [0, 1]$ and use the function $f(x) = -\log x - 1$ again. Then all of the inequalities above become equalities.

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3.3. Proof of Theorem 1.3. Fix $\lambda > 0$. Arguing as in the proof of (1.3), we see that it suffices to prove that for any $I \subseteq I^0$ and any $f : I \to \mathbb{R}$ satisfying $f_I = 0$ and $\|f\|_{\mathrm{BLO}(I)} \leq 1$, the inequality (2.3), with $V(x) = \chi_{\{|y| \geq \lambda\}}$ and $c = P(\lambda)$, is valid. Note that the function V is not continuous, so to enable the application of Theorem 2.2, we fix $\varepsilon > 0$ and use the function $V_{\varepsilon,\lambda}$ given by

$$V_{\varepsilon,\lambda}(x) = \begin{cases} 0 & \text{if } |x| < \lambda, \\ \dfrac{|x| - \lambda}{\varepsilon} & \text{if } \lambda \leq |x| \leq \lambda + \varepsilon, \\ 1 & \text{if } |x| > \lambda + \varepsilon. \end{cases}$$

We consider the cases $\lambda \leq \frac{1}{2}$, $\frac{1}{2} < \lambda < 1$ and $\lambda \geq 1$ separately.

The case $\lambda \leq \frac{1}{2}$. The inequality (1.4) is trivial; to see that the constant 1 cannot be improved, pick $I = I^0 = [0, 1]$ and consider the function $f = (\chi_{[0,1/2)} - \chi_{[1/2,1)})/2$. Then $\|f\|_{\mathrm{BLO}(I^0)} \leq 1$, since for any interval $I \subseteq [0, 1]$ we have $f_I \leq \frac{1}{2}$ and $\operatorname{ess\,inf}_I f \geq -\frac{1}{2}$; furthermore, the left-hand side of (1.4) is equal to 1.

The case $\frac{1}{2} < \lambda < 1$. Assume that $\varepsilon < 1 - \lambda$ and let $U_\lambda : \mathcal{D} \to \mathbb{R}$ be given by

$$U_\lambda(x, y) = \begin{cases} 1 - (x + \lambda)(1 - e^{1-2\lambda}) & \text{if } y \leq -\lambda < x, \\ (x - y)e^{y-\lambda+1} & \text{if } -\lambda < y \leq \lambda - 1, \\ \dfrac{x - y}{\lambda - y} & \text{if } \lambda - 1 < y \leq x < \lambda, \\ 1 & \text{if } |x| \geq \lambda. \end{cases}$$

Since $U_\lambda(x, x) = \chi_{\{|x| \geq \lambda\}}$, the majorisation $U_\lambda(x, x) \geq V_{\varepsilon,\lambda}(x)$ holds true. The concavity of $U(\cdot, y)$ is also straightforward. (This function is either linear, or linear on two subintervals of $[y, y + 1]$ and the appropriate bounds for one-sided derivatives are valid.) To prove (2.6), observe that U_λ is continuous and the partial derivatives with respect to y of the expressions appearing in the definition of U_λ are nonpositive. Finally, both sides of (2.7) are equal. Thus, by Theorem 2.2,

$$\frac{1}{|I|} \int_I V_{\varepsilon,\lambda}(f(x))\, dx \leq 1 - \lambda(1 - e^{1-2\lambda}).$$

But $V_{\varepsilon,\lambda}(x) \geq \chi_{|x| \geq \lambda+\varepsilon\}}$, so the above bound implies

$$\frac{1}{|I|} |\{x \in I : |f(x)| \geq \lambda + \varepsilon\}| \leq 1 - \lambda(1 - e^{1-2\lambda}).$$

We have proved this estimate for arbitrary $\lambda \in (\frac{1}{2}, 1)$ and $\varepsilon \in (0, 1 - \lambda)$. Any number $\mu \in (\frac{1}{2}, 1)$ can be written as a sum of two such parameters, with ε as small as we wish. Thus,

$$\frac{1}{|I|} |\{x \in I : |f(x)| \geq \mu\}| \leq 1 - (\mu - \varepsilon)(1 - e^{1-2\mu+2\varepsilon}),$$

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and letting $\varepsilon \to 0$ yields (1.5). To see that the bound is sharp, take $I = I^0 = [0, e^{\lambda-1}/\lambda]$ and consider the function $f : I^0 \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \lambda & \text{if } 0 \le x < e^{-\lambda}, \\ -\log x - 1 & \text{if } e^{-\lambda} \le x < e^{\lambda-1}, \\ -\lambda & \text{if } e^{\lambda-1} \le x \le e^{\lambda-1}/\lambda. \end{cases}$$

Then $\|f\|_{\mathrm{BLO}(I^0)} = 1$. To check this, observe that f is nonincreasing, so the appropriate version of (2.11) holds true. If $b \le e^{-\lambda}$, then $f_{[0,b]} - \operatorname{ess\,inf}_{[0,b]} f = 0$; if $e^{-\lambda} < b \le e^{\lambda-1}$, then $f_{[0,b]} - \operatorname{ess\,inf}_{[0,b]} f = 1$; finally, if $b \in (e^{\lambda-1}, e^{\lambda-1}/\lambda]$, then

$$f_{[0,b]} \le f_{[0,e^{\lambda-1}]} \quad \text{and} \quad \operatorname*{ess\,inf}_{[0,b]} f = \operatorname*{ess\,inf}_{[0,e^{\lambda-1}]} f,$$

so $f_{[0,b]} - \operatorname{ess\,inf}_{[0,b]} f \le f_{[0,e^{\lambda-1}]} - \operatorname{ess\,inf}_{[0,e^{\lambda-1}]} f = 1$, and thus the BLO norm of f equals 1. Furthermore, it is easy to check that $f_{[0,e^{\lambda-1}/\lambda]} = 0$ and

$$\frac{1}{|I|}|\{x \in I : |f(x)| \ge \lambda\}| = \frac{e^{-\lambda} + e^{\lambda-1}(\lambda^{-1} - 1)}{e^{\lambda-1}/\lambda} = P(\lambda),$$

so the constant $P(\lambda)$ cannot be improved.

The case $\lambda \ge 1$. Here the analysis is very similar to that from the previous case (and the calculations are somewhat easier), so we will only present the special function U_λ and the extremal example which gives equality in (1.5). We introduce $U_\lambda : \mathcal{D} \to \mathbb{R}$ by the formula

$$U_\lambda(x, y) = \begin{cases} (x - y)e^{y-\lambda+1} & \text{if } y \le \lambda - 1, \\ \dfrac{x - y}{\lambda - y} & \text{if } \lambda - 1 < y \le x < \lambda, \\ 1 & \text{if } |x| \ge \lambda. \end{cases}$$

The optimality of the constant $e^{-\lambda}$ can be extracted from the function $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \lambda$ if $x \le e^{-\lambda}$ and $f(x) = -\log x - 1$ otherwise.

4. Estimates for anisotropic BLO

The above methodology concerned the one-dimensional case, but it can be used in higher dimensions if one changes the original definition of BMO and BLO, and works with the anisotropic versions of these spaces. Let $d \ge 2$. Recall that f belongs to the anisotropic space $A\mathrm{BMO}(\mathbb{R}^d)$ (respectively, anisotropic $A\mathrm{BLO}(\mathbb{R}^d)$), if the inequality (1.1) (respectively, (1.2)) holds true for all bounded rectangles $Q \subset \mathbb{R}^d$ (the products of intervals of possibly different lengths). As previously, we will study the less restrictive, 'local' case in which the rectangles Q are assumed to be contained in a given bounded rectangle Q^0. First we show the version of Theorem 2.2.

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THEOREM 4.1. *Suppose that there is a function satisfying the conditions (2.4)–(2.7). Then, for any bounded rectangle $Q \subset \mathbb{R}^d$ and any function $f : Q \to \mathbb{R}$ satisfying $f_Q = 0$ and $\|f\|_{\mathrm{ABLO}(Q)} \le 1$,*

$$\frac{1}{|Q|} \int_Q V(f(x))\, dx \le c.$$

We need the following extension of Lemma 2.1.

LEMMA 4.2. *Pick $Q = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ and $\ell \in \{1, 2, \ldots, d\}$. Suppose that $\varepsilon \in (0, \frac{1}{2})$ is a fixed number and let $f : Q \to \mathbb{R}$ be an arbitrary function satisfying $\|f\|_{\mathrm{ABLO}(Q)} < 1 - \varepsilon$. Then there exists a splitting of Q into two rectangles Q_- and Q_+ along a hyperplane orthogonal to the ℓth axis, for which*

$$x_- := \frac{1}{|Q_-|} \int_{Q_-} f(x)\, dx - \operatorname*{ess\,inf}_Q f < 1,$$
$$x_+ := \frac{1}{|Q_+|} \int_{Q_+} f(x)\, dx - \operatorname*{ess\,inf}_Q f < 1$$

and such that the splitting parameters $\alpha_\pm = |Q_\pm|/|Q|$ belong to $[\varepsilon, 1 - \varepsilon]$.

The proof is essentially the same as in the one-dimensional setting and is omitted.

PROOF OF THEOREM 4.1. The reasoning is similar to that in the proof of Theorem 2.2, so we will be brief. We may assume that $\|f\|_{\mathrm{ABLO}(Q)} < 1 - \varepsilon$, for some $\varepsilon \in (0, \frac{1}{2})$. The only essential change there is how to construct the sequence $\{\mathcal{Q}^n\}_{n\ge 0}$ of partitions of Q. We use the following inductive procedure. Set $\mathcal{Q}^0 = \{Q\}$ and suppose that we have defined $\mathcal{Q}^n = \{Q^{n,1}, Q^{n,2}, \ldots, Q^{n,2^n}\}$. We split each $Q^{n,k}$ along the hyperplane orthogonal to the longest side of the rectangle, according to Lemma 4.2 (if there are two or more longest sides of $Q^{n,k}$, we pick any of them); then put

$$\mathcal{Q}^{n+1} = \{Q_-^{n,1}, Q_+^{n,1}, Q_-^{n,2}, Q_+^{n,2}, \ldots, Q_-^{n,2^n}, Q_-^{n,2^n}\}.$$

This sequence of partitions has two important properties: first, the diameter of $\mathcal{Q}^n$ (defined as $\sup_{1\le k\le 2^n} \operatorname{diam} Q^{n,k}$) converges to 0 as $n \to \infty$; second, for any k, the ratio of the lengths of any two sides of $Q^{n,k}$ is bounded by a number depending only on ε and the lengths of the initial rectangle Q. This makes Lebesgue's differentiation theorem applicable and, hence, having proved

$$\frac{1}{|Q|} \int_Q V(f_n(x))\, dx \le \frac{1}{|Q|} \int_Q U(f_n(x), g_n(x))\, dx \le \frac{1}{|Q|} \int_Q U(f_0(x), g_0(x))\, dx \le c,$$

it suffices to let $n \to \infty$ and use Fatou's lemma to get the claim. □

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Now we establish the analogues of the results formulated in the Introduction.

THEOREM 4.3. *Suppose that* $Q^0 \subset \mathbb{R}$ *is a bounded rectangle and* $f \in \mathrm{ABLO}(Q^0)$*. Then for any* $1 \le p < \infty$, $a > 0$ *and* $\lambda > 0$,

$$||f||_{\mathrm{ABMO}_p(Q^0)} \le C_p ||f||_{\mathrm{ABLO}(Q^0)}, \tag{4.1}$$

$$\sup_{Q \subseteq Q^0} \frac{1}{|Q|} \int_Q \exp(a|f(x) - f_Q|)\, dx \le K(a||f||_{\mathrm{ABLO}(Q^0)}), \tag{4.2}$$

$$\sup_{Q \subseteq Q^0} \frac{1}{|Q|} |\{x \in Q : |f(x) - f_Q| \ge \lambda\}| \le P(\lambda/||f||_{\mathrm{ABLO}(Q^0)}) \tag{4.3}$$

and all of the estimates are sharp.

PROOF. The validity of the estimates follows from Theorem 4.1. To see the sharpness of (4.1), we use the extremal example from (1.3) in the following way: put $Q = Q^0 = [0, 1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ and $f(x) = -\log x_1 - 1$, $x = (x_1, x_2, \ldots, x_d) \in Q$. This choice gives equality in (4.1). The bounds (4.2) and (4.3) are handled similarly. □

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