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SQUARE ROOTS OF HYPONORMAL OPERATORS

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Abstract. An operator $T \in \mathcal{L}(H)$ is called a *square root* of a hyponormal operator if T^2 is hyponormal. In this paper, we prove the following results: Let S and T be square roots of hyponormal operators.

(1) If $\sigma(T) \cap [-\sigma(T)] = \phi$ or $\{0\}$, then T is isoloid (i.e., every isolated point of $\sigma(T)$ is an eigenvalue of T).

(2) If S and T commute, then ST is Weyl if and only if S and T are both Weyl.

(3) If $\sigma(T) \cap [-\sigma(T)] = \phi$ or $\{0\}$, then Weyl's theorem holds for T.

(4) If $\sigma(T) \cap [-\sigma(T)] = \phi$, then T is subscalar. As a corollary, we get that T has a nontrivial invariant subspace if $\sigma(T)$ has non-empty interior. (See [**3**].)

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1. Introduction. Let H and K be separable complex Hilbert spaces, and let $\mathcal{L}(H, K)$ denote the space of all bounded linear operators from H to K. If $H = K$, we write $\mathcal{L}(H)$ in place of $\mathcal{L}(H, K)$.

An operator T is called *hyponormal* if $T^*T \geq TT^*$, or equivalently if $\|Th\| \geq \|T^*h\|$, for all $h \in H$. We say that an operator $T \in \mathcal{L}(H)$ is a *square root* of a hyponormal operator if T^2 is hyponormal. In general, T^2 can be hyponormal without T being hyponormal. For example, if T is any nilpotent operator of order 2 (i.e., $T^2 = 0$), then T is not necessary a hyponormal operator, but is a square root of a hyponormal operator.

A bounded linear operator S on H is called *scalar of order m* if it possesses a spectral distribution of order m; i.e., if there is a continuous unital morphism

$$\Phi : C_0^m(C) \to \mathcal{L}(H)$$

such that $\Phi(z) = S$, where z stands for the identity function on $\mathbf{C}$ and $C_0^m(\mathbf{C})$ for the space of compactly supported functions on $\mathbf{C}$, continuously differentiable of order m, where $0 \leq m \leq \infty$. An operator is *subscalar* if it is similar to the restriction of a scalar operator to an invariant subspace.

2. Preliminaries. An operator $T \in \mathcal{L}(H)$ is said to be *Fredholm* if ran T is closed and both ker T and $H/\text{ran}\,T(= \ker T^*)$ are finite dimensional. The *index* of a Fredholm operator is defined as

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$$\begin{aligned}\text{index } T &= \dim \ker T - \dim (H/\text{ran } T)\\ &= \dim \ker T - \dim \ker T^*.\end{aligned}$$

The *spectrum* of T is defined by

$$\sigma(T) = \{\lambda \in \mathbf{C} : T - \lambda \text{ is not invertible}\}$$

and $\rho(T) = \mathbf{C} \backslash \sigma(T)$ is called the resolvent set of T.

The *Weyl spectrum* of T is defined by

$$\omega(T) = \{\lambda \in \mathbf{C} : T - \lambda \text{ is not Fredholm of index } 0\}.$$

Clearly, $\omega(T) \subset \sigma(T)$. Furthermore, if T is Fredholm of index 0, we say that T is Weyl.

Let iso $\sigma(T)$ denote the set of isolated points of $\sigma(T)$, $\sigma_p(T)$ the set of eigenvalues of T, and $\pi_{00}(T)$ the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. If $\omega(T) = \sigma(T) - \pi_{00}(T)$, or equivalently, if $\sigma(T) - \omega(T) = \pi_{00}(T)$, we say that Weyl's theorem holds for T. It is known that Weyl's theorem holds for any hyponormal operator. (See [**2**].)

An operator $T \in \mathcal{L}(H)$ is said to satisfy the single valued extension property if for any open subset U in $\mathbf{C}$, the function

$$z - T : \mathcal{O}(U, H) \to \mathcal{O}(U, H),$$

defined by the obvious pointwise multiplication, is one-to-one, where $\mathcal{O}(U, H)$ denotes the Fréchet space of H-valued analytic functions on U with respect to the uniform topology. If, in addition, the function $z - T$ above has closed range on $\mathcal{O}(U, H)$, then T satisfies Bishop's condition (β).

Lemma 2.1 [**10**, Introduction]. *Every subscalar operator has property* (β).

Let z be the coordinate in $\mathbf{C}$ and let $d\mu(z)$, or simply $d\mu$, denote planar Lebesgue measure. Fix a separable, complex Hilbert space H and a bounded (connected) open subset U of $\mathbf{C}$. We shall denote by $L^2(u, H)$ the Hilbert space of measurable functions $f : U \to H$, such that

$$\|f\|_{2,U} = \left\{ \int_U \|f(z)\|^2 d\mu(z) \right\}^{\frac{1}{2}} < \infty.$$

The space of functions $f \in L^2(U, H)$ that are analytic in U (i.e., $\bar{\partial} f = 0$) is denoted by

$$A^2(U, H) = L^2(U, H) \cap \mathcal{O}(U, H).$$

$A^2(U, H)$ is called the *Bergman space* for U.

Let us define now a special Sobolev type space. Let U again be a bounded open subset of $\mathbf{C}$ and m a fixed non-negative integer. The vector valued Sobolev space $W^m(U, H)$ with respect to $\bar{\partial}$ and of order m will be the space of those functions

$f \in L^2(U, H)$ whose derivatives $\bar{\partial}f, \cdots, \bar{\partial}^m f$ in the sense of distributions also belong to $L^2(U, H)$. Endowed with the norm

$$\|f\|_{W^m}^2 = \sum_{i=0}^{m} \|\bar{\partial}^i f\|_{2,U}^2$$

$W^m(U, H)$ becomes a Hilbert space contained continuously in $L^2(U, H)$.

The linear operator M of multiplication by z on $W^m(U, H)$ is continuous and it has a spectral distribution of order m, defined by the relation

$$\Phi_M : C_0^m(\mathbf{C}) \to \mathcal{L}(W^m(U, H)), \Phi_M(f) = M_f.$$

Therefore, M is a scalar operator of order m.

3. Weyl's theorem. In this section, we show that if T is a square root of a hyponormal operator with the property that $\sigma(T) \cap [-\sigma(T)] = \phi$ or $\{0\}$, then Weyl's theorem holds for T.

THEOREM 3.1. *If T is a square root of a hyponormal operator with the property that $\sigma(T) \cap [-\sigma(T)] = \phi$ or $\{0\}$, then T is isoloid* (*i.e.*, iso $\sigma(T) \subset \sigma_p(T)$).

Proof. If $\lambda \in$ iso $\sigma(T)$, then $\lambda^2 \in$ iso $\sigma(T)^2$. Since iso $\sigma(T)^2 =$ iso $\sigma(T^2)$, by the spectral mapping theorem, $\lambda^2 \in$ iso $\sigma(T^2)$. Since T^2 is hyponormal, it is isoloid by [**11**]. Therefore, $\lambda^2 \in \sigma_p(T^2)$. Hence $\lambda^2 \in \sigma_p(T)^2$ by [**4**, Problem 74].

If $\lambda = 0$, it is clear that $0 \in \sigma_p(T)$.

If $\lambda \neq 0$, then either $\lambda \in \sigma_p(T)$ or $-\lambda \in \sigma_p(T)$. Since $\lambda \in$ iso $\sigma(T)$ and $\sigma(T) \cap [-\sigma(T)] = \phi$, we know that $-\lambda \notin \sigma(T)$. Therefore $\lambda \in \sigma_p(T)$. This completes the proof. □

COROLLARY 3.2. *If T is a square root of a hyponormal operator with the property that $\sigma(T) \cap [-\sigma(T)] = \phi$ or $\{0\}$, then for any polynomial p we have*

$$\sigma(p(T)) - \pi_{00}(p(T)) = p(\sigma(T) - \pi_{00}(T)).$$

Proof. This is clear from Theorem 3.1 and [**9**, Proposition 1].

LEMMA 3.3. ([**5**], [**12**]). *Let S and T be commuting operators.*

(a) *ST is Fredholm if and only if S and T are both Fredholm.*
(b) *If S and T are both Fredholm, then* index $ST =$ index $S +$ index T.

THEOREM 3.4. *If S and T are commuting square roots of hyponormal operators, then ST is Weyl if and only if S and T are both Weyl.*

Proof. If ST is Weyl, then ST is Fredholm of index 0. By Lemma 3.3, we know that $(ST)^2$ is Fredholm and index $(ST)^2 =$ index $ST +$ index $ST = 0$. Since

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$(ST)^2 = S^2T^2$ is Fredholm, S^2 and T^2 are both Fredholm, by Lemma 3.3. Since S^2 and T^2 are hyponormal, index $S^2 \leq 0$ and index $T^2 \leq 0$. Since index $(ST)^2 = 0$, Lemma 3.3 implies that

$$\begin{aligned} 0 &= \text{index } (ST)^2 \\ &= \text{index } S^2 + \text{index } T^2. \end{aligned}$$

Therefore, index $S^2 = 0 = $ index T^2.

Since S^2 and T^2 are Fredholm, it follows from Lemma 3.3 that S and T are Fredholm and

$$0 = \text{index } S^2 = \text{index } S + \text{index } S.$$

Therefore, index $S = 0$. Similarly, index $T = 0$. We conclude that S and T are both Weyl.

Conversely, if S and T are both Weyl, then S and T are both Fredholm of index 0. By Lemma 3.3, ST is Fredholm and

$$\text{index } ST = \text{index } S + \text{index } T = 0.$$

Therefore, ST is Weyl. □

LEMMA 3.5. *Let T be a square root of a hyponormal operator with the property that $\sigma(T) \cap [-\sigma(T)] = \phi$ or $\{0\}$. If $\lambda \in \pi_{00}(T)$, then $\lambda^2 \in \pi_{00}(T^2)$.*

Proof. By Corollary 3.2 with $p(t) = t^2$, we have

$$\sigma(T^2) - \pi_{00}(T^2) = (\sigma(T) - \pi_{00}(T))^2.$$

Let $\lambda \in \pi_{00}(T)$. If $\lambda^2 \notin \pi_{00}(T^2)$, then $\lambda^2 \in \sigma(T^2) - \pi_{00}(T^2)$. It follows that $\lambda^2 \in (\sigma(T) -\pi_{00}(T))^2$.

If $\lambda = 0$, it is clear that $0 \in \sigma(T) - \pi_{00}(T)$ and so we have a contradiction.

If $\lambda \neq 0$, then either $\lambda \in \sigma(T) - \pi_{00}(T)$ or $-\lambda \in \sigma(T) - \pi_{00}(T)$. Since $\lambda \in \pi_{00}(T)$ and $\sigma(T) \cap [-\sigma(T)] = \phi$, $-\lambda \notin \sigma(T)$. Therefore, $\lambda \in \sigma(T) - \pi_{00}(T)$ and we have a contradiction. Thus $\lambda^2 \in \pi_{00}(T^2)$. □

THEOREM 3.6. *If T is a square root of a hyponormal operator with the property that $\sigma(T) \cap [-\sigma(T)] = \phi$ or $\{0\}$, then Weyl's theorem holds for T.*

Proof. It suffices to show that $\lambda \in \sigma(T) - \pi_{00}(T)$ if and only if $\lambda \in \omega(T)$.

If $\lambda \in \sigma(T) - \pi_{00}(T)$, then $\lambda^2 \in (\sigma(T) - \pi_{00}(T))^2$. By Corollary 3.2, we have

$$\lambda^2 \in \sigma(T^2) - \pi_{00}(T^2).$$

Since T^2 is hyponormal, a theorem of Coburn [**2**] implies that

$$\lambda^2 \in \sigma(T^2) - \pi_{00}(T^2) = \omega(T^2).$$

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Since $\omega(T^2) \subset \omega(T)^2$ by [**1**], $\lambda^2 \in (T)^2$.

If $\lambda = 0$, it is clear that $0 \in \omega(T)$.

If $\lambda \neq 0$, then either $\lambda \in \omega(T)$ or $-\lambda \in \omega(T)$. Since $\lambda \in \sigma(T), \omega(T) \subset \sigma(T)$, and $\sigma(T) \cap [-\sigma(T)] = \phi$, $-\lambda \notin \omega(T)$. Therefore, $\lambda \in \omega(T)$.

Conversely, let $\lambda \in \omega(T)$.

Claim. If $\lambda \notin \sigma(T) - \pi_{00}(T)$, *then* $\lambda^2 \notin (\sigma(T) - \pi_{00}(T))^2$.

We verify the claim above. If $\lambda = 0$, it is clear. Let $\lambda \neq 0$. If $\lambda^2 \in (\sigma(T) - \pi_{00}(T))^2$, then either $\lambda \in \sigma(T) - \pi_{00}(T)$ or $-\lambda \in \sigma(T) - \pi_{00}(T)$. Since $\lambda \notin \sigma(T) - \pi_{00}(T)$, $-\lambda \in \sigma(T) - \pi_{00}(T)$. But $-\lambda \notin \sigma(T)$, since $\lambda \in \omega(T) \subset \sigma(T)$ and $\sigma(T) \cap [-\sigma(T)] = \phi$. Hence we have a contradiction.

Let us come back now to the proof of Theorem 3.6. If $\lambda \notin \sigma(T) - \pi_{00}(T)$, then $\lambda^2 \notin (\sigma(T) - \pi_{00}(T))^2$ by the Claim. Now Corollary 3.2 implies that

$$\lambda^2 \notin \sigma(T^2) - \pi_{00}(T^2).$$

Since T^2 is hyponormal, a theorem of Coburn [**2**] implies that $\lambda^2 \notin \sigma(T^2) - \pi_{00}(T^2) = \omega(T^2)$. Therefore, $\lambda^2 \in \sigma(T^2) - \omega(T^2)$. Thus $T^2 - \lambda^2$ is Weyl.

If $\lambda = 0$, then T^2 is Weyl. Hence T is Weyl by Theorem 3.4. Thus $0 \notin \omega(T)$, and so we have a contradiction.

Let $\lambda \neq 0$. By Lemma 3.3, $T + \lambda$ and $T - \lambda$ are both Fredholm. Since $\lambda \in \omega(T) \subset \sigma(T)$ and $\sigma(T) \cap [-\sigma(T)] = \phi$, $-\lambda \notin \sigma(T)$. Therefore, $T + \lambda$ is invertible and so $T + \lambda$ is Weyl. By Lemma 3.3, we have

$$0 = \text{index } (T^2 - \lambda^2) = \text{index } (T + \lambda) + \text{index } (T - \lambda).$$

Therefore, index $(T - \lambda) = 0$. Hence $T - \lambda$ is Weyl. Thus $\lambda \notin \omega(T)$, and so we have a contradiction. Thus $\lambda \in \sigma(T) - \pi_{00}(T)$.

Corollary 3.7. *If T is a square root of a hyponormal operator with the property that $\sigma(T) \cap [-\sigma(T)] = \phi$ or $\{0\}$ and N is a nilpotent operator commuting with T, then Weyl's theorem holds for $T + N$.*

Proof. It follows from Theorem 3.6 and [**9**, Theorem 3]. □

4. Subscalarity. In this section, we show that if T is a square root of a hyponormal operator with the property that $\sigma(T) \cap [-\sigma(T)] = \phi$ then T is subscalar.

Lemma 4.1. ([**10**], Proposition 2.1). *For every bounded disk D in $\mathbf{C}$ there is a constant C_D, such that for an arbitrary operator $T \in \mathcal{L}(H)$ and $f \in W^2(D, H)$ we have*

$$\|(I - P)f\|_{2,D} \leq C_D\big(\|(T - z)^* \bar{\partial} f\|_{2,D} + \|(T - z)^* \bar{\partial}^2 f\|_{2,D}\big),$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$.

LEMMA 4.2. *Let T be a square root of a hyponormal operator with the property that $\sigma(T) \cap [-\sigma(T)] = \phi$ and let D be a bounded disk which contains $\sigma(T)$. Then the operator $V : H \to H(D)$, defined by*

$$Vh = 1 \otimes h + \overline{(z - T)W^2(D, H)} \qquad (= 1\tilde{\otimes} h),$$

is one-to-one and has closed range, where $H(D) = W^2(D, H)/\overline{(z - T)W^2(D, H)}$ and $1 \otimes h$ denotes the constant function sending any $z \in D$ to h.

Proof. Let $h_i \in H$ and $f_i \in W^2(D, H)$ be sequences such that

$$\lim_{i\to\infty} \|(z - T)f_i + 1 \otimes h_i\|_{w^2} = 0. \tag{1}$$

Then, by the definition of the norm of a Sobolev space, (1) implies that

$$\lim_{i\to\infty} \|(z - T)\bar{\partial}^j f_i\|_{2,D} = 0 \tag{2}$$

for $j = 1, 2$. From (2), we get

$$\lim_{i\to\infty} \|(z^2 - T^2)\bar{\partial}^j f_i\|_{2,D} = 0$$

for $j = 1, 2$. Since T^2 is hyponormal,

$$\lim_{i\to\infty} \|(\bar{z}^2 - T^{*2})\bar{\partial}^j f_i\|_{2,D} = 0. \tag{3}$$

Since $z - T$ is invertible for $z \in D\backslash\sigma(T)$, the equation (2) implies that

$$\lim_{i\to\infty} \|\bar{\partial}^j f_i\|_{2,D\backslash\sigma(T)} = 0.$$

Therefore,

$$\lim_{i\to\infty} \|(\bar{z} - T^*)\bar{\partial}^j f_i\|_{2,D\backslash\sigma(T)} = 0. \tag{4}$$

Since $\sigma(T) \cap [-\sigma(T)] = \phi$ and $\sigma(T)^* = \sigma(T^*)$, it is clear that $T^* + \bar{z}$ is invertible for $z \in \sigma(T)$. Therefore, from the equation (3) we have

$$\lim_{i\to\infty} \|(\bar{z} - T^*)\bar{\partial}^j f_i\|_{2,\sigma(T)} = 0. \tag{5}$$

Hence, from (4) and (5) we obtain

$$\lim_{i\to\infty} \|(\bar{z} - T^*)\bar{\partial}^j f_i\|_{2,D} = 0. \tag{6}$$

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Then, by Lemma 4.1, we have

$$\lim_{i\to\infty} \|(I-P)f_i\|_{2,D} = 0, \tag{7}$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto $A^2(D, H)$. By (1) and (7), we have

$$\lim_{i\to\infty} \|(z-T)Pf_i + 1\otimes h_i\|_{2,D} = 0.$$

Let Γ be a circle in D such that $\sigma(T)$ lies inside Γ. Assume that Γ is described once counterclockwise. Then

$$\lim_{i\to\infty} \|Pf_i(z) + (z-T)^{-1}(1\otimes h_i)\| = 0$$

uniformly for z in Γ. Hence, by the Riesz functional calculus,

$$\lim_{i\to\infty} \|\frac{1}{2\pi i}\int_\Gamma Pf_i(z)dz + h_i\| = 0.$$

But $\int_\Gamma Pf_i(z)dz = 0$. Hence, $\lim_{i\to\infty} h_i = 0$.

Theorem 4.3. *If T is a square root of a hyponormal operator with the property that $\sigma(T)\cap[-\sigma(T)] = \phi$, then T is subscalar of order 2.*

Proof. Consider an arbitrary bounded open disk D in $\mathbf{C}$ that contains $\sigma(T)$ and the quotient space

$$H(D) = W^2(D, H)/\overline{(z-T)W^2(D, H)}$$

endowed with the Hilbert space norm. Let $M(=M_z)$ be the multiplication operator by z on $W^2(D, H)$. Then M is a scalar operator of order 2 and its spectral distribution is

$$\Phi : C_0^2(\mathbf{C}) \to \mathcal{L}\big(W^2(D, H)\big), \ \ \Phi(f) = M_f,$$

where M_f is the operator of multiplication by f. Since M commutes with $z-T$, $\tilde{M}$ on $H(D)$ is still a scalar operator of order 2, with $\tilde{\Phi}$ as a spectral distribution.

Let V be the operator

$$Vh = 1\otimes h (= 1\otimes h + \overline{(z-T)W^2(D, H)}),$$

from H into $H(D)$, denoting by $1\otimes h$ the constant function h. Then $VT = \tilde{M}V$. Since V is one-to-one and has closed range by Lemma 4.2, T is subscalar of order 2. □

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COROLLARY 4.4. *Let T be a square root of a hyponormal operator with the property that $\sigma(T) \cap [-\sigma(T)] = \phi$. If $\sigma(T)$ has interior in the plane, then T has a nontrivial invariant subspace.*

Proof. It follows from Theorem 4.3 and [**3**]. □

COROLLARY 4.5. *If T is a square root of a hyponormal operator with the property that $\sigma(T) \cap [-\sigma(T)] = \phi$, then T has the property (β).*

Proof. It follows from Theorem 4.3 and Lemma 2.1. □

COROLLARY 4.6. *Let T be a square root of a hyponormal operator with the property that $\sigma(T) \cap [-\sigma(T)] = \phi$. If A is any quasiaffine transform of T (i.e., there exists a one-to-one X with dense range such that $XA = TX$), then $\sigma(T) \subset \sigma(A)$.*

Proof. It is clear from Corollary 4.5 and [**8**]. □

REFERENCES

1. S. K. Berberian, The Weyl spectrum of an operator, *Indiana Univ. Math. J.* **20** (1970), 529–544.

2. L. A. Coburn, Weyl's theorem for nonnormal operators, *Michigan Math. J.* **13** (1966), 285–288.

3. J. Eschmeier and B. Prunaru, Invariant subspaces for operators with Bishop's property (β) and thick spectrum, *J. Funct. Anal.* **94** (1990), 196–222.

4. P. R. Halmos, *A Hilbert space problem book* (Springer-Verlag, 1982).

5. R. Harte, *Invertibility and singularity for bounded linear operators* (Marcel Dekker, New York, 1988).

6. E. Ko, Algebraic and triangular n-hyponormal operators, *Proc. Amer. Math. Soc.* **123** (1995), 3473–3481.

7. E. Ko, k-Quasihyponormal operators are subscalar, *Integral Equations Operator Theory* **28** (1997), 492–499.

8. E. Ko, On a Clary theorem, *Bull Korean Math. Soc.* **33** (1996), 29–33.

9. K. Oberai, On the Weyl spectrum II, *Illinois J. Math.* **21** (1977), 84–90.

10. M. Putinar, Hyponormal operators are subscalar, *J. Operator Theory* **12** (1984), 385–395.

11. J. G. Stampfli, Hyponormal operators, *Pacific J. Math.* **12** (1962), 1453–1458.

12. A. E. Taylor and D. C. Lay, *Introduction to functional analysis* (John Wiley and Sons, New York, 1980).