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A CONDITION FOR ARTINIAN RINGS TO BE NOETHERIAN

ICHIRO MURASE

1. Introduction. Throughout this paper the word "Artinian (Noetherian) ring" means an associative ring with minimum (maximum) condition on *left* ideals. According to C. Hopkins, an Artinian ring is Noetherian if it contains a left or right identity [**3**, p. 728]. However we shall consider Artinian rings without the assumption of existence of such an identity, and the theorem of Hopkins will be reproved.

Let A be an Artinian ring, and N its Jacobson radical. As is well known, L. Fuchs proved that A is Noetherian if and only if the additive group of A contains no subgroup of type $C(p^\infty)$ [**2**, p. 285]. Recently Y. -H. Xǔ obtained another theorem [**6**, p. 274], and H. Tominaga reproved and restated it as follows: A is Noetherian if and only if the factor module N/AN is finite [**5**]. We shall investigate relation between these theorems and show that the theorem of Fuchs is connected with that of Xǔ-Tominaga by the following theorem: In case A is nilpotent, A is Noetherian if and only if A is finite. In case A is nonnilpotent, consider an idempotent e of A lifted from the identity of the semisimple ring A/N. Let R_e be the right annihilators of e in A. Then A is Noetherian if and only if R_e is finite. The connection is based on the fact that the additive group of R_e is an Artinian torsion group.

In the way of investigation or as an application we shall get some related theorems which are as follows. First, let A be a nonnilpotent Artinian ring. Then for every left ideal M of A one has $M = AM$ if and only if A contains a left identity. Also for every right ideal M of A one has $M = MA$ if and only if A contains a right identity. Next, if an algebra A over an infinite field is a nonnilpotent Artinian ring, then A contains a left identity. Further, there does not exist an algebra over a field of characteristic 0 which is a nilpotent Artinian ring.

2. Theorem of Xǔ-Tominaga. Let A be an Artinian ring and N its Jacobson radical. We observe the following series.

(1) $A \supseteq N \supseteq AN \supseteq N^2 \supseteq \cdots \supseteq AN^{\rho-1} \supseteq N^\rho = 0.$

As is well known, A is Noetherian if and only if this series can be refined to a composition series for the left ideals of A. We begin by reproving anew the following theorems.

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THEOREM 1. (Xŭ). *An Artinian ring A is Noetherian if and only if $k_i N^i \subseteq AN^i$ for some positive integer $k_i (i = 1, 2, \cdots \rho - 1)$, where $k_i N^i = \{k_i a | a \in N^i\}$ and $k_i a = a + a + \cdots + a$ (k_i summands).*

THEOREM 2 (Tominaga). *An Artinian ring A is Noetherian if and only if the factor module N/AN is finite.*

Note that the factor module N/AN is considered merely as an additive group, because it is trivial as a left A-module. First we show the equivalence of the conditions in Theorem 1 and Theorem 2.

LEMMA 1. *If an additive Abelian group G of bounded order satisfies the minimum condition on subgroups, then G is finite.*

Proof. As is well known, an additive Abelian group of bounded order is a direct sum of cyclic groups [**2**, p. 44]. Since G moreover satisfies the minimum condition, G is a direct sum of a finite number of cyclic subgroups. Hence clearly G is finite.

Proof of the equivalence. Assume $kN \subseteq AN$ for some positive integer k. Then the orders of the elements of N/AN is bounded. Moreover the additive group N/AN satisfies the minimum condition on subgroups, because every subgroup of N/AN is a homomorphic image of a left ideal of A and the left ideals of A obey the minimum condition. Hence N/AN is finite by Lemma 1.

Assume conversely that N/AN is finite. Then $kN \subseteq AN$ for some positive integer k. Therefore clearly $kN^i \subseteq AN^i$ for every $i = 1, 2, \cdots \rho - 1$. By the same argument as above, then every N^i/AN^i is finite.

LEMMA 2. *If an Artinian ring A with radical N is Noetherian, then every factor module N^i/AN^i is finite ($i = 1, 2, \cdots \rho - 1$).*

Proof. Let $G = N^i/AN^i$. Then the additive group G satisfies both maximum condition and minimum condition on subgroups. By the maximum condition G is finitely generated. Recall the fundamental theorem of finitely generated Abelian groups. Then by the minimum condition G is a direct sum of a finite number of cyclic subgroups. Hence G is finite.

Proof of Theorem 2. Assume that A is Noetherian. Then N/AN is finite by Lemma 2.

Assume conversely that N/AN is finite. Then first, every N^i/AN^i is finite, as previously noted. Next, every left A-module AN^i/N^{i+1} is completely reducible. It can be shown by the classical argument as follows.

Let $\bar{A} = A/N$. Then $\bar{A}$ is a semisimple ring. As can be easily seen, the left A-module AN^i/N^{i+1} can be regarded as a unital left $\bar{A}$-module. Moreover it satisfies the minimum condition on submodules. Hence it is completely reducible.

Therefore the series (1) can be refined to a composition series for the left ideals of A. Hence A is Noetherian.

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THEOREM 3. *A nilpotent Artinian ring A is Noetherian if and only if A is finite.*

Proof. By assumption we have $A = N$, and so the series (1) becomes

$$A = N \supset N^2 \supset \cdots \supset N^{\rho-1} \supset N^\rho = 0.$$

If A is Noetherian, then every N^i/N^{i+1} is finite by Lemma 2. Hence A is finite. The converse is trivial.

THEOREM 4 (Hopkins). *If an Artinian ring A contains a left identity, then A is Noetherian.*

Proof. In this case we have $N = AN$. Therefore the series (1) for this case is

$$A \supset N \supset N^2 \supset \cdots \supset N^{\rho-1} \supset N^\rho = 0,$$

because $N^i = AN^i$ for all $i = 1, 2, \cdots \rho - 1$. Hence every N^i/N^{i+1} is completely reducible.

3. A condition for existence of a left (right) identity. Assume that an Artinian ring A is nonnilpotent, and let e be any nonzero idempotent element of A. Then we have the Peirce decompositions:

(2) $A = Ae + L_e$ and $A = eA + R_e$,

where $L_e = \{x \in A | xe = 0\}$ and $R_e = \{x \in A | ex = 0\}$.

Lift the identity element of A/N to an idempotent of A, and let this idempotent be e. Then both L_e and R_e are contained in the radical N. Therefore we have further

(3) $N = Ne + L_e$ and $N = eN + R_e$.

This idempotent e will be called a *principal idempotent.*

THEOREM 5. *Let A be a nonnilpotent Artinian ring with radical N. Then $N = AN$ if and only if A contains a left identity. Also $N = NA$ if and only if A contains a right identity.*

Proof. Let e be a principal idempotent of A. Then by (2)

$$AN = (eA + R_e)N = eAN + R_eN.$$

Since $eAN = eN$, it can be rewritten as

(4) $AN = eN + R_eN.$

Compare this with $N = eN + R_e$. Then it follows that $N = AN$ if and only if $R_e = R_eN$. It implies that

$$R_e = R_eN = R_eN^2 = \cdots = R_eN^\rho = 0.$$

Then $A = eA$, and so e is a left identity of A.

Similarly, $N = NA$ if and only if $L_e = 0$, i.e. e is a right identity.

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Clearly this theorem can be restated as follows.

THEOREM 6. *Let A be a nonnilpotent Artinian ring. Then for every left ideal M of A one has $M = AM$ if and only if A contains a left identity. Also, for every right ideal M of A one has $M = MA$ if and only if A contains a right identity.*

4. A condition for Artinian rings to be Noetherian. Let A be a nonnilpotent Artinian ring with radical N, and e be a principal idempotent of A. Consider a mapping $\varphi: R_e \rightarrow N/AN$ defined by

$$x\varphi = x + AN \quad \text{for} \quad x \in R_e.$$

Then the mapping φ yields a homomorphism on the additive group of R_e, and it induces an isomorphism on the factor group $R_e/\text{Ker } \varphi$. Recall (4) and (3). Then it can be easily seen that $\text{Ker } \varphi = R_eN$ and $\text{Im } \varphi = N/AN$. Hence we have

$$(5) \quad N/AN \cong R_e/R_eN.$$

Generally also for $i = 2, 3, \cdots \rho - 1$, we have

$$N^i = (eN + R_e)N^{i-1} = eN^i + R_eN^{i-1},$$
$$(6) \quad AN^i = (eA + R_e)N^i = eN^i + R_eN^i,$$
$$N^i/AN^i \cong R_eN^{i-1}/R_eN^i.$$

THEOREM 7. *A nonnilpotent Artinian ring A is Noetherian if and only if R_e is finite.*

Proof. If R_e is finite, then N/AN is finite, as is clear by (5). Therefore A is Noetherian by Theorem 2.

Assume conversely that A is Noetherian. Then N^i/AN^i for every i is finite by Lemma 2. Now consider the series

$$(7) \quad R_e \supset R_eN \supseteq R_eN^2 \supseteq \cdots \supseteq R_eN^{\rho-1} = 0.$$

Looking through this series from the left to the right, let $R_eN^{j-1} \supset R_eN^j$ be the last proper inclusion. Then we have

$$[R_e: 0] = [R_e: R_eN][R_eN: R_eN^2] \cdots [R_eN^{j-1}: 0]$$
$$= [N: AN][N^2: AN^2] \cdots [N^j: AN^j]$$

This is equal to the number of elements of R_e, and so R_e is finite.

5. Some properties of R_e.

PROPOSITION 8. *The additive group of R_e satisfies the minimum condition on subgroups.*

Proof. Any module with minimum condition on submodules is said to be Artinian. We claim that R_e is an Artinian module. Consider again (7) and (6).

Then R_eN^{i-1}/R_eN^i is an Artinian module, because N^i/AN^i is so. We first consider $R_eN^{\rho-2}$. Every submodule M of $R_eN^{\rho-2}$ is a left ideal of A, because

$$AM = (Ae + L_e)M = L_eM \subseteq N^\rho = 0.$$

Therefore $R_eN^{\rho-2}$ is an Artinian module. Now, recall the well-known theorem: Let B be a submodule of a module A. Then A is Artinian if and only if B and A/B are Artinian [**4**, p. 22]. Then it follows at once that every module R_eN^i and R_e are Artinian modules.

Remark. The criterium is applied to **Z**-modules. The above proof goes through—by dropping R_e—if $A = N$. That is, if $A = N$, then the additive group of A is Artinian.

PROPOSITION 9. *The additive group of R_e is a torsion group.*

Proof. Let u be a nonzero element of R_e. We claim that $ku = 0$ for some positive integer k.

Consider a set S of left ideals of A generated by a multiple of u. Then by the minimum condition there exists a minimal ideal in S. Let us write it as $(m\mathbf{Z})u + Au$. Then for every positive integer r we have

$$(rm\mathbf{Z})u + Au = (m\mathbf{Z})u + Au.$$

It implies that for some positive integer s we have $mu - (rms)u \in Au$. Let $m_1 = (rs - 1)m$. Then

$$m_1u \in Au = (Ae + L_e)u = L_eu \subseteq N^2.$$

If $m_1u \neq 0$, then we apply the same argument to $u_1 = m_1u$ and find a positive integer m_2 such that

$$m_2u_1 \in Au_1 = (Ae + L_e)u_1 = L_eu_1 \subseteq N^3.$$

By a repetition of this argument we can find a positive integer k such that $ku = 0$.

PROPOSITION 10 (Hopkins [**3**, p. 727]). *The number of the elements of R_ee is finite.*

Proof. Decompose Ae into a direct sum of indecomposable left ideals. Let it be $Ae = L_1 + L_2 + \cdots + L_n$. Then we have mutually orthogonal primitive idempotents $e_1, e_2 \cdots e_n$ such that $e = e_1 + e_2 + \cdots + e_n$ and $L_i = Ae_i$ for all $i = 1, 2, \cdots n$. Accordingly

$$R_ee = R_ee_1 + R_ee_2 + \cdots + R_ee_n,$$

and $R_ee_i = R_e(e_iAe_i)$ for all $i = 1, 2, \cdots n$.

Note that every e_iAe_i is a completely primary ring. Therefore every element of e_iAe_i not belonging to e_iNe_i has a multiplicative inverse [**1**, p. 97].

We first prove that if $R_e e_i \neq 0$ then $m_i(R_e e_i) = 0$ for some positive integer m_i. We need only show that $m_i e_i = 0$, because then we have $m_i x = m_i(xe_i) = x(m_i e_i) = 0$ for all elements x of $R_e e_i$.

Let u be a nonzero element of $R_e e_i$. Since the additive group of $R_e e_i$ is a torsion group, $mu = 0$ for some positive integer m. If $me_i \neq 0$ and $me_i \notin e_i N e_i$, then there is an element v of $e_i A e_i$ such that $(me_i)v = e_i$. Then

$$u = ue_i = u\{(me_i)v\} = \{u(me_i)\}v = (mu)v = 0,$$

contradictory to the assumption $u \neq 0$. Therefore $me_i = 0$ or $me_i \in e_i N e_i$. But if $me_i \in e_i N e_i$, then $(me_i)^j = m^j e_i = 0$ for some positive integer j. In any case there is a positive integer m_i such that $m_i e_i = 0$.

Therefore we have a positive integer k such that $k(R_e e) = 0$. Hence the additive group of $R_e e$ is of bounded order. Moreover it satisfies the minimum condition on subgroups. Therefore $R_e e$ is finite by Lemma 1.

THEOREM 11 (Hopkins). *If an Artinian ring A contains a right identity, then A is Noetherian.*

Proof. Let e be the right identity. Then e is clearly a principal idempotent. We have $R_e = R_e e$. Hence R_e is finite, and so A is Noetherian by Theorem 7.

6. The theorem of Fuchs.

THEOREM 12 (Szele-Fuchs [**2**, p. 280]). *If an Artinian ring A is nilpotent, then the additive group of A is an Artinian torsion module.*

Proof. By assumption $A = N$. First, the additive group of N satisfies the minimum condition on subgroups. It is remarked at the end of the proof of Proposition 8. Next, the additive group of N is a torsion group. It can be proved similarly to Proposition 9.

THEOREM 13 (Fuchs). *An Artinian ring A is Noetherian if and only if the additive group of A contains no subgroup of type $C(p^\infty)$.*

Proof 1. The case where A is nilpotent. By Theorem 3, A is Noetherian if and only if A is finite. Therefore we claim that A is finite if and only if A contains no subgroup of type $C(p^\infty)$, i.e. no quasicyclic subgroup.

Recall the theorem of Kuroš which is as follows. The subgroups of an additive Abelian group G satisfy the minimum condition if and only if G is a direct sum of a finite number of quasicyclic and/or cyclic p-groups [**2**, p. 65].

Note that our additive group A satisfies the minimum condition by Theorem 12. Then the theorem of Kuroš completes the proof.

Proof 2. The case where A is nonnilpotent. Let e be any principal idempotent and consider R_e. Then by Theorem 7, A is Noetherian if and only if R_e is finite. Therefore we claim that R_e is finite if and only if A contains no quasicyclic subgroup.

Assume that A contains no quasicyclic subgroup. Then of course R_e contains no quasicyclic subgroup. By Proposition 8 the additive group of R_e satisfies the minimum condition on subgroups. Hence by the theorem of Kuroš R_e is finite.

Assume conversely that R_e is finite. Then R_e contains no quasicyclic subgroup by the theorem of Kuroš. Here we have to cite the theorem of Fuchs that every quasicyclic subgroup belongs to the annihilator of A [**2**, p. 281]. Then any quasicyclic subgroup of A must be contained in R_e. However R_e contains no quasicyclic subgroup. Hence A also contains no such subgroup.

7. Application to algebras. In this section we consider an algebra A over a field K merely as a ring. Then a left ideal L of A need not satisfy the condition:

(8) if $a \in L$ and $\gamma \in K$, then $\gamma a \in L$.

This condition is imposed upon A only.

THEOREM 14. *Let A be an algebra over any infinite field K. If A is a nonnilpotent Artinian ring, then A contains a left identity.*

Proof. Consider first the case where the characteristic of K is 0. Let e be a principal idempotent of A. Then the additive group of R_e is a torsion group by Proposition 9. However, in this case A clearly contains no torsion element. Hence $R_e = 0$, and so e is a left identity.

Consider next the case where the characteristic of K is $p \neq 0$. Then $pR_e = 0$, i.e. the additive group of R_e is of bounded order. Besides, the additive group R_e satisfies the minimum condition on subgroups. Hence by Lemma 1 R_e is finite.

Suppose $R_e \neq 0$, and let v be a nonzero element of R_e. Let γ be a nonzero element of K. Then $\gamma v \neq 0$ and $\gamma v \in R_e$, because $e(\gamma v) = \gamma(ev) = 0$. Therefore R_e must contain an infinite number of elements. This is a contradiction. Hence $R_e = 0$, and e is a left identity.

THEOREM 15. *Let A be an algebra of finite rank over any infinite field K. Then A is a nonnilpotent Artinian ring if and only if A contains a left identity.*

Proof. Because of Theorem 14, it remains only to prove the "if" part. Assume that A contains a left identity e. Let L be any left ideal of A. Then the condition (8) is necessarily satisfied, because

$$\gamma a = \gamma(ea) = (\gamma e)a \in AL \subseteq L.$$

Therefore L is a left K-module. Since A is a left K-module of finite rank, it is obvious that A satisfies the minimum condition on left ideals.

THEOREM 16. *Let K be a field of characteristic* 0. *Then there does not exist an algebra over K which is a nilpotent Artinian ring.*

Proof. Suppose that an algebra A over the field K is a nilpotent Artinian ring. Then the additive group of A is a torsion group by Theorem 12. However, A clearly contains no torsion element. This is a contradiction.

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Japan Women's University,
Mejirodai, Bunkyo-ku, Tokyo 112, Japan