An Elementary Proof of Morley's Trisector Theorem

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Morley's theorem states that if ABC be any triangle, and if those trisectors of the angles B and C adjacent to BC meet in L, and M, N be similarly constructed, then the triangle LMN is equilateral.

LMN is in fact one of eighteen equilateral triangles formed by meets of trisectors of the angles of ABC, when we consider the angle contained by a given pair of lines to be determined only in the form $\theta + 2n\pi$, where n is an arbitrary integer. This extension has been concisely treated in trigonometrical terms.[1] To Morley himself is due the most elegant treatment of the subject, in which the sides of the equilateral triangles arise as the locus of centres of cardioids inscribed in ABC.[2]

The following proof requires only a knowledge of elementary geometry as far as Euclid, Book III. Such a proof has already been given,[3] but it seems unfortunate that in the construction there adopted the symmetry intrinsic to the beauty of the proposition should play no part.

We shall prove the theorem in the equivalent form:— If P_1 P_2 P_3 be any equilateral triangle and α_1, α_2, α_3 any three angles of sum $\pi/3$, then a triangle may be constructed having angles, of size $3\alpha_1$, $3\alpha_2$, $3\alpha_3$, the trisectors of which meet in P_1, P_2, P_3.

The theorem as originally stated follows immediately from this, since if the angles of ABC be taken as $3\alpha_1$, $3\alpha_2$, $3\alpha_3$ we obtain a figure similar to the original one, whence LMN is equilateral.

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[1] H. Lob and H. W. Richmond, *Proc. London Math. Soc.* (2), 31 (1930), 355-369.

[2] F. Morley, *American J. of Math.*, 51 (1929), 465-472 (469).

[3] J. M. Child, *Math. Gazette*, 11 (1923), 171.

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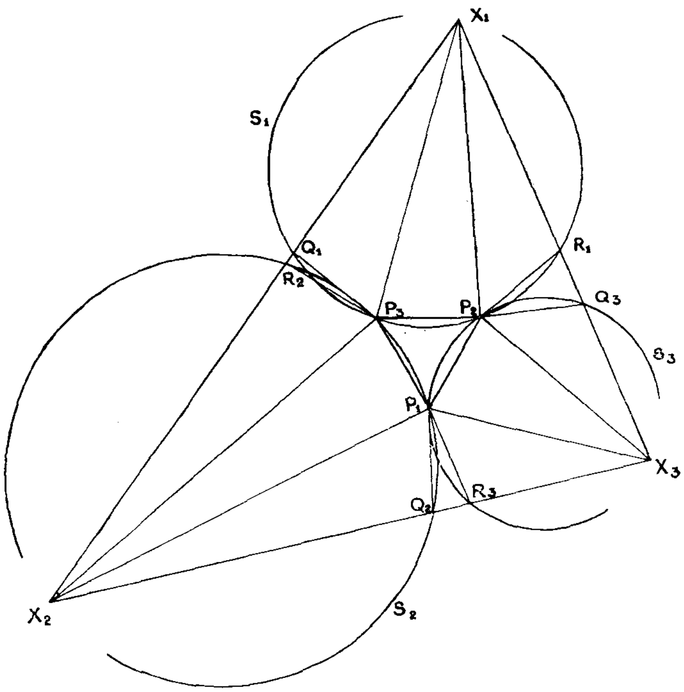

Construct through P_j, P_k that circle, S_i, at whose circumference $P_j P_k$ subtends an angle α_i on the side opposite to P_i, the convention $ijk = 123$, 231 and 312 being obeyed throughout.

Let Q_i, R_i be those points of S_i distinct from P_j, P_k such that the chords $Q_i\,P_k$, $P_j\,R_i$ are equal in length to $P_j\,P_k$. Let $Q_k\,R_i$, $R_j\,Q_i$ meet in X_i. We shall show that $X_1\,X_2\,X_3$ is the required triangle.

It is an immediate consequence of the construction that $Q_i\,\widehat{P_k}\,P_j = P_k\,\widehat{P_j}\,R_i = \pi - 2\alpha_i$. Hence, adding the angles at P_i and using $\Sigma\,\alpha_i = \pi/3$, we have $R_k\,\widehat{P_i}\,Q_j = \pi/3 - 2\alpha_i$, whence $P_i\,\widehat{Q_j}\,X_j = P_i\,\widehat{R_k}\,X_k = 2\pi/3 - \alpha_i$.

Now the angles of a pentagon have sum 3π, so that $Q_i\,\widehat{X_i}\,R_i = 3\alpha_i$. But this is the angle subtended at the circumference of S_i by $Q_i\,R_i$, whence X_i lies on S_i. It follows that $Q_i\,\widehat{X_i}\,R_i$ is trisected by $X_i\,P_j$, $X_i\,P_k$.

Thus $X_1\,X_2\,X_3$ is a triangle having angles, of size $3\alpha_1$, $3\alpha_2$, $3\alpha_3$, the trisectors of which meet in P_1, P_2, P_3.