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PAPER

On planarity of graphs in homotopy type theory

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Abstract

In this paper, we present a constructive and proof-relevant development of graph theory, including the notion of maps, their faces and maps of graphs embedded in the sphere, in homotopy type theory (HoTT). This allows us to provide an elementary characterisation of planarity for locally directed finite and connected multigraphs that takes inspiration from topological graph theory, particularly from combinatorial embeddings of graphs into surfaces. A graph is planar if it has a map and an outer face with which any walk in the embedded graph is walk-homotopic to another. A result is that this type of planar maps forms a homotopy set for a graph. As a way to construct examples of planar graphs inductively, extensions of planar maps are introduced. We formalise the essential parts of this work in the proof assistant Agda with support for HoTT.

Keywords: Planarity; combinatorial maps; univalent mathematics; formalisation of mathematics

1. Introduction

Topological graph theory investigates the embedding of graphs into diverse surfaces such as the plane, sphere and torus (Archdeacon 1996; Gross and Tucker 1987; Stahl 1978). The simplest case, graph map into the plane, has generated numerous intriguing characterisations and mathematical results. Kuratowski's theorem and Wagner's theorem (Diestel 2012; Rahman 2017) are two such characterisations, both defining planarity by excluding two forbidden minors, K_5 and $K_{3,3}$. Alternative approaches include algebraic methods like MacLane's theorem (MacLane 1937) and Schnyder's theorem (Baur 2012, Section 3.3).

One of the most powerful tools in topological graph theory is the *combinatorial representation* of graph embeddings, called graph maps, also known as rotation systems (Gross and Tucker 1987). These representations encode what the embedding looks like around each node, characterising the embedding up to isotopy. It is known that for a suitable general class of embeddings into closed surfaces – namely, the cellular ones – the embedding is characterised by the cyclic order of outgoing edges from each node as they lie around the node on the surface.

In this paper, we present a constructive and proof-relevant definition of these combinatorial representations of graph embeddings in homotopy type theory (HoTT for short) (Univalent Foundations Program 2013). HoTT is a variation of dependent-type theory which emphasises the higher-dimensional structure of types. In HoTT, equalities within a type are seen as paths, and the type of all equalities between two elements – the identity type – is thought of as a path space. In this way, HoTT takes seriously the notion of proof-relevancy, and interesting questions arise when considering what the equality between two proofs is.

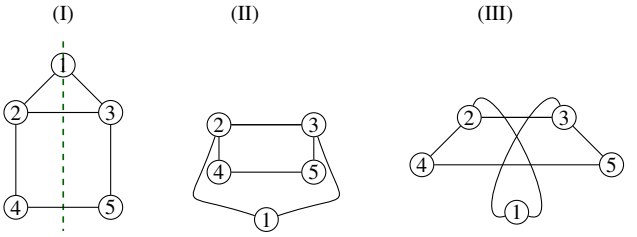

Figure 1. Different visual representations for the same graph map of the house graph given in Example 1.1. Note how the cyclic order of edges around each node is preserved consistently across all representations. The first two representations correspond to *drawings* – the result of planar maps for the house graph, while the last representation does not, as it features an edge crossing, so it is not an embedding.

In this context, we present planarity as a structure imposed on a graph, rather than a simple property of it – as is the case in classical graph theory. The intuitive explanation is that a proof of a graph's planarity is its embedding into the plane.

The question is then, when are two such embeddings considered equal? A plausible response is that the proofs should be regarded as equal when the embeddings are isotopic, meaning they can be continuously deformed into one another without edge crossing. This will hold true for the concept proposed in this paper. However, to reach a type of graph maps where the identity type corresponds to isotopy, a lot of care must be taken with respect to the definition of embeddings and planarity.

In short, a planar graph will be defined as a graph with a combinatorial embedding into the sphere, along with a designated face for puncturing. The intuition is that an embedding into the plane can be obtained from an embedding into the sphere by puncturing the sphere at a point symbolising infinity (in any direction) on the plane. Up to isotopy, the important data when choosing a point puncture is which face the points lies in.

In contrast to previous works in planarity of graphs and related formal verification, our development adopts Voevodsky's Univalence Axiom (UA) from HoTT. As a result, isomorphic graphs are equal and share the same structures and properties. This correspondence is crucial for formalising mathematics, as it allows us to understand a graph's symmetry through its identity type, as in standard mathematical practice. Any automorphism of a graph gives rise to an inhabitant of its identity type and vice versa. By studying its identity type, we can describe the group structure of the set of automorphisms for a graph.

To conclude, let us consider a familiar example to gain a clearer understanding of the concepts presented in this paper.

Example 1.1. *Consider the house graph G depicted in Fig. 1. This graph consists of five nodes and six edges:* $(1, 2)$, $(1, 3)$, $(2, 3)$, $(2, 4)$, $(3, 5)$, *and* $(4, 5)$.

As previously hinted, a graph map assigns to each node a counterclockwise cycle of its adjacent nodes. Consider m *as a graph map for our house graph induced from Fig. 1 (I). At node* 2*, the graph map results in the sequence* $[1, 4, 3]$. *This sequence not only lists the adjacent nodes but also specifies a counterclockwise order among the connecting edges. Thus, edge* $(2, 1)$ *is followed by* $(2, 4)$ *and then by* $(2, 3)$ *in this established order, see Fig. 2.*

On the planarity of the house graph, we notice that it has six planar drawings split into two sets based on (I) and (II) in Fig. 1, respectively. With the graph map m, there are three options for the outer face, as illustrated in Fig. 3. The absence of edge crossings and the existence of a graph map with an outer face confirm the graph's planarity, at least for now. To be able to prove this kind of claim in the context of this paper, we must develop our planarity criteria, as detailed in Section 6.

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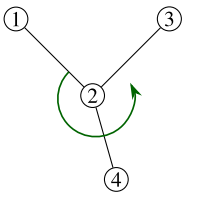
(a)

Node	Adjacent nodes	Graph map
1	2, 3	[2, 3]
2	1, 3, 4	[1, 4, 3]
3	1, 2, 5	[1, 2, 5]
4	2, 5	[2, 5]
5	3, 4	[3, 4]

Related information to the map m.

(b)



The graph map m at node 2.

Figure 2. Graph map m for the house graph G depicted in Fig. 1 (I).

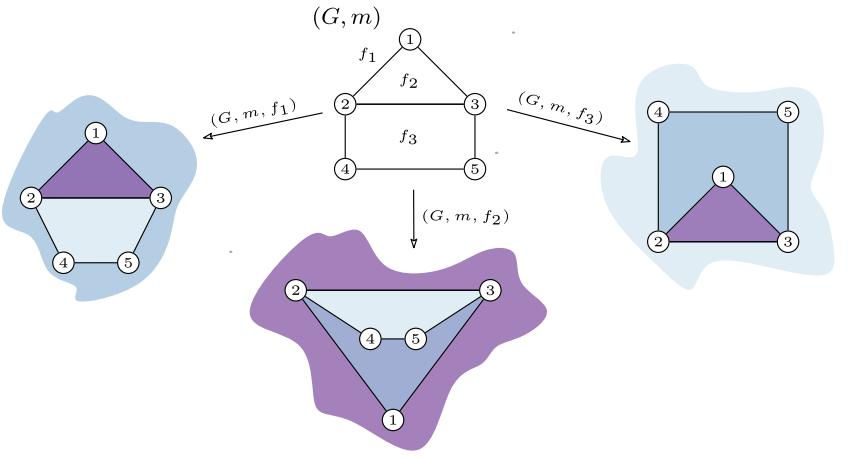


Figure 3. The house graph G and its planar maps. The three distinct planar drawings (G, m, f_i) for m are presented. Each drawing corresponds to an individually selected outer face: f_1, f_2 and f_3. These faces, enclosed by a pentagon, triangle and rectangle, respectively, are differentiated by distinct shading. The unbounded region of the plane, represented as a splashed area, denotes the outer face in each planar drawing.

1.1 Outline

The paper is structured as follows: Section 2 introduces the basic terminology and notation. Next, the category of graphs, along with pertinent examples, is described in Section 3. In Section 4, we present different types for graph-theoretic concepts, which allows us to define planar maps and, consequently, planar graphs in HoTT. The construction of larger planar graphs, including the proof of planarity for cyclic graphs and graph extensions, is detailed in Section 6. Connections between this work and other developments are explored in Section 7. Finally, Section 8 concludes the paper with a discussion on future work and some concluding remarks.

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1.2 Formalisation

Working with systems like HoTT brings the opportunity to produce machine-verified proofs (Harrison 2008). We employ Agda, a proof assistant rooted in Martin-Löf type theory (The Agda Development Team 2023), for verification of the fundamental constructions in this paper. Agda, a robust dependently typed programming language, facilitates working at an abstraction-level equivalent to our paper-based mathematical reasoning. This rigorous approach instills confidence and enables us to formalise mathematical concepts and proofs.

Machine-verified proofs not only provide insights into new proofs and theorems (Avigad and Harrison 2014) but also help identify overlooked flaws and corner cases. Therefore, special attention must be given to definitions and theorems, being the primary input for these systems. The process of formalising on a computer is both exciting and challenging, replete with intricate details and technical issues (Appel and Haken 1986; Gonthier 2008).

We use Agda v2.6.2.2-442c76b for type-checking the formalisation (Prieto-Cubides 2022*a*) of this paper's essential parts. The flags `without-K` (Cockx et al. 2016) and `exact-split` are used to ensure compatibility with HoTT and to guarantee that all clauses in a definition are definitional equalities, respectively.

2. Mathematical Foundation

In this paper, we work with HoTT, a Martin-Löf intensional type theory extended with the UA (Awodey 2018; Voevodsky 2010), and some higher inductive types (HITs), such as propositional truncation (Escardó 2018; Univalent Foundations Program 2013). The presentation of our constructions is informal, in a similar style as in the HoTT book (Univalent Foundations Program 2013).

HoTT emphasises the role of the identity type as a path type. The intended interpretation is that elements, $a, a' : A$, are *points* and that a witness of an equality $p : a = a'$ is a *path* from a to a' in A. Since the identity type is again a type, we can iterate the process, which gives each type the structure of an ∞-groupoid (Awodey 2012).

This may at first seem of little relevance when working with finite combinatorics, as one would expect only types with trivial path types (sets) to show up in combinatorics. However, we will see that types with non-trivial path types do arise naturally in combinatorics – which should come as no surprise to anyone familiar with the role of groups and groupoids in this field, such as Joyal's work on combinatorial species (Baez et al. 2009 – 08; Yorgey 2014) – and that the paths in these types are often various forms of permutations.

2.1 Notation

An informal type theoretical notation derived from the HoTT book (Univalent Foundations Program 2013) and the formal system Agda (Norrell 2007) is used throughout this paper. The following list summarises the most important conventions and notations used in this paper.

- Definitions are introduced by $(:\equiv)$, while judgemental equalities use $(\equiv)$.
- The type $\mathcal{U}$ is a *univalent* universe.
- The notation $A : \mathcal{U}$ indicates that A is a type. A term a of type A is denoted by $a : A$ and A is referred to as a type *inhabited*.
- The equality sign of the identity type of A is denoted by $(=_A)$. The constructor of the identity type $x =_A x$ is denoted by $\mathsf{refl}(x)$ for $x : A$. If the type A can be inferred from the context, we simply write $(=)$. The equalities between $x, y : A$ are of type $x = y$.
- The type of non-dependent functions between A and B is denoted by $A \to B$.
- Type equivalences are denoted by $(\simeq)$. The canonical map for types is the function $\mathsf{idtoequiv}$ of type $A = B \to A \simeq B$ and its inverse function is called ua. Given the equivalence $e : A \simeq B$, the application, $\mathsf{ua}(e)$ is denoted by $\bar{e}$, while the underlying function of the equivalence e of

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type $A \to B$ can be also denoted by e. Moreover, the coercion along a path $p : A = B$ is the function denoted by $\mathsf{coe}(p)$ of type $A \to B$.

- ▹ The point-wise equality for functions (also known as *homotopy*) is denoted by $(\sim)$. The function happly is of type $f = g \to f \sim g$ and its inverse function is called funext.
- ▹ The coproduct of two types A and B is denoted by $A + B$. The corresponding data constructors are the functions $\mathsf{inl} : A \to A + B$ and $\mathsf{inr} : B \to A + B$.
- ▹ Dependent product types (Π-types) are denoted by $\Pi_{x:A} B(x)$ for a type A and a type family $B : A \to \mathcal{U}$, while dependent sum types (Σ-types) are denoted by $\Sigma_{x:A} B(x)$. If $x : A$ and $y : B(x)$, then the pair (x, y) is of type $\Sigma_{x:A} B(x)$. The corresponding projection functions for a pair are denoted by π_1 and π_2 so that $\pi_1(x, y) :\equiv x$ and $\pi_2(x, y) :\equiv y$. If the type family B over A is constant, then we may denote the type $\Sigma_{x:A} B(x)$ by $A \times B$, and the $\Pi_{x:A} B(x)$ by $A \to B$.
- ▹ The empty type and the unit type are denoted by $\mathbb{0}$ and $\mathbb{1}$, respectively.
- ▹ The type $x \neq y$ denotes the function type $(x = y) \to \mathbb{0}$.
- ▹ Natural numbers are of type $\mathbb{N}$. $0 : \mathbb{N}$. The successor of $n : \mathbb{N}$ is denoted by $S(n)$ or $n + 1$. The variable n is of type $\mathbb{N}$, unless stated otherwise.
- ▹ Given $n : \mathbb{N}$, the standard type with n elements is denoted by $[\![n]\!]$.
- ▹ The universe $\mathcal{U}$ is closed under the type formers considered above.
- ▹ The function transport/substitution is denoted by tr of type $\Pi_{u:x=x'} B(x) \to B(x')$, where $x, x' : A$ and $B : A \to \mathcal{U}$. Furthermore, we denote by tr_2 the function of type $\Pi_{p:a_1=a_2} \mathsf{tr}^B(p, b_1) = b_2 \to C(a_1, b_1) \to C(a_2, b_2)$, where the type family B is indexed by the type A, $a_1, a_2 : A$, $b_1 : B(a_1)$, $b_2 : B(a_2)$, and the type C is of type $\Pi_{x:A}\ (B(x) \to \mathcal{U})$.

In the next sections, we will use variables A, B and X to denote types, unless stated otherwise. To define some inductive types, we adopt a similar notation as in Agda, including the keyword data and the curly braces for implicit arguments, for example, $\{a : A\}$ denotes a is of type A, and it is an implicit variable. The type may be omitted in the former notation, as they can usually be inferred from the context.

2.2 Homotopy levels

The following establishes a level hierarchy for types with respect to the non-trivial homotopy structure of the identity type.

Definition 2.1. *Let n be an integer such that $n \geq -2$. One states that a type A is an n-*type *and that it has homotopy level n if the type* $\mathsf{is\text{-}level}(n, A)$ *is inhabited:*

$$\mathsf{is\text{-}level}(-2, A) :\equiv \sum_{(c\,:\,A)} \prod_{(x\,:\,A)} (c = x),$$

$$\mathsf{is\text{-}level}(n + 1, A) :\equiv \prod_{(x,y\,:\,A)} \mathsf{is\text{-}level}(n, x = y).$$

For this document, the first four homotopy levels are enough to express the mathematical objects we want to construct. They are referred to in order, starting from -2, as contractible types, propositions, sets and groupoids. For convenience, we use the following predicates:

- ▹ $\mathsf{isContr}(A) :\equiv \mathsf{is\text{-}level}(-2, A)$,
- ▹ $\mathsf{isProp}(A) :\equiv \mathsf{is\text{-}level}(-1, A)$,
- ▹ $\mathsf{isSet}(A) :\equiv \mathsf{is\text{-}level}(0, A)$, and
- ▹ $\mathsf{isGroupoid}(A) :\equiv \mathsf{is\text{-}level}(1, A)$.

Types that are propositions are of type hProp and similarly with the other levels. If A is an inhabited proposition, then we say that A *holds*. Additionally, it is possible to have an n-type

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out of any type A for $n \geq -2$. This can be done using the construction of a HIT called n-truncation (Univalent Foundations Program 2013, Section 7.3) denoted by $\|A\|_n$. The case for (-1)-truncation is called *propositional truncation* (or *reflection*) and is often simply denoted by $\|A\|$.

Definition 2.2. Propositional truncation *of a type A denoted by $\|A\|_{-1}$ is the* universal solution *to the problem of mapping A to a proposition P. The elimination principle of this construction gives rise to a map of type $\|A\| \to P$, which requires a map $f : A \to P$ and a proof that P is a proposition.*

Propositional truncation allows us to model the *mere* existence of inhabitants of type A. We state that x is *merely* equal to y when $\|x = y\|$ for $x, y : A$. If $\|A\|$ is inhabited, then we say that type A is *non-empty*.

Definition 2.3. *Given $x : A$, the* connected component *of x in A is the type $\Sigma_{y:A} \|y = x\|$.*

Definition 2.4. *The type A is called* connected *if $\|A\|$ holds and each $x : A$ belongs to the same connected component.*

Theorem 2.5. *Let $P : A \to \mathsf{hProp}$ and $x, y : A$. If $\|y = x\|$, then $P(x) \simeq P(y)$. Thus, terms in the same connected component share the same propositional properties.*

Proof. The proof is established by constructing a term of type $\|x = y\| \to \Sigma_{f:P(x)\to P(y)} \mathsf{isEquiv}(f)$. The type $\mathsf{isEquiv(g)}$ is the proposition that f is an equivalence; see the HoTT book (Univalent Foundations Program 2013, Section 4). We apply the elimination principle of propositional truncation to obtain this map, given that its codomain is a proposition, as it is a Σ-type of propositions. Further, we apply path induction over a path of type $x = y$, setting a new goal to find an equivalence of type $P(x) \simeq P(x)$, which is the trivial provided by the identity function. □

2.3 Finite types

In the following, we make precise the intuition that a type is finite when it is equivalent to $[\![n]\!]$ for some $n : \mathbb{N}$. The type $[\![n]\!]$ is the standard type with n elements, which can be defined as the following Σ-type:

$$[\![n]\!] :\equiv \sum_{(m\,:\,\mathbb{N})} m < n, \tag{1}$$

where the binary relation $(<)$ can be defined by cases, that is, $0 < m + 1$ for all m and for all n if $m < n$ then $m + 1 < n + 1$.

Definition 2.6. *A type X is* finite *if the type $\mathsf{isFinite}(X)$ in (2) is inhabited:*

$$\mathsf{isFinite}(X) :\equiv \sum_{(n\,:\,\mathbb{N})} \left\| X \simeq [\![n]\!] \right\|. \tag{2}$$

The finiteness of a type A is the existence of a bijection between A and the type $[\![n]\!]$ for some $n : \mathbb{N}$. However, this description is not a structure on A, which provides it with a specific equivalence $A \simeq [\![n]\!]$, but rather a property, a mere proposition. This ensures that the identity type on the total type of finite types is free to permute the elements, without having to respect a chosen equivalence.

Theorem 2.7. *The type $\mathsf{isFinite}(X)$ is a proposition.*

Proof. Let $(n, p), (m, q) : \mathsf{isFinite}(X)$, which we want to prove equal. Since p and q are elements of a family of propositions, it is sufficient to show that $n = m$. This equation is a proposition, so we can apply the truncation-elimination principle to get $X \simeq [\![n]\!]$ and $X \simeq [\![m]\!]$. Thus, from $[\![n]\!] \simeq [\![m]\!]$ follows that $n = m$ by a well-known result on finite sets. □

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The natural number n in (2) is referred to as the cardinal number of X, which is also denoted by $\#X$. If X and Y are finite and the identity type $X = Y$ is inhabited, then both types have the same cardinal number and Y is a permutation of X. Furthermore, Definition 2.6 is equivalent to the type $\exists_{n:\mathbb{N}}(X = [\![n]\!])$. However, the former definition makes it easier to obtain the cardinal number n by projecting on the first coordinate. This is more practical for certain proofs, such as Theorem 2.15. Additionally, any property of $[\![n]\!]$, like 'being a set' and 'being discrete' can be transferred to any finite type.

Theorem 2.8 (Hedberg's theorem). *Any type A with decidable equality, that is, $x = y + x \neq y$ for all $x, y : A$, is a set. Types like A are below referred to as* discrete *sets.*

Theorem 2.9. *Finite sets are closed under (co) products, type equivalences, Σ-types, Π-types and propositional truncation.*

2.4 Cyclic types

We want to define a notion of *cyclic type* to capture the idea of a finite type together with a permutation within orbiting freely over the whole type. To do so, we use the pred function which generates a cyclic subgroup (of order n) of the group of permutations on $[\![n]\!]$. An equivalent cyclic subgroup can be defined by means of the suc function, where the function suc is the inverse of pred.

Definition 2.10. *Let* pred *be a function from $[\![n+1]\!]$ to itself defined by induction on n and the following equations. If $n = 0$, then* pred *is the trivial function. If $n > 0$, then,*

$$\begin{aligned}&\mathsf{pred} : [\![n+1]\!] \to [\![n+1]\!].\\ &\mathsf{pred}((0, !)) :\equiv (n, p).\\ &\mathsf{pred}((m+1, q)) :\equiv (m, r).\end{aligned}$$

Where p is a proof that $n < n+1$ and r is a proof that $m < n+1$ using q, which is a proof that $m+1 < n+1$.

Definition 2.11. $\mathsf{Cyclic}(A)$ *defines the type of cyclic structures on type A:*

$$\mathsf{Cyclic}(A) :\equiv \sum_{(\varphi\,:\,A\to A)} \sum_{(n\,:\,\mathbb{N})} \left\| \sum_{(e\,:\,A\simeq [\![n]\!])} (e \circ \varphi = \mathsf{pred} \circ e) \right\|. \tag{3}$$

Notice that the type $\mathsf{Cyclic}(A)$ mirrors the structure of $[\![n]\!]$ given by pred for any finite type A along with an endomap $\varphi : A \to A$. This is reflected in (3) by establishing a structure-preserving map between (A, φ) and $([\![n]\!], \mathsf{pred})$. Therefore, a type A with cyclic structure is a triple such as $\langle A, f, n\rangle$ where $(f, n, \text{-}) : \mathsf{Cyclic}(A)$. Given such a triple, we refer to A as an *n-cyclic* and f as the corresponding *cyclic function*. As a notation, if $p : \mathsf{Cyclic}(A)$ and $x : A$, then $p(x)$ is the image of x under the cyclic function f.

Theorem 2.12. *Let P be a family of propositions of type $\Pi_{X:\mathcal{U}}(X \to X) \to \mathsf{hProp}$ and an n-cyclic structure $\langle A, f, n\rangle$. If $P([\![n]\!], \mathsf{pred})$, then $P(A, f)$.*

Proof. It follows from Theorem 2.5. Note that being cyclic for a type is equivalent to saying (A, f) and $([\![n]\!], \mathsf{pred})$ are connected in $\Sigma_{X:\mathcal{U}}(X \to X)$. □

Theorem 2.13. *Let P be a family of propositions of type $\mathcal{U} \to \mathsf{hProp}$ and an n-cyclic structure $\langle A, f, n\rangle$. If $P([\![n]\!])$, then $P(A)$.*

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By Theorems 2.12 and 2.13, one could prove that any n-cyclic type $\langle A, f, n\rangle$ is a finite set and that the function f is a bijection. For convenience, we denote f by pred and its inverse by suc. To define the functions pred and suc for a cyclic structure $\langle A, f, n\rangle$, we borrow the notation from group theory, expressing permutations as products of cycles. For example, a permutation in $[\![3]\!]$ can be defined as the product of two cycles: $\mathsf{pred} :\equiv (0)(12)$, meaning that 0 is fixed and the elements 1 and 2 are swapped.

Theorem 2.14. *Let A be a type. If $\mathsf{Cyclic}(A)$ is inhabited, then A is a finite set.*

Proof. Let A be an n-cyclic type. The conclusion follows immediately from Theorem 2.13 and the fact that the standard finite type $[\![n]\!]$ is a finite set. □

In any finite type, every element is searchable. In particular, given an n-cyclic type $\langle A, f, n\rangle$, one can search any element by iterating the function f on any other element at most n times.

Theorem 2.15. *If A is an n-cyclic type, then for every a and b in A, there exists a unique number k with $k < n$ such that $\mathsf{pred}_A^k(a) = b$.*

The total type, $\Sigma_{A:\mathcal{U}} \mathsf{Cyclic}(A)$, is the classifying type of finite cyclic groups (Bezem et al. 2022, Section 4.6-7). Let us now compute the identity type between two finite cyclic types that we use, for example, in Example 5.12 to enumerate the maps of the bouquet graph B_2.

Theorem 2.16. *Given two cyclic types, $\mathscr{A}$ and $\mathscr{B}$, defined by $\langle A, f, n\rangle$ and $\langle B, g, m\rangle$, respectively, the identity type between them is given by the following equivalence:*

$$(\mathscr{A} = \mathscr{B}) \simeq \sum_{(\alpha\,:\,A = B)} (\mathsf{coe}\,(\alpha) \circ f = g \circ \mathsf{coe}\,(\alpha)\,). \qquad \begin{array}{ccc} A & \xrightarrow{\mathsf{coe}(\alpha)} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[\mathsf{coe}(\alpha)]{} & B \end{array}$$

Proof. We show the equivalence via calculation (4). In (4b), we unfold the cycle-type definitions for $\mathscr{A}$ and $\mathscr{B}$. The numbers n and m are the cardinalities of the types A and B, respectively, and p and q are propositions of the truncation appearing in the type in (3). The type in the equivalence in (4c) follows from the characterisation of the identity type between pairs in a Σ-type (Univalent Foundations Program 2013, Section 3.7). In (4c), we have the product of two propositions, the identity types, $n = m$ and $p = q$. These two types are, in fact, contractible, therefore, equivalent to the one-point type. The numbers n and m are equal because A and B are finite and equal by α, and p and q are equal because truncation of any type is also a proposition. We can then simplify the inner Σ-type to its base in (4d) to obtain by the equivalence $\Sigma_{x:A}\mathbb{1} \simeq A$ in (4e):

$$(\mathscr{A} = \mathscr{B}) \equiv \tag{4a}$$

$$((A, (f, n, p)) = (B, (g, m, q))) \simeq \tag{4b}$$

$$\sum_{(\alpha\,:\,A = B)} \sum_{(\beta\,:\,\mathsf{tr}^{\lambda X.X\to X}(\alpha,f) = g)} (n = m) \times (p = q) \simeq \tag{4c}$$

$$\sum_{(\alpha\,:\,A = B)} \sum_{(\beta\,:\,\mathsf{tr}^{\lambda X.X\to X}(\alpha,f) = g)} \mathbb{1} \simeq \tag{4d}$$

$$\sum_{(\alpha\,:\,A = B)} \mathsf{tr}^{\lambda X.X\to X}(\alpha, f) = g \simeq \tag{4e}$$

$$\sum_{(\alpha\,:\,A = B)} \mathsf{coe}\,(\alpha) \circ f = g \circ \mathsf{coe}\,(\alpha)\,. \tag{4f}$$

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Finally, as a consequence of transporting functions along the equality α, we obtain the type in (4f). The conclusion is that the identity type $\mathscr{A} = \mathscr{B}$ is equivalent to the type of equalities between A and B along with a proof that the structure of f is preserved in the structure of g. □

Theorem 2.17. $\mathsf{Cyclic}(A)$ *is a finite set for any type* A.

Proof. We unfold the definition of $\mathsf{Cyclic}(A)$ to obtain the type $\Sigma_{\varphi\,:\,A\to A}\ \Sigma_{n\,:\,\mathbb{N}}\ \|\ P(A,n)\ \|$ where $P(A,n) :\equiv \Sigma_{e:A\simeq[\![n]\!]}(e\circ\varphi = \mathsf{pred}\circ e)$.

Given the finiteness of type A, it follows that $A \to A$ is finite. We now aim to show that $\Sigma_{n:\mathbb{N}}\|P(A,n)\|$ is finite. We can show this by establishing the equivalence:

$$\sum_{(n:\mathbb{N})} \|P(A,n)\| \simeq \|P(A,\#A)\| \tag{5}$$

and demonstrating that the type $P(A,\#A)$ is finite. Once established, we can conclude that the equivalence preserves the finiteness of the type $\|P(A,\#A)\|$, by the closure property of finite types under Σ-types and propositional truncation.

To establish the equivalence in (5), as both types are propositions, we only need to construct two functions f and g as follows using the propositional truncation elimination principle:

$$\begin{aligned} &f : \textstyle\sum_{(n:\mathbb{N})} \|P(A,n)\| \to \|P(A,\#A)\|.\\ &f((n,|p|)) :\equiv |p|.\\ &\\ &g : \|P(A,\#A)\| \to \textstyle\sum_{(n:\mathbb{N})} \|P(A,n)\|.\\ &g(|r|) :\equiv (\#A,|r|). \end{aligned}$$

The Σ-type, $P(A,\#A)$, is finite given that the base type is an equivalence between two finite types, A and $[\![\#A]\!]$, and each fibre is an identity type over a finite type, which is finite. This leads us to conclude that the type $\Sigma_{n:\mathbb{N}}\|P(A,n)\|$ is finite, thereby implying that $\mathsf{Cyclic}(A)$ is finite. □

3. Notions of Graph Theory

Graphs are a fundamental mathematical concept that has found widespread applications in various fields, including mathematics and computer science. They are used to modelling relationships between objects or entities, making them a versatile tool for analysing complex systems. However, the definition of a graph can vary depending on the context in which it is used. The choice of a specific notion of a graph in a given context depends on the application, such as power graphs in computational biology, quivers in category theory, and networks in network theory. In some cases, graphs are undirected, while in others, they are directed. Additionally, the inclusion of self-edges may be allowed or prohibited.

3.1 The type of graphs

The following is our working definition of graphs. We later introduce concepts such as graph homomorphism, finite graphs, and cyclic graphs.

Definition 3.1. *A graph is an term of type* Graph. *The corresponding data of a graph consists of a set N whose elements are referred to as points, vertices or nodes. Additionally, for every pair of nodes a and b, there is a family of sets E, each of which corresponds to the edges connecting a and b. The elements of these sets are called edges:*

$$\mathsf{Graph} :\equiv \sum_{(\mathsf{N}\,:\,\mathcal{U})}\ \sum_{(\mathsf{E}\,:\,\mathsf{N}\to\mathsf{N}\to\mathcal{U})} \mathsf{isSet}(\mathsf{N}) \times \prod_{(x,y\,:\,\mathsf{N})} \mathsf{isSet}(\mathsf{E}(x,y)).$$

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Given a graph G, for brevity, the set of nodes and the family of edges are denoted by N_G and E_G, respectively. In this way, the graph G is defined as $(\mathsf{N}_G, \mathsf{E}_G, (p_G, q_G))$ where $p_G : \mathsf{isSet}(\mathsf{N}_G)$ and $q_G : \prod_{x,y:\mathsf{N}_G} \mathsf{isSet}(\mathsf{E}_G(x,y))$. We may refer to G only as the pair $(\mathsf{N}_G, \mathsf{E}_G)$, unless we require showing the remaining data, the propositions p_G and q_G. For example, we define the *empty* graph and the *unit* graph, respectively, as $(\mathbb{0}, \lambda\, u\, v.\mathbb{0})$ and $(\mathbb{1}, \lambda\, u\, v.\mathbb{0})$. We will use variables G and H as graphs, and variables x, y, and z as nodes in G, unless otherwise specified.

Remark 3.2. Our primary objective is to provide a comprehensive characterisation of graph planarity. To achieve this, we utilise a set-level concept of graphs, which includes directed multigraphs and those with self-edges, diverging from the traditional focus on undirected graphs. The choice of a set-level structure is based on the common use of sets in the objects and relations studied within graph theory. However, this constraint can be easily modified for different applications.

Definition 3.3. *A* graph homomorphism *from G to H is a pair of functions (α, β) such that $\alpha : \mathsf{N}_G \to \mathsf{N}_H$ and $\beta : \prod_{x,y:\mathsf{N}_G} \mathsf{E}_G(x,y) \to \mathsf{E}_H(\alpha(x), \alpha(y))$. We denote by $\mathsf{Hom}(G,H)$ the type of these pairs.*

We denote by id_G, for any graph G, the identity graph homomorphism where the corresponding α and $\beta(x,y)$ are the corresponding identity functions.

Theorem 3.4. *The type $\mathsf{Hom}(G,H)$ forms a set.*

Proof. Since sets are closed under Π- and Σ-types, and given that both $\mathsf{N}_G \to \mathsf{N}_H$ and $\prod_{x,y:\mathsf{N}_G} \mathsf{E}_G(x,y) \to \mathsf{E}_H(\alpha(x), \alpha(y))$ are function types with set codomains, it follows that $\mathsf{Hom}(G,H)$, being comprised of these types, is a set. □

3.2 The category of graphs

Graphs as objects and graph homomorphisms as the corresponding arrows form a small precategory. In fact, the type of graphs is a small univalent category in the sense of the HoTT book (Univalent Foundations Program 2013, Section 9.1.1). This fact follows from Theorem 3.7 and, morally, because the Graph type is a set-level structure.

In a (pre-) category, an isomorphism is a morphism which has an inverse. In the particular case of graphs, this can be formulated in terms of the underlying maps being equivalences.

Theorem 3.5. *Let h be a graph homomorphism given by the pair-function (α, β). The claim h is an* isomorphism*, denoted by $\mathsf{isIso}(h)$, is a proposition equivalent to stating that the functions α and $\beta(x,y)$ for all $x, y : \mathsf{N}_G$, are all bijections:*

$$\mathsf{isIso}(h) :\equiv \mathrm{isEquiv}\,(\alpha) \times \prod_{(x,y:\mathsf{N}_G)} \mathrm{isEquiv}\,(\beta(x,y)).$$

The type of all isomorphisms between G and H is denoted by $G \cong H$ and defined as:

$$G \cong H :\equiv \sum_{(h:\mathsf{Hom}(G,H))} \mathsf{isIso}(h), \tag{6}$$

or equivalently, as the following type,

$$\sum_{(\alpha:\mathsf{N}_G \simeq \mathsf{N}_H)} \prod_{(x,y:\mathsf{N}_G)} \mathsf{E}_G(x,y) \simeq \mathsf{E}_H(\alpha(x), \alpha(y)). \tag{7}$$

If the type $G \cong H$ is inhabited, it is said that G and H are *isomorphic*.

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Theorem 3.6. *The type $G \cong H$ forms a set.*

Proof. Given $G \cong H$ as a subtype of $\mathsf{Hom}(G, H)$, and by Theorem 3.4 asserting that $\mathsf{Hom}(G, H)$ is a set, it immediately follows from (6) that $G \cong H$ inherits the set structure. □

We define a type to compare the sameness in graphs in Theorem 3.5; the type of graph isomorphisms. In HoTT, the identity type (=) serves the same purpose, and one expects the two notions to coincide (Coquand and Danielsson 2013). In Theorem 3.7, we prove that they are, in fact, homotopy equivalent. The same correspondence for graphs also arises for many other structures, for example, groups and topological spaces (Ahrens and North 2019; Ahrens et al. 2020).

Theorem 3.7 (Equivalence principle). *The canonical map*

$$\mathsf{idtoiso} : (G = H) \to (G \cong H)$$

is an equivalence and its inverse function is denoted by $\mathsf{isotoid}$.

Proof. It is sufficient to show that $(G = H) \simeq (G \cong H)$. Remember that being an equivalence for a function constitutes a proposition. We consider the following type families to shorten the presentation:

- ▷ $F_1(X) :\equiv X \to X \to \mathcal{U}$ and
- ▷ $F_2(X, R) :\equiv \Pi_{x,y:X} \mathsf{isSet}(R(x, y))$ where R is of type $F_1(X)$.

The required equivalence follows from the calculation below in (8):

$$(G = H) \equiv \tag{8a}$$

$$((\mathsf{N}_G, \mathsf{E}_G, (p_G, q_G)) = (\mathsf{N}_H, \mathsf{E}_H, (s_H, t_H))) \simeq \tag{8b}$$

$$\sum_{(\alpha:\mathsf{N}_G=\mathsf{N}_H)} \sum_{(\beta:\mathsf{tr}^{F_1}(\alpha,\mathsf{E}_G)=\mathsf{E}_H)} (\mathsf{tr}^{\mathsf{isSet}}(\alpha, p_G) = s_H) \times (\mathsf{tr}_2^{F_2}(\alpha, \beta, q_G) = t_H) \simeq \tag{8c}$$

$$\sum_{(\alpha:\mathsf{N}_G=\mathsf{N}_H)} \sum_{(\beta:\mathsf{tr}^{F_1}(\alpha,\mathsf{E}_G)=\mathsf{E}_H)} \mathbb{1} \times \mathbb{1} \simeq \tag{8d}$$

$$\sum_{(\alpha:\mathsf{N}_G=\mathsf{N}_H)} \mathsf{tr}^{F_1}(\alpha, \mathsf{E}_G) = \mathsf{E}_H \simeq \tag{8e}$$

$$\sum_{(\alpha:\mathsf{N}_G=\mathsf{N}_H)} \prod_{(x,y:\mathsf{N}_G)} \mathsf{E}_G(x, y) = \mathsf{E}_H(\mathsf{coe}(\alpha)(x), \mathsf{coe}(\alpha)(y)) \simeq \tag{8f}$$

$$\sum_{(\alpha:\mathsf{N}_G\simeq\mathsf{N}_H)} \prod_{(x,y:\mathsf{N}_G)} \mathsf{E}_G(x, y) \simeq \mathsf{E}_H(\alpha(x), \alpha(y)) \simeq \tag{8g}$$

$$(G \cong H). \tag{8h}$$

We first unfold definitions in (8b). The equivalence in (8c) follows from the characterisation of the identity type between pairs in a Σ-type (Lemma 3.7 in HoTT book). The equivalence in (8d) stems from the fact that being a set is a mere proposition and, thus, equations between proofs of such are contractible, similarly as in (2.16). To get (8f), we apply function extensionality twice in the inner equality in (8e). By the UA, we replace in (8g) equalities by equivalences. Finally, (8h) follows from (3.5) completing the calculation from which the conclusion follows. □

Theorem 3.8. *The type of graphs is a groupoid.*

Proof. Consider graphs G and H. We want to show that the identity type $G = H$ is a set, for which we apply Theorem 3.7. This yields an equivalence between the type $G = H$ and the set of isomorphisms $G \cong H$ (refer to Theorem 3.6). Since equivalences preserve set structures, it follows that $G = H$ is indeed a set. □

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3.3 Subtypes and structures on graphs

In graph theory, graphs are often classified according to their structure in different *graph classes*. This can be mirrored in type theory by considering type families over the type Graph. These type families result in a subtype of graphs if they are propositions; otherwise, they might provide a structure on graphs.

A notable example of such a structure is our characterisation of *planar* graphs. We define a type family Planar over Graph and establish that Planar(G) is a set, not a proposition, for any graph G. Here are some informal examples of graph subtypes that one can define in type theory.

- ▷ *Simple* graphs: The edge relation is propositional.
- ▷ *Undirected* graphs: The edge relation is symmetric.
- ▷ *Connected* graphs: A walk exists between any two nodes.
- ▷ *Complete* graphs: Each node is connected to every other node by an edge.
- ▷ *Trees*: These are connected graphs without *cycles*.
- ▷ *Regular* graphs: Each node has the same number of connected edges.
- ▷ *Bipartite graphs*: Nodes can be split into two disjoint sets with all edges connecting a node in one set to a node in the other.

In HoTT, constructions preserve the structure of their constituents; thus, graph subtypes are stable under isomorphisms. Theorem 3.7 enables property transport across isomorphic graphs, affirming that they share any property – a manifestation of the *Leibniz principle* for graphs. For further discussion on a related principle, *equivalence induction*, see Escardó (2019, Section 3.15).

3.4 Finite graphs

A graph is *finite* if its node set and each edge set are finite sets, as stated in Definition 3.9. Like finite types, a finite graph has an associated cardinal number for the count of nodes and edges. Hence, we can demonstrate that equality is decidable on both the node set and each edge set for finite graphs.

Definition 3.9. *A graph G is said to be* finite *when the following proposition* isFiniteGraph(G) *holds.*

$$\mathsf{isFiniteGraph}(G) :\equiv \mathsf{isFinite}(\mathsf{N}_G) \times \mathsf{isFinite}\left(\sum_{(x,y\,:\,\mathsf{N}_G)} \mathsf{E}_G(x,y)\right).$$

For a finite graph G, the cardinality of the node set and edge set are represented as $\#\mathsf{N}_G$ and $\#\mathsf{E}_G$, respectively.

3.5 Walks and strongly connected graphs

A graph G is considered to be *strongly connected* or (*connected* for short) when for any pair of nodes x and y, there is a walk from x to y in G. Intuitively, a *walk* in a graph is a sequence of edges that forms a chain, of the type stated in Definition 3.10.

Definition 3.10. *A* walk *in G from x to y is a sequence of connected edges that we construct using the following inductive data type:*

$$\begin{aligned} \textbf{\textit{data}}\ \mathsf{W} &: \mathsf{N}_G \to \mathsf{N}_G \to \mathcal{U} \\ \langle\_\rangle &: (x : \mathsf{N}_G) \to \mathsf{W}(x,x) \\ (\_\odot\_) &: \Pi\,\{x\,y\,z : \mathsf{N}_G\}\,.\,\mathsf{E}_G(x,y) \to \mathsf{W}(y,z) \to \mathsf{W}(x,z). \end{aligned}$$

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Consider w as a walk from x to y, that is, a term of type $\mathsf{W}_G(x, y)$. Here, x and y are the *head* and *end* of w, respectively. A *trivial* or *one-point* walk is denoted by $\langle x \rangle$. If w takes the form $(e \odot \langle x \rangle)$, it represents a *one-edge* walk e. Walks of the form $(e \odot w)$ are non-trivial, and a *loop* signifies a walk with identical head and end. The notion of walk can also be understood as a path, as suggested in Remark 3.14.

Theorem 3.11. *The type of walks for any graph forms a set.*

Proof. Consider the type of walks $\mathsf{W}(x, y)$ for any graph G and nodes x and y. One can show that such a type is equivalent to $\Sigma_{n:\mathbb{N}} \hat{W}(n, x, y)$ with $\hat{W}$ defined as follows:

$$\hat{W} : \mathbb{N} \to \mathsf{N}_G \to \mathsf{N}_G \to \mathcal{U}. \tag{9a}$$

$$\hat{W}(0, x, y) :\equiv (x = y). \tag{9b}$$

$$\hat{W}(S(n), x, y) :\equiv \sum_{(k:\mathsf{N}_G)} \mathsf{E}_G(x, k) \times \hat{W}(n, k, y). \tag{9c}$$

It suffices to show that the type $\hat{W}(n, x, y)$ forms a set for $n : \mathbb{N}$, which will be proven by induction on n. If $n = 0$, one obtains the proposition $x = y$, which is a set. Consequently, we must now show that the type in (9c) is a set. By the graph definition, the base type N_G and E_G are both sets. Thus, one only requires that $\hat{W}(n, k, y)$ forms a set, which is precisely the induction hypothesis. □

Definition 3.12. *A graph G is said to be connected when the proposition* $\mathsf{Connected}(G)$ *holds.*

$$\mathsf{Connected}(G) :\equiv \prod_{(x,y\,:\,\mathsf{N}_G)} \|\mathsf{E}_{W(G)}(x, y)\|.$$

3.6 Graph families

Let us define some graph families indexed by the type of natural numbers.

Definition 3.13. *The* path graph *with n nodes is the non-connected graph P_n, defined as:*

$$P_n :\equiv (\llbracket n \rrbracket, \lambda\, u\, v.\mathsf{toNat}(u) + 1 = \mathsf{toNat}(v)),$$

where

$$\mathsf{toNat} : \llbracket n \rrbracket \to \mathbb{N}.$$
$$\mathsf{toNat}\,(k, !) :\equiv k.$$

The length of path graph P_n is defined as the number of edges in P_n. Graphs P_0 and P_1 have zero length, and P_2 has one edge. Therefore, for $n > 0$, P_n has length $n - 1$.

Remark 3.14. The path graph definition allows us to alternatively define graph walks. Specifically, a walk in a connected graph G of length n between nodes a and b can be defined as a graph homomorphism from P_{n+1} to G for $n > 0$. This homomorphism maps node 0 to a and n to b. A trivial walk is a graph homomorphism from P_1 to G, selecting only one node a in G. If a equals b, the walk is *closed*. Closed walks, also known as cycles, are introduced using an alternative definition in Definition 3.18 that reflects cyclic types.

Definition 3.15. *An n-cycle graph denoted by C_n is a graph with n edges defined as:*

$$C_n :\equiv (\llbracket n \rrbracket, \lambda\, u\, v.u = \mathsf{pred}(v)),$$

when $n \geq 1$. Otherwise, C_0 is the one-point graph with one trivial loop. The function pred *is defined in Definition 2.10. Similarly to path graphs, the length of an n-cycle graph is n.*

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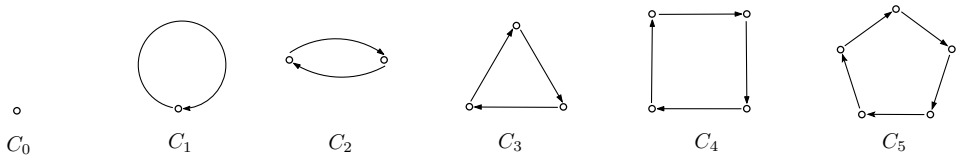
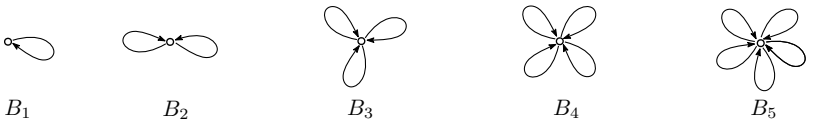


In the treatment of embeddings of graphs on surfaces, we found that bouquet graphs, besides their simple structure, have non-trivial embeddings.

Definition 3.16. *The family of* bouquet *graphs B_n, given by:*

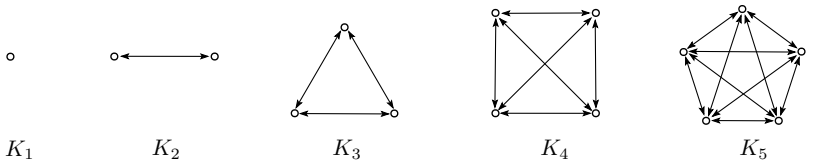
$$B_n :\equiv (\mathbb{1}, \lambda\, u\, v.[\![n]\!]),$$

consists of graphs obtained by considering a single point with n self-loops.



Definition 3.17. *A graph of n nodes is called* complete *when every pair of distinct nodes is joined by an edge. The complete standard graph with node set $[\![n]\!]$ is denoted by K_n:*

$$K_n :\equiv ([\![n]\!], \lambda\, u\, v.u \neq v).$$



For brevity, we will use a double arrow in the pictures from now on to denote a pair of edges in opposite directions.

3.7 Cyclic graphs

Similarly, as for cyclic types, we introduce a type of graphs with a cyclic structure. A graph is *cyclic* when it is in the connected component of an n-cycle graph in the Graph type.

Let us consider the homomorphism $\mathsf{rot} : \mathsf{Hom}(C_n, C_n)$ that acts similarly as the function pred in Definition 2.11. The homomorphism rot is an isomorphism on C_n, and then we can iterate it k times to obtain the isomorphism denoted by rot^k. Any of these isomorphisms can be used to define what it means for a graph to be cyclic.

In particular, the cyclic structure for graphs can be defined as the property of preserving the structure in C_n induced by the morphism rot. We will make use of the same notation as for cyclic sets to refer to cyclic graphs.

Definition 3.18. *A graph G is considered to be* cyclic *if the type* $\mathsf{CyclicGraph}(G)$ *is inhabited:*

$$\mathsf{CyclicGraph}(G) :\equiv \sum_{(\varphi\,:\,\mathsf{Hom}(G,G))} \sum_{(n\,:\,\mathbb{N})} \mathsf{isCyclic}(G, \varphi, n),$$

where $\mathsf{isCyclic}(G, \varphi, n) :\equiv \|(G, \varphi) = (C_n, \mathsf{rot})\|$.

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3.8 The identity type on graphs

For any element, x of a groupoid type, X, the type $\mathsf{Aut}_X(x) :\equiv (x = x)$ has a group structure given by reflexivity, symmetry and path composition. Applying this definition to the groupoid of graphs, the equivalence principle of Theorem 3.7 gives that for any graph G, we identify $\mathsf{Aut}(G)$ with its automorphisms, $G \cong G$. This allows us to compute $\mathsf{Aut}(G) :\equiv G \cong G$ in the examples below:

(1) $\mathsf{Aut}(B_2)$ is the group of two elements. With only two edges in B_2 and one node, we can only have, besides the identity function, the function that swaps the two edges. In general, the identity type $B_n = B_n$ is equivalent to the group S_n, the group which contains the permutations of n elements.
(2) Any isomorphism in $\mathsf{Aut}(C_n)$ is completely determined by how it acts on a fixed node in C_n, stated in the following.

Theorem 3.19. *Let $n : \mathbb{N}$. If $n > 0$, then there exists an equivalence between the type $\mathsf{Aut}(C_n)$ and the type $[\![n]\!]$.*

Proof. The result follows from considering the isomorphism rot as introduced in Definition 3.18 and the isomorphisms rot^k for $k < n$. The equivalence between the type $[\![n]\!]$ and the collection of isomorphisms $C_n \cong C_n$ is then given by the following function f and its inverse g.

$$f : [\![n]\!] \to (C_n \cong C_n). \qquad g : (C_n \cong C_n) \to [\![n]\!].$$
$$f(k, !) :\equiv (\mathsf{rot}^k, p). \qquad g(h, !) :\equiv (r, s).$$

The term p used to define f is the proof that rot^k is an isomorphism. The term r is the solution to the equation $\mathsf{rot}^r = h$, and s is the proof that $r < n$. Now, since $[\![n]\!]$ is a set, we obtain a homotopy $g \circ f \sim \mathsf{id}_{[\![n]\!]}$. The other homotopy condition, that is, $f \circ g \sim \mathsf{id}_{(C_n \cong C_n)}$, can be derived from the intermediate result, stating that if $\mathsf{rot}^p = \mathsf{rot}^q$ and $p, q < n$, then $p = q$. □

The family of graphs C_n is presented intentionally, serving as a crucial component in defining the type of faces of a combinatorial map, referenced in Section 5. The previous result contributes to the proof that the type of faces of a given map for a graph forms a set, elaborated in Theorem 5.7.

4. Graph Maps

We explore the use of graph maps as an alternative approach to directly working with surfaces on which graphs are embedded. Our aim is to characterise graphs with no edge crossing in the two-dimensional plane without needing to represent the surface explicitly. This is motivated by the fact that the concept of surface is not well defined in HoTT, and for our purpose, working with real numbers can be laborious, as discussed in Yamamoto et al. (1995).

To avoid the complexities associated with the explicit notion of the surface in type theory, we focus on representing the drawings of graphs in a more abstract way, which is defining the type of graph maps, also called cellular embeddings, using their combinatorial characterisation (Stahl 1978). By leveraging the power of combinatorial representation of graph maps, we provide a more comprehensive framework for analysing graph planarity, rather than focusing exclusively on the geometric properties and how two-edges cross in the plane, which can be more challenging to study.

4.1 Symmetrisation of graphs

Here, we introduce the symmetrisation construction which allows us to establish two key concepts related to graph maps, stars and faces. The symmetrisation of a graph G, denoted by $\mathsf{Sym}(G)$, is

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one solution used here to encode how the edges are oriented in a graph map. This construction is similar to the concept of *half-edges* for signed rotation maps in the literature of embedded undirected graphs (Ellis-Monaghan and Moffatt 2013, Section 1.1.8).

Definition 4.1. *The* symmetrisation *of a graph G is the graph* $\mathsf{Sym}(G)$ *defined as follows:*

$$\begin{aligned}&\mathsf{Sym} : \mathsf{Graph} \to \mathsf{Graph}.\\&\mathsf{Sym}(G) :\equiv (\mathsf{N}_G, \lambda xy.\mathsf{E}_G(x,y) + \mathsf{E}_G(y,x), p_G, r(q_G)),\end{aligned}$$

where r is a proof that the coproduct $\mathsf{E}_G(x,y) + \mathsf{E}_G(y,x)$ is a set using q_G as a proof that $\mathsf{E}_G(x,y)$ is a set for all $x, y : \mathsf{N}_G$.

Every edge $a : \mathsf{E}_G(x,y)$ in G induces two edges in $\mathsf{Sym}(G)$. The first is $\mathsf{inl}(a)$ keeping the same direction as a. This edge is denoted by $\overleftarrow{a}$ for short. The second is $\mathsf{inr}(a)$, which goes in the opposite direction of a. This edge is denoted by $\overrightarrow{a}$ for short. Since the nodes of $\mathsf{Sym}(G)$ are the same as the nodes of G, we will use the same notation for the nodes of both graphs. The following is an immediate consequence of the induced edges in $\mathsf{Sym}(G)$ by the edges in G.

Theorem 4.2. *Consider a graph G. For every walk w in G, we can induce a corresponding walk in the symmetrisation* $\mathsf{Sym}(G)$*, denoted by* $\mathsf{sym}(w)$.

Proof. The function sym in (10) generates the induced walk in $\mathsf{Sym}(G)$ from a walk w in G:

$$\begin{aligned}&\mathsf{sym} : \prod_{(x,y:\mathsf{N}_G)} \mathsf{W}_G(x,y) \to \mathsf{W}_{\mathsf{Sym}(G)}(x,y).\\&\mathsf{sym}(x, \_, \langle x\rangle) :\equiv \langle x\rangle.\\&\mathsf{sym}(x, y, e \odot w) :\equiv \mathsf{inl}(e) \odot \mathsf{sym}(\_, y, w).\end{aligned} \tag{10}$$

□

Theorem 4.3. *The* Sym *operation on a graph G preserves the following properties:*

- ▹ *connectedness of G and*
- ▹ *finiteness of G.*

Proof. Let us begin by proving the first property. Assume that G is connected, and our objective is to show that $\mathsf{Sym}(G)$ is also connected. This can be established by showing the existence of a function of type:

$$\left\| \prod_{(x,y:\mathsf{N}_G)} \mathsf{W}_G(x,y) \right\| \to \left\| \prod_{(x,y:\mathsf{N}_{\mathsf{Sym}(G)})} \mathsf{W}_{\mathsf{Sym}(G)}(x,y) \right\|.$$

Since the fact that G is connected is a proposition, we can construct such a function using the elimination rule for propositional truncation and the function sym defined in Theorem 4.2 when applied to a walk in G. In general, for A and B types, a function of type $A \to B$ can be lifted $\|A\| \to \|B\|$ by similar reasoning.

On the other hand, to prove that $\mathsf{Sym}(G)$ is finite when G is finite, we only need to consider the family of edges in $\mathsf{Sym}(G)$. This family consists of finite coproducts, as it is the coproduct of two finite sets. Furthermore, the set of nodes in $\mathsf{Sym}(G)$ is identical to the set of nodes in G, which is finite by assumption. □

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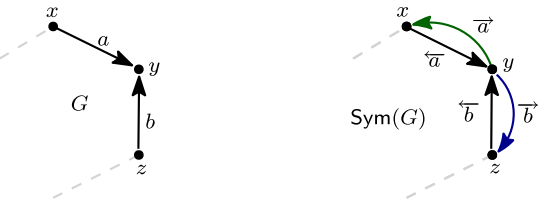

Figure 4. On the left we show a part of a graph G with two distinguished edges, a and b. On the right we show the corresponding symmetrisation, Sym(G), including the two edges, $\overleftarrow{a}$ and $\overrightarrow{a}$ induced by a, and similarly, $\overleftarrow{b}$ and $\overrightarrow{b}$ induced by b. For brevity, we will only draw a segment representing related edges in the symmetrisation, as in Fig. 5(b).

4.2 Stars

Definition 4.4. *Given a node x in a graph G, its* star *is defined as the type $\mathsf{Star}_G(x)$ consisting of all edges incident to x:*

$$\mathsf{Star}_G(x) :\equiv \sum_{(y\,:\,\mathsf{N}_G)} \mathsf{E}_{\mathsf{Sym}(G)}(x, y). \tag{11}$$

Let y be a node in G. If $e : \mathsf{E}_G(x, y)$, then the pair $(y, \mathsf{inl}(e))$ is referred to as an *outgoing edge* in the start at x. Similarly, if $e : \mathsf{E}_G(y, x)$, then the pair $(y, \mathsf{inr}(e))$ is referred to as an *incoming edge* in the start at x. An *incident* edge of x is either an outgoing or an incoming edge in the star at x. The cardinality of the set of incident edges at x is known as the *valency* of x.

Example 4.5. *The graph C_n is a basic example of a planar graph and a building block to construct more complex planar graphs. To enable this construction, we need to characterise the stars at any node in C_n for $n > 0$. The case when n is zero is trivial, as the star at any node in the empty graph is empty.*

As C_n is a graph consisting of n nodes in $[\![n]\!]$ arranged in a polygon/cycle, one can associate the previous and the next node in the cycle, $\mathsf{pred}(x)$ *and* $\mathsf{suc}(x)$*, for each node x in C_n, respectively. We will prove that the valency of any node in C_n is two by proving that there exists an equivalence f_x from $\mathsf{Star}_{C_n}(x)$ to $[\![2]\!]$ for every node x in C_n. The candidate to be the inverse of f_x is the function g_x defined below:*

$$\begin{aligned} &f_x : \mathsf{Star}_{C_n}(x) \to [\![2]\!]. \quad g_x : [\![2]\!] \to \mathsf{Star}_{C_n}(x). \\ &f_x\,(y, \mathsf{inl}(p)) :\equiv (0, !). \quad g_x\,(0, !) :\equiv (\mathsf{suc}(x), \mathsf{inl}(a^+)). \\ &f_x\,(y, \mathsf{inr}(p)) :\equiv (1, !). \quad g_x\,(1, !) :\equiv (\mathsf{pred}(x), \mathsf{inr}(a)). \end{aligned} \tag{12}$$

One can easily prove that both $\mathsf{E}_{C_n}(\mathsf{pred}(x), x)$ and $\mathsf{E}_{C_n}(x, \mathsf{suc}(x))$ are contractible types. Therefore, without loss of generality, we write a^+ to denote the edge from x to $\mathsf{suc}(x)$ *and a to denote the edge from* $\mathsf{pred}(x)$ *to x in C_n.*

To complete the proof that f_x is an equivalence, we need to show that $f_x \circ g_x \sim \mathsf{id}_{[\![2]\!]}$ and $g_x \circ f_x \sim \mathsf{id}_{\mathsf{Star}_{C_n}(x)}$. The first is immediate by case analysis. For example, $(f_x \circ g_x)((0, !)) \equiv f_x(g_x((0, !))) \equiv f_x(\mathsf{suc}(x), \mathsf{inl}(p)) \equiv (0, !)$, and one can similarly show that $f_x \circ g_x((1, !)) = (1, !)$.

To prove the second part, we show that $g_x \circ f_x \sim \mathsf{id}_{\mathsf{Star}C_n(x)}$ by performing a case analysis on the second component of a term $(y, z) : \mathsf{Star}_{C_n}(x)$. Specifically, we consider whether z is either $\mathsf{inl}(u)$ *or* $\mathsf{inr}(v)$*. For the first case, we need to prove that $g_x(f_x((y, \mathsf{inl}(u)))) = (y, \mathsf{inl}(u))$. Evaluating the expression of the composite, we obtain an equality with the question mark below, which we need to show one can inhabit:*

$$g_x(f_x((y, \mathsf{inl}(u)))) \equiv g_x((0, !)) \equiv (\mathsf{suc}(x), \mathsf{inl}(a^+)) \stackrel{?}{=} (y, \mathsf{inl}(u)).$$

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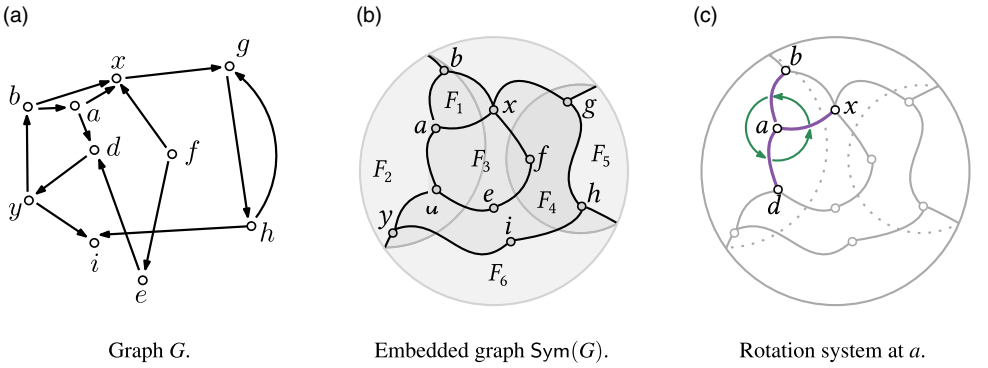

Figure 5. We show in (a) the drawing of a graph G with edge crossings. A representation of the graph G embedded in the sphere is shown in (b). The corresponding faces of the graph map shaded in (b) are named F_i for i from 1 to 6. It is shown in (c) with fuchsia colour the incident edges at the node a in $\mathsf{Sym}(G)$. The rotation system at a, that is, the cyclic set denoted by $(ba\ ad\ ax)$, is shown in green colour. The dashed lines represent edges not visible to the view.

However, we can establish the required equality by noting that $\mathsf{E}_{C_n}(x, \mathsf{suc}(x))$ *is contractible. This implies that* $\mathsf{E}_{\mathsf{Sym}(C_n)}(x, \mathsf{suc}(x))$ *is a proposition, which in turn implies that* $a^+ = u$ *and that we have* $y = \mathsf{suc}(x)$*. Similarly, we can show that* $g_x(f_x((y, \mathsf{inr}(v)))) = (y, \mathsf{inr}(v))$*. This completes the proof that* f_x *is an equivalence and shows that* $\mathsf{Star}_{C_n}(x)$ *has only two elements.*

Theorem 4.6. *If G is a (finite) graph, then the type* $\mathsf{Star}_G(x)$ *is a (finite) set.*

Proof. The conclusion follows since the base type in Definition 4.4 is the set of edges in the graph, and each of the fibres of the Σ-type is a set since they are coproducts of sets. In particular, if the graph is finite, then all the types appearing in the type $\mathsf{Star}_G(x)$ are finite sets, and then our conclusion follows. □

4.3 The type of graph maps

A combinatorial map is a specific type of data structure that is used to represent a graph that is embedded in a surface. This data structure offers a powerful substitute for traditional analytic/geometric techniques for representing such embeddings. Unlike geometric methods, combinatorial maps allow us to represent the combinatorial structure of the topological embedding without the need to explicitly work with the surface in which the graph is embedded.

In this work, we focus on defining the type of combinatorial maps in type theory; see Definition 4.7. We then turn our attention to a particular kind of embedding, *cellular* embeddings. The reason for this focus is that all graph maps in the two-dimensional plane are cellular embeddings. Therefore, drawing graphs in the plane without edge crossings can be represented by cellular embeddings.

Cellular embeddings are particularly interesting because they can be characterised combinatorially up to isotopy by the cyclic order they induce in the set of nodes around each node in the graph (Gross and Tucker 1987), as illustrated in Fig. 5(b). This characterisation is minimal, as no additional information is required beyond the cyclic orders.

One observation is that not all finite graphs can be drawn in the plane, but all finite graphs can be drawn on some orientable surface (Stahl 1978). The literature in graph theory has proven that a graph cannot have a cellular embedding on any surface if it has at least one node of infinite valency (Mohar 1988, Proposition Section S3.2). As our focus is on cellular embeddings, we will only examine locally finite graphs throughout the document.

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Definition 4.7. $\mathsf{Map}(G)$ *is the type of combinatorial maps (maps for short) for a graph* G *defined as follows:*

$$\mathsf{Map}(G) :\equiv \prod_{(x\,:\,\mathsf{N}_G)} \mathsf{Cyclic}(\mathsf{Star}_G(x)).$$

Definition 4.8. *A graph* G *is* locally finite *if the set of* incident *edges at the* star *at any node* x *in* G*, is a finite set.*

Theorem 4.9. *If the type* $\mathsf{Map}(G)$ *is inhabited, then the graph* G *is locally finite.*

Proof. A map of G provides each node a cyclic order on its star. Since these orders are finite by Theorem 2.17, the local finiteness of G follows. □

Theorem 4.10. *The type of maps for a (finite) graph forms a (finite) set.*

Proof. The type $\mathsf{Map}(G)$ is a set using the closure property of Π-types under (finite) sets. The type $\mathsf{Cyclic}(\mathsf{Star}_G(x))$ is a finite set by Theorem 2.17. □

For brevity, we use from now the variable $\mathscr{M}$ to denote a map of the graph G.

Example 4.11. *The possible maps for the cycle* C_n *for* $n > 0$ *can be listed considering the cyclic structures of the two-point type. These correspond to the cyclic structures of the stars of* C_n*, see the correspondence exhibited in Example 4.5. The two maps are given by the following functions:*

- $c_1 :\equiv \langle [\![2]\!], \mathsf{pred}, 2\rangle$ *and*
- $c_2 :\equiv \langle [\![2]\!], \mathsf{suc}, 2\rangle$.

5. The Type of Faces

In the context of cellular embeddings, faces correspond to regions homeomorphic, to the open disk. Combinatorially, a face associated with a graph map consists of a cyclic walk in the embedded graph where no edges are inside the cycle, and no node occurs twice. Definition 5.3 is our attempt to make this intuition formal.

The first component of a face, as in Definition 5.3, captures the concept that its edges form a cyclic walk in the embedded graph. While working with such walks would typically necessitate a fixed starting point, this point does not contribute to the face's combinatorial structure. Hence, we can employ a cyclic graph to represent all such cyclic walks, thereby obviating the need for any distinguished starting point in such walks.

The second component, the *map-compatibility* property, explicitly defines the 'no edges on the inside' criterion for a face. This criterion is captured by the fact that each pair of consecutive edges on the face is a *successor-predecessor* pair in the cyclic order of the edges around their common node. In other words, when we move along the edges of the face either clockwise or counterclockwise, we will never come across an edge that goes through the inside of the face. As our graphs are directed, we must traverse the edges in the symmetrisation of the graph rather than the graph itself.

The following two definitions are used in the definition of the type of faces.

Definition 5.1. *A graph homomorphism* h *from* G *to* H *given by* (α, β) *is* edge-injective*, denoted by* $\mathsf{isEdgeInj}(h)$*, if the function* f *defined below is an embedding:*

$$f : \sum_{(x,y\,:\,\mathsf{N}_G)} \mathsf{E}_G(x,y) \to \sum_{(x,y\,:\,\mathsf{N}_H)} \mathsf{E}_H(x,y).$$
$$f\ (x,y,e) :\equiv (\alpha(x), \alpha(y), \beta(x,y,e)).$$

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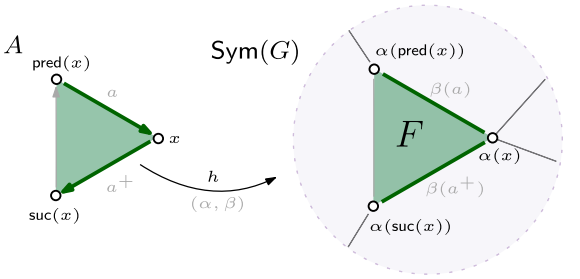

Figure 6. On the right side, we shade the face F of the graph G embedded in the sphere given in Fig. 5. We have the cycle graph C_3 and $h : \mathsf{Hom}(C_3, \mathsf{Sym}(G))$ given by (α, β) on the left side. C_3 and h can be used to define the face F using C_3 as the graph A in Definition 5.3.

Definition 5.2. *The function* flip *changes the direction of an edge in* $\mathsf{Sym}(G)$*:*

$$\begin{aligned}
&\mathsf{flip} : \prod_{(x,y:\mathsf{N}_G)} \mathsf{E}_{\mathsf{Sym}(G)}(x,y) \to \mathsf{E}_{\mathsf{Sym}(G)}(y,x).\\
&\mathsf{flip}\,(x,y,\mathsf{inl}(e)) :\equiv \mathsf{inr}(e).\\
&\mathsf{flip}\,(x,y,\mathsf{inr}(e)) :\equiv \mathsf{inl}(e).
\end{aligned}$$

Since the first two arguments of the function flip *are inferrable from the third argument, we will omit them below.*

Definition 5.3. *The type* $\mathsf{Face}(G, \mathscr{M})$ *is the type of faces of a combinatorial map* $\mathscr{M}$ *of a graph* G*. A face of type* $\mathsf{Face}(G, \mathscr{M})$ *consists of:*

(1) a cyclic graph A*,*
(2) a graph homomorphism h *given by* (α, β) *of type* $\mathsf{Hom}(A, \mathsf{Sym}(G))$*, such that*
  a. h *is* edge-injective*,*
  b. h *is* map-compatible*, denoted by* $\mathsf{isMapComp}(h)$*, meaning that* h *is star-compatible and corner-preserving, properties defined below, respectively.*
    ⋆ h *is* star-compatible*, if the condition in* (13) *holds for every* $x : \mathsf{N}_A$*,*

$$\mathsf{isStarComp}(h)(x) :\equiv \|\mathsf{Star}_G(\alpha(x))\| \to \|\mathsf{Star}_A(x)\|. \tag{13}$$

    ⋆ h *is* corner-compatible*, if there is evidence that* h *is compatible with the edge-ordering given by the map* $\mathscr{M}$ *at the node* $\alpha(x)$ *and the edge-ordering coming from the star at that node* x *in* A*. To state this property, let us consider the following notation.*
      ⋆ *The* previous edge *at* x *is the edge* $a : \mathsf{E}_{\mathsf{N}_A}(\mathsf{pred}(x), x)$*,*
      ⋆ *the edge* after a_x *is the edge denoted by* a_x^+
      of type $\mathsf{E}_{\mathsf{N}_A}(x, \mathsf{suc}(x))$*, as illustrated in Fig. 6, and*
      ⋆ *since* $\mathscr{M}(\alpha(x))$ *is a triple like* $\langle f, m, ! \rangle$ *of type*

$$\mathsf{Cyclic}(\mathsf{Star}_G(\alpha(x)))$$

    for some function $f : \mathsf{Star}_G(\alpha(x)) \to \mathsf{Star}_G(\alpha(x))$ *and some number* m *(the cardinality of the star at* $\alpha(x)$*), we abuse notation and use* $\mathscr{M}(\alpha(x))$ *to denote the function* f*. See more on the cyclic type in Definition 2.11:*

$$\begin{aligned}
\mathsf{isCornerComp}(h)(x) :\equiv &\mathscr{M}(\alpha(x))((\alpha(\mathsf{pred}(x)), \mathsf{flip}(\beta(\mathsf{pred}(x), x, a))))\\
&=_{\mathsf{Star}_G(\alpha(x))} (\,\alpha(\mathsf{suc}(x))\,,\ \beta(x, \mathsf{suc}(x), a^+)\,).
\end{aligned} \tag{14}$$

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It should be noted that the truncation in (13) is intentional. By incorporating this, we aim to emphasise that if the graph G has at least one edge at a given node, then a face covering that node, represented by the cyclic graph A, must have at least one edge at the corresponding node as well. Without this condition, the type of faces could be inhabited with *empty faces* using A as the cyclic graph without edges (C_0) at every node of the graph G. In Fig. 6, we illustrate a portion of the required data to define a face F_1 for the map of graph G given in Fig. 5(b).

Theorem 5.4. *For a graph homomorphism, being edge-injective is a proposition.*

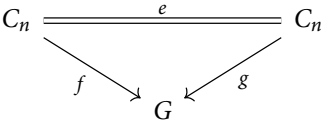
Proof. Edge-injectivity is a proposition by iteratively applying the closure of Π-types to propositions. Ultimately, we need to show that for any two terms (x, y, e_1) and (x', y', e_2) in $\Sigma_{x,y\,:\,\mathsf{N}_G}\mathsf{E}_G(x, y)$, the identity type $(x, y, e_1) = (x', y', e_2)$ is a proposition. This is true because the Σ-type in question is a set, and sets are closed under Σ-types, given that both N_G and $\mathsf{E}_G(x, y)$ are sets. □

Theorem 5.5. *For a graph homomorphism, being map-compatible is a proposition.*

Proof. For a graph homomorphism h, map-compatibility decomposes into star-compatibility and corner-compatibility. We must show each type in this product is a proposition. Star-compatibility is a proposition as it involves a function type with a propositional codomain – the propositional truncation of a set. Corner-compatibility is also a proposition, being a function type whose codomain is the identity type on $\mathsf{Star}_G(\alpha(x))$ at $\alpha(x)$. This identity type is a proposition since stars are sets, as established in Theorem 4.6. □

We devote the rest of this section to proving that the type of faces forms a set in Theorem 5.7. This claim rests on the fact that (i) the type of cyclic graphs forms a set, (ii) the type of graph homomorphisms forms a set, and (iii) the conditions, edge-injective and map-compatible in, Definition 5.3 are propositions. One might suspect that this type forms a groupoid based on previous facts. However, the edge-injectivity property of the underlying graph homomorphism of each face suffices to show that the type of faces is a set.

Theorem 5.6. *Let f and g be edge-injective graph homomorphisms from C_n to a graph G and $n > 0$. Then the type $\Sigma_{e:C_n=C_n}\,(\mathsf{tr}^{\lambda X.\mathsf{Hom}(X,G)}(e, f) = g)$ is a proposition.*



Proof. The result follows from the proof that the Σ-type in question is equivalent to a proposition. The corresponding equivalence is given in (15), in which we use some known results about Univalence and Theorem 3.19, as in the very last step:

$$\sum_{(e\,:\,C_n\,=\,C_n)} (\mathsf{tr}^{\lambda X.\mathsf{Hom}(X,G)}(e, f) = g) \simeq \sum_{(e\,:\,C_n\,=\,C_n)} (f = g \circ \mathsf{coe}\,(e)\,) \tag{15a}$$

$$\simeq \sum_{(e\,:\,C_n\,\simeq\,C_n)} (f = g \circ e) \tag{15b}$$

$$\simeq \sum_{(k\,:\,[\![n]\!])} (f = g \circ \mathsf{rot}^k). \tag{15c}$$

It remains to show that the last equivalent type is a proposition. Let (k_1, p_1) and (k_2, p_2) be of type $\Sigma_{k:[\![n]\!]}(f = g \circ \mathsf{rot}^k)$. We must show that (k_1, p_1) is equal to (k_2, p_2). Since $\mathsf{Hom}(C_n, G)$ is a set, we only need to prove that k_1 is equal to k_2. To show that, Theorem 3.19 is used in the

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proof. By computing the identity type of graph isomorphisms, we obtain that $p_1^{-1} \cdot p_2$ of type $g \circ \mathsf{rot}^{k_1} = g \circ \mathsf{rot}^{k_2}$ is equivalent to having two equalities:

- $p : \pi_1(g \circ \mathsf{rot}^{k_1}) = \pi_1(g \circ \mathsf{rot}^{k_2})$ and
- $q : \mathsf{tr}^{\lambda e.\, \prod_{x,y\,:\,\mathsf{N}_{C_n}} \mathsf{E}_{C_n}(x,y) \to \mathsf{E}_G(e(x),e(y))}(,)p\pi_2(g \circ \mathsf{rot}^{k_1}) = \pi_2(g \circ \mathsf{rot}^{k_2})$.

By characterising the identity of the Σ-types and with the previous equalities, p and q, one can get another equality r of the type in (16) for $x, y : \mathsf{N}_{C_n}$ and $e : \mathsf{E}_{C_n}(x, y)$:

$$\begin{aligned}&((\pi_1(g \circ \mathsf{rot}^{k_i}))(x), \quad (\pi_1(g \circ \mathsf{rot}^{k_i}))(y), \quad (\pi_2(g \circ \mathsf{rot}^{k_i}))(x, y, e)) = \\ &((((\pi_1(g))(\pi_1(\mathsf{rot}^{k_i})))(x)), (((\pi_1(g))(\pi_1(\mathsf{rot}^{k_i})))(y)), (((\pi_2(g))(\pi_2(\mathsf{rot}^{k_i})))(x, y, e))).\end{aligned} \tag{16}$$

Now since the graph homomorphism g is edge-injective, applying Definition 5.1 to the equality r, one gets an equality r' of the type below in (17). By applying Theorem 3.19 to r', we conclude that k_1 is equal to k_2 from which the required conclusion follows:

$$\begin{aligned}&((\pi_1(\mathsf{rot}^{k_1}))(x), (\pi_1(\mathsf{rot}^{k_1}))(y), (\pi_2(\mathsf{rot}^{k_1}))(x, y, e)) = \\ &((\pi_1(\mathsf{rot}^{k_2}))(x), (\pi_1(\mathsf{rot}^{k_2}))(y), (\pi_2(\mathsf{rot}^{k_2}))(x, y, e)).\end{aligned} \tag{17}$$

□

Theorem 5.7. *The type of faces for a graph map forms a set.*

Proof. Let F_1 and F_2 be two faces of a map $\mathscr{M}$. We will show that the type $F_1 = F_2$ is a proposition in (18), with the following conventions.

- $\mathscr{A}$ is the cyclic graph related to the face F_1:
$$\mathscr{A} :\equiv (A, (\varphi_A, n, \mathsf{isCyclic}(A, \varphi_A, n))).$$
- $\mathscr{B}$ is the cyclic graph related to the face F_2:
$$\mathscr{B} :\equiv (B, (\varphi_B, m, \mathsf{isCyclic}(B, \varphi_B, m))).$$

We first unfold the definitions of F_1 and F_2 in (18a) and simplify the propositions in (18b), namely isEdgeInj, isMapComp and isCyclic. Then, by expanding the definitions of $\mathscr{A}$ and $\mathscr{B}$ in (18c) and simplifying the propositions in terms such as being a cyclic graph, one gets (18d). Next, we reorder in (18d) the tuple equalities to create an opportunity for path induction towards the application of Theorem 5.6. Now, since we want to prove that the type of faces is a set, and that itself is a proposition, the truncation elimination principle is applied to the propositions $\mathsf{isCyclic}(A, \varphi_A, n)$ and $\mathsf{isCyclic}(A, \varphi_A, n)$. Then, the graphs A and B become, respectively, C_n and C_m in (18e). The step in (18f) follows from the characterisation of the identity type between tuples in a nested Σ-type:

$$(F_1 = F_2) \equiv$$
$$((\mathscr{A}, f, \mathsf{isEdgeInj}(f), \mathsf{isMapComp}(f)) = (\mathscr{B}, g, \mathsf{isEdgeInj}(g), \mathsf{isMapComp}(g))) \simeq \tag{18a}$$
$$((\mathscr{A}, f) = (\mathscr{B}, g)) \equiv \tag{18b}$$
$$((A, (\varphi_A, n, \mathsf{isCyclic}(A, \varphi_A, n))), f) = ((B, (\varphi_B, m, \mathsf{isCyclic}(B, \varphi_B, m))), g) \simeq \tag{18c}$$
$$((A, (\varphi_A, n)), f) = ((B, (\varphi_B, m)), g) \simeq \tag{18d}$$
$$((n, ((C_n, f), \varphi_{C_n})) = (m, ((C_m, g), \varphi_{C_m}))) \simeq \tag{18e}$$
$$\sum_{(p:n=m)} \sum_{\left((e',-):\sum_{(e:C_n=C_m)} \mathsf{tr}^{\lambda X.\mathsf{Hom}(X,\mathsf{Sym}(G))}(e,f)=g\right)} \mathsf{tr}^{\lambda X.\mathsf{Hom}(X,X)}(e', \varphi_{C_n}) = \varphi_{C_m}. \tag{18f}$$

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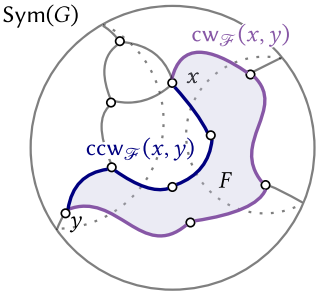

Figure 7. The graph embedding Sym(G), as depicted in Fig. 5, is associated with a face $\mathcal{F}$ defined by $\langle A, f\rangle$. The underlying cyclic graph A contains two highlighted walks between distinct nodes x and y. These walks correspond to clockwise and counterclockwise closed walks in Sym(G), represented as $\mathsf{cw}_{\mathcal{F}}(x, y)$ and $\mathsf{ccw}_{\mathcal{F}}(x, y)$, respectively.

It only remains to show that the type in (18f) is a proposition. We show this by proving that each type in (18f) is a proposition. First, we unfold the cyclic graph definition for C_n and C_m, using Definition 3.18. Second, a case analysis on n and m is performed. This approach creates four cases where n and m can be zero or positive. However, we only keep the cases where n and m are structurally equal. One can show that the other cases are imposible with an equality between n and m.

(1) If n and m are zero, then, by definition, C_n and C_m are the one-point graph. In this case, the conclusion follows easily. The base type $n = m$ of the total space in (18f) is a proposition because $\mathbb{N}$ is a set. The type $C_0 = C_0$ is a proposition, since it is contractible. The identity graph homomorphism is the unique automorphism of C_0. Lastly, because $\mathsf{Hom}(C_n, C_n)$ is a set, the remaining type of the Σ-type is a proposition, completing the proof obligations.
(2) If n and m are positive, we reason similarly. The type $n = m$ is a proposition. By path induction on $p : n = m$, the second base type of the Σ-type becomes the type in (19):

$$\sum_{(e\,:\,C_n = C_n)} (\mathsf{tr}^{\lambda X.\mathsf{Hom}(X,\mathsf{Sym}(G))}(e, f) = g), \tag{19}$$

which is a proposition by Theorem 5.6. The remaining type of the Σ-type is a proposition, because $\mathsf{Hom}(C_n, C_n)$ is a set. Therefore, the Σ-type in (18f) is a proposition as required. □

5.1 The boundary of a face

Each face $\mathcal{F}$ of a map $\mathcal{M}$ consisting of a cyclic graph A, a homomorphism h and some extra data as described in Definition 5.3 induced a closed walk that follows the edges of its defining polygon, which we refer to as its *boundary*.

Definition 5.8. *Let $\mathcal{F}$ be a face for a map of the graph G, the boundary of $\partial\mathcal{F}$ is the subgraph of the image of the associated function, h, given in the definition of the type of $\mathcal{F}$:*

$$\partial\mathcal{F} \equiv \partial((A, (h, -))) :\equiv \mathsf{Img}(h).$$

Here, $\mathsf{Img}(h)$ is the induced subgraph of G by the image of h. More specifically, it is defined as:

$$\mathsf{Img}(h) :\equiv (\Sigma_{x:\mathsf{N}_A}, \pi_1(h)(x), \lambda x.\lambda y.\lambda e.\pi_2(h)(x, y, e)).$$

The *degree* of a face $\mathcal{F}$ is the length of $\partial\mathcal{F}$, which is the number of nodes in A. The boundary $\partial\mathcal{F}$ can be walked in two directions with respect to the orientation given by its map.

As illustrated by Fig. 7, given two different nodes x and y in $\partial\mathcal{F}$, we can connect x to y using the walk in the clockwise direction, $\mathsf{cw}_{\mathcal{F}}(x, y)$. Similarly, one can connect x to y using the walk

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in the counterclockwise direction, $\mathsf{ccw}_{\mathscr{F}}(x, y)$. Such walks are induced by the walks in the cyclic graph A, see Theorem 5.10.

For brevity, we omit the proofs of Theorems 5.9 and 5.10. These lemmas refer to properties inherent in the construction of C_n and its symmetrisation, see Fig. 4. Thus, the proofs are straightforward applications of definitions.

Theorem 5.9. *Supposing $x, y : \mathsf{N}_{C_n}$, the following claims hold for the cycle graph C_n.*

(1) The type $\mathsf{E}_{C_n}(x, y)$ is a proposition.
(2) For $n > 0$, there exists an edge of type $\mathsf{E}_{C_n}(\mathsf{pred}(x), x)$ and an edge of type $\mathsf{E}_{C_n}(x, \mathsf{suc}(x))$.
(3) For $n > 0$, there exists a walk going in the clockwise direction denoted by $\mathsf{cw}_{C_n}(x, y)$ from x to y.

Theorem 5.10. *Supposing $x, y : \mathsf{N}_{C_n}$, the following claims hold for the graph $\mathsf{Sym}(C_n)$.*

(1) If $n > 1$, then the type $\mathsf{E}_{\mathsf{Sym}(C_n)}(x, y)$ is a proposition.
(2) There exists an edge of type $\mathsf{E}_{\mathsf{Sym}(C_n)}(\mathsf{pred}(x), x)$ and of type $\mathsf{E}_{\mathsf{Sym}(C_n)}(x, \mathsf{suc}(x))$.
(3) There exist two walks from x to y in $\mathsf{Sym}(C_n)$, denoted by $\mathsf{cw}_{\mathsf{Sym}(C_n)}(x, y)$ and $\mathsf{ccw}_{\mathsf{Sym}(C_n)}(x, y)$, respectively.
   a. The walk $\mathsf{cw}_{\mathsf{Sym}(C_n)}(x, y)$ represents the walk in the clockwise direction from x to y.
   b. On the other hand, the walk $\mathsf{ccw}_{\mathsf{Sym}(C_n)}(x, y)$ represents the walk in the counterclockwise direction from x to y. In case $x = y$, the walk $\mathsf{ccw}_{\mathsf{Sym}(C_n)}(x, y)$ corresponds to the trivial walk $\langle x \rangle$.

Example 5.11. *For cycle graphs C_n, only one combinatorial map exists. Cyclic structures of two-point type c_1 and c_2, defined in Example 4.5, precisely induce the maps of C_n. In other words, one can obtain a map $\mathscr{M}$ using c_1 by (20) and*

$$(\mathsf{pred}, 2, |(\mathsf{ideqv}, \mathsf{refl}_{\mathsf{pred}})|) : \mathsf{Cyclic}(\llbracket 2 \rrbracket).$$

Moreover, using function extensionality, Theorem 2.16 implies that the map induced by c_2 and the map $\mathscr{M}$ are equal.

$$\begin{aligned} \mathsf{Map}(C_n) &\equiv \prod_{(x:\llbracket n \rrbracket)} \mathsf{Cyclic}(\mathsf{Star}_{C_n}(x)) \\ &\simeq \prod_{(x:\llbracket n \rrbracket)} \mathsf{Cyclic}(\llbracket 2 \rrbracket). \end{aligned} \tag{20}$$

Example 5.12. *Recall that a graph consisting of a single node and n loop edges is referred to as an n-bouquet, denoted by B_n. To enumerate the maps of B_2, we can label the edges of its sole star as ($\overrightarrow{x}$, $\overleftarrow{x}$, $\overrightarrow{y}$, and $\overleftarrow{y}$). It is important to note that reflection is not treated as symmetry here. Consequently, we identify six distinct combinatorial maps for B_2, each distinct cyclic permutation of the set of edges creates a map in this case, as depicted in Fig. 8.*

6. Planar Maps

In this section, we examine the type of graphs with an embedding in the two-dimensional plane. Such embeddings are called *planar embeddings* or *planar maps*. A graph is *planar* if it has a planar map and the graph embedded is called a *plane* graph. To discuss the notion of planar embeddings, we take inspiration from topological graph theory (Gross and Tucker 1987, Section 3). Then one can work with combinatorial maps that represent graph maps into a surface – up to isotopy.

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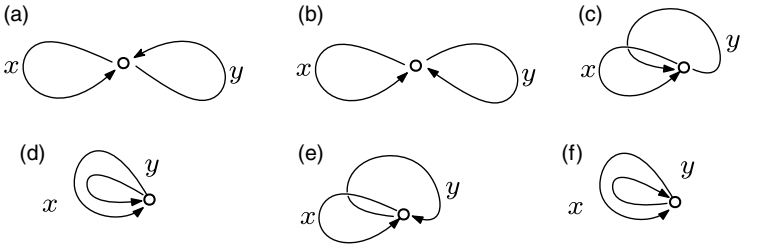

Figure 8. The six possible maps of the bouquet B_2. Respectively, they are denoted and defined as follows: $a :\equiv (\overrightarrow{x}, \overleftarrow{x}, \overrightarrow{y}, \overleftarrow{y})$, $b : (\overrightarrow{x}, \overleftarrow{x}, \overleftarrow{y}, \overrightarrow{y})$, $c : (\overrightarrow{x}, \overrightarrow{y}, \overleftarrow{x}, \overleftarrow{y})$, $d : (\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{y}, \overleftarrow{x})$, $e : (\overrightarrow{x}, \overleftarrow{y}, \overleftarrow{x}, \overrightarrow{y})$ and $f : (\overrightarrow{x}, \overleftarrow{y}, \overrightarrow{y}, \overleftarrow{x})$.

In the following, we focus on describing embeddings of graphs in the sphere called spherical maps. These graph maps are used later to establish the type of planar embeddings for a given graph.

6.1 Spherical maps and homotopy for walks

Any graph map gives rise to an implicit surface. For planar embeddings, this surface is a space homeomorphic to the sphere. In particular, any embedding in the sphere induces an embedding in the plane. To see this, for a graph embedded in the sphere, one can puncture the sphere at some distinguished point, and subsequently, apply the *stereographic projection* to it.

The sphere in topology has two main invariants: path-connectedness and simply-connectedness. The former states that a path connects any pair of points in the sphere, and the latter states that any two paths with the same endpoints in the sphere can be deformed into one another.

If we now consider a walk as a path in the corresponding space induced by the map, then the path-connectedness property coincides with being connected for the graph embedded. However, if we want to address simply connectedness for the surface induced by a graph embedding, then we need to have an equivalent notion to saying how a pair of walks can be deformed into one the other. One proposal of such a notion is *homotopy for walks* in directed multigraphs (Prieto-Cubides 2022*b*).

In this subsection, we present a binary relation, denoted by $(\sim_{\mathcal{M}})$, on the set of walks between fixed endpoints in a graph as in introduced in Prieto-Cubides (2022*b*). This relation is designed to capture the behaviour of walks in an embedded graph in a surface such as the two-dimensional plane, where all the walks can be deformed one into another along the faces of the graph map in use.

Definition 6.1. *Let w_1, w_2 be two walks from x to y in* $\mathsf{Sym}(G)$. *The expression $w_1 \sim_{\mathcal{M}} w_2$ denotes that one can* deform *w_1 into w_2 along the faces of $\mathcal{M}$. We acknowledge evidence of this deformation as a walk homotopy between w_1 and w_2, of type $w_1 \sim_{\mathcal{M}} w_2$.*

The relation $(\sim_{\mathcal{M}})$ has four constructors, as follows. The first three constructors are functions to indicate that homotopy for walks is an equivalence relation; they are hrefl, hsym *and* htrans. *Let $x, y : \mathsf{N}_G$.*

$$\begin{aligned}
\mathsf{hrefl} &: \prod_{(w_1 : \mathsf{W}_{\mathsf{Sym}(G)}(x,y))} w_1 \sim_{\mathcal{M}} w_1.\\
\mathsf{hsym} &: \prod_{(w_1, w_2 : \mathsf{W}_{\mathsf{Sym}(G)}(x,y))} w_1 \sim_{\mathcal{M}} w_2 \to w_2 \sim_{\mathcal{M}} w_1.\\
\mathsf{htrans} &: \prod_{(w_1, w_2, w_3 : \mathsf{W}_{\mathsf{Sym}(G)}(x,y))} w_1 \sim_{\mathcal{M}} w_2 \to w_2 \sim_{\mathcal{M}} w_3 \to w_1 \sim_{\mathcal{M}} w_3.
\end{aligned} \tag{21}$$

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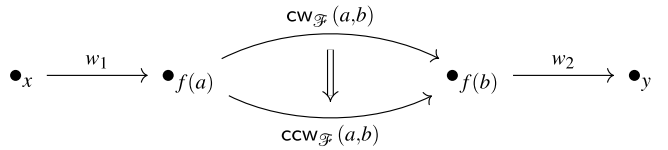


Figure 9. Given a face $\mathscr{F}$ of a map $\mathscr{M}$, we illustrate here hcollapse, one of the four constructors of the homotopy relation on walks in Definition 6.1. The arrow ($\Downarrow$) represents a homotopy of walks.

The fourth constructor, illustrated in Fig. 9, is the hcollapse *function that establishes the walk homotopy:*

$$(w_1 \cdot \mathsf{ccw}_{\mathscr{F}}(a,b) \cdot w_2) \sim_{\mathscr{M}} (w_1 \cdot \mathsf{cw}_{\mathscr{F}}(a,b) \cdot w_2),$$

supposing one has the following,

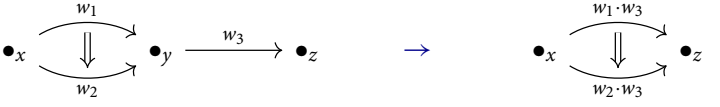
(i) a face $\mathscr{F}$ given by $\langle A, f\rangle$ of the map $\mathscr{M}$,
(ii) a walk w_1 of type $\mathsf{W}_{\mathsf{Sym}(G)}(x, f(a))$ for a node x in G with a node a in A, and
(iii) a walk w_2 of type $\mathsf{W}_{\mathsf{Sym}(G)}(f(b), y)$ for a node b in A with a node y in G.

One consequence of Definition 6.1 is that, in each face $\mathscr{F}$, there is a walk homotopy between $\mathsf{ccw}_{\mathscr{F}}(x,y)$ and $\mathsf{cw}_{\mathscr{F}}(x,y)$ using the constructor hcollapse.

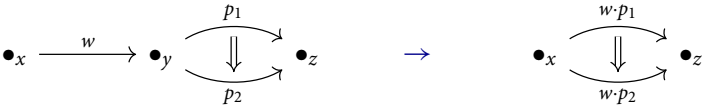
The following shows how to compose walk homotopies horizontally and vertically. We consider a map $\mathscr{M}$ for a graph G and distinguishable nodes, x, y, and z where w, w_1 and w_2 are walks from x to y.

Theorem 6.2. *The following claims hold for the homotopy relation on walks.*

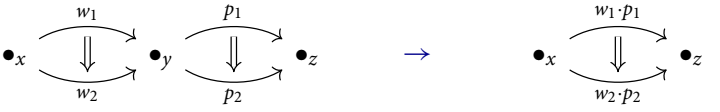
(1) (Right whiskering) Let w_3 be a walk of type $\mathsf{W}_{\mathsf{Sym}(G)}(y,z)$. If $w_1 \sim_{\mathscr{M}} w_2$ then $(w_1 \cdot w_3) \sim_{\mathscr{M}} (w_2 \cdot w_3)$.



(2) (Left whiskering) Let p_1, p_2 be walks of type $\mathsf{W}_{\mathsf{Sym}(G)}(y,z)$. If $p_1 \sim_{\mathscr{M}} p_2$, then $(w \cdot p_1) \sim_{\mathscr{M}} (w \cdot p_2)$.



(3) (Horizontal composition) Let p_1, p_2 be walks of type $\mathsf{W}_{\mathsf{Sym}(G)}(y,z)$. If $w_1 \sim_{\mathscr{M}} w_2$ and $p_1 \sim_{\mathscr{M}} p_2$, then $(w_1 \cdot p_1) \sim_{\mathscr{M}} (w_2 \cdot p_2)$.



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6.2 The type of spherical maps

In topology, the property of being simply connected to the sphere states that one can freely deform/contract any walk on the sphere into another whenever they share the same endpoints. This property of the sphere leads to the predicate in Definition 6.3, which sets the criteria for a graph to be embeddable in the 2-sphere.

Definition 6.3. *Given a graph G, a map $\mathscr{M}$ for G is said to be spherical if the type in (22) is inhabited:*

$$\mathsf{isSpherical}(\mathscr{M}) :\equiv \prod_{(x,y:\mathsf{N}_{\mathsf{Sym}(G)})} \prod_{(w_1,w_2:\mathsf{W}_{\mathsf{Sym}(G)}(x,y))} \| w_1 \sim_{\mathscr{M}} w_2 \| . \tag{22}$$

Theorem 6.4. *Being spherical for a map is a proposition.*

Proof. Since the codomain of the function type in (22) is a propositional truncation of a set of walk homotopies, it follows that it is indeed a proposition. □

Theorem 6.5. *The collection of all spherical maps for a graph forms a set.*

Proof. Spherical maps constitute a subtype within the set of all graph maps by Theorem 6.4, thus forming a set. □

Remark 1. When examining the definition of spherical maps in Definition 6.3, it becomes clear that showing that a map is spherical is not a simple task. To label a map as spherical, as per (22), one must evaluate all potential walk pairs for every node pair. This task can be daunting unless the set of walks exhibits a particular property. As a result, we suggest an alternative formulation for spherical maps based on a loop reduction procedure within a graph with a discrete node set.

Specifically, employing walk homotopies within a graph with a discrete node set and a pre-existing spherical map allows the reduction of any walk to an inner loop-free form, termed the *normal form* of a walk. The idea stems from the redundancy of loops, or the potential to simplify a loop into a point. Eradicating these loops results in a more tractable, yet equivalent, definition for spherical maps based on a loop reduction procedure for walks. Each walk is walk-homotopic to its normal form. It is also worth noting that one can determine if a map is spherical for graphs with a discrete node set. Additionally, the number of spherical maps for a graph is finite. For these results and their proofs, which exceed the scope of this text, we refer the reader to Prieto-Cubides (2022*b*).

6.3 The type of planar maps

Our goal is to characterise graph planarity within the framework of HoTT, guided by the intuitive idea that edges on a plane should not intersect. Defining the concept of edge intersection with precision poses a significant challenge. If we align with the geometric essence of this intuitive description, using the two-dimensional plane as our basis, it requires the use of real numbers (Univalent Foundations Program 2013, Section 10). This implies defining edge intersections in terms of points on the $\mathbb{R}^2$ plane and considering edges as curves within it. However, given its complexity, we opt for a combinatorial approach instead. As previously discussed, this method enables us to represent graph maps on the plane or, equivalently, a punctured 2-sphere, without any mention of real numbers.

Definition 6.6. *A connected and locally finite graph G is* planar *if the type* $\mathsf{Planar}(G)$ *is inhabited. Elements of* $\mathsf{Planar}(G)$ *are called* planar maps *of G:*

$$\mathsf{Planar}(G) :\equiv \sum_{(\mathscr{M} \,:\, \mathsf{Map}(G))} \mathsf{isSpherical}(\mathscr{M}) \times \underbrace{\mathsf{Face}(G,\mathscr{M})}_{\text{outerface}} .$$

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We define the type $\mathsf{Planar}(G)$ to represent all possible embeddings of G into the plane, specifically focusing on plane graphs. Although $\mathsf{Planar}(G)$ is not a planarity test in itself, it can be used to determine if a finite graph is planar or not by generating all the maps of the graph and subsequently verifying their spherical nature and the presence of an outer face, see Theorem 6.4.

Theorem 6.7. *The type of all planar maps of a graph forms a set.*

Proof. The type of planar maps in Definition 6.6 is not a proposition. It encompasses two sets: the set of combinatorial maps, see Theorem 4.10, and the set of faces, see Theorem 5.7. Since being spherical for a map is a mere proposition, one concludes that the Σ-type collecting all planar maps of a graph forms a set. □

In addition to their simple structure, cyclic graphs, and in particular C_n graphs, are building blocks in a few relevant constructions in formal systems related to the study of the planarity of graphs, such as planar triangulations and the characterisation of all 2-connected planar graphs.

Example 6.8. *To show the planarity of C_n, we begin with the base case $n = 0$. The graph C_0 is a unit graph, a graph with a single node $\star$ and no edges. Without edges, the type of functions mapping this node to any cyclic order of its star is a contractible type, yielding a unique, trivially spherical map. The map is spherical since the only walk to consider is the empty walk, which is trivially homotopic to itself. Planarity follows as C_0 is connected by definition and possesses an outer face. To define this face, we use as the base cyclic graph, the graph C_0 itself along with identity graph homomorphism h, see that $\mathsf{Sym}(C_0) \cong C_0$. The other conditions to inhabit the type of faces for our map are thus trivially satisfied.*

For $n > 0$, C_n is connected and locally finite as shown by Theorem 5.9. Its planarity is supported by Example 5.11, which confirms the existence of a unique map $\mathcal{M}$ for C_n. To show this map is spherical, it suffices to show that any two walks w_1 and w_2 with identical endpoints are homotopic. Inner loops in walks can be ignored since they are irrelevant to walk homotopy, as shown in Prieto-Cubides (2022b). Let us now consider the following cases. For $n = 1$, the only walk is the trivial one, which is self-homotopic. For $n > 1$, when examining nodes x and y in C_n, we have:

- ▹ *If $x \neq y$, the relevant walks are $\mathsf{ccw}_{\mathsf{Sym}(C_n)}(x, y)$ and $\mathsf{cw}_{\mathsf{Sym}(C_n)}(x, y)$, as per Theorem 5.10. These walks are homotopic via $\mathsf{hcollapse}(\mathcal{F}, x, y, x, y, \langle x \rangle, \langle y \rangle)$, where $\mathcal{F}$ denotes the face associated with $\mathsf{Sym}(C_n)$ where these walks form the boundary of $\mathcal{F}$.*
- ▹ *If $x = y$, the walks under consideration are the trivial walk at x and $\mathsf{cw}_{\mathsf{Sym}(C_n)}(x, x)$. Similarly to the previous case, these walks are homotopic via* hcollapse.

Finally, the outer face of $\mathcal{M}$ is naturally induced by C_n, which satisfies Definition 5.3 by construction. In fact, the definition of faces in Definition 5.3 was informed by the structure of C_n. Hence, we conclude that C_n is planar for all n.

In order to expand our collection of planar map examples, we will now explore the concept of planar extensions in the context of graph maps. This approach will provide a deeper understanding and additional instances of planar structures in graph theory.

6.4 Planar extensions

This subsection outlines the construction of planar maps from existing ones using the path addition operation. The inspiration for this construction derives from ear decompositions (Bang-Jensen and Gutin 2009, Section 5.3), reliable networks, extensions of planar graphs for undirected graphs (Gross et al. 2018, Section 5.2,7.3) and the characterisation of 2-connected graphs (Whitney 1932).

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Definition 6.9. *Let G be a graph with nodes u, v and P_n denote a path graph of n nodes as defined in Definition 3.13. The (simple) path addition of P_n to G at nodes u and v in G is a new graph constructed using the function* path-addition *with arguments G, u, v, n and r showing that n is positive, as illustrated in Fig. 13(a). For short, this new graph is denoted by $G \bullet_{u,v} P_n$. Here, u and v are referred to as the* endpoints of the addition*:*

$$\mathsf{path\text{-}addition} : \prod_{(G:\mathsf{Graph})} \prod_{(u,\, v:\mathsf{N}_G)} \prod_{(n:\mathbb{N})} (0 < n) \to \mathsf{Graph}.$$

$$\mathsf{path\text{-}addition}\ (G, u, v, n, r) :\equiv (N', E', h_1, h_2).$$

The types of nodes N' and the family of edges E' are defined below. The functions h_1 and h_2 are well defined, although not elaborated here, see Prieto-Cubides (2022a) for details on these functions, their properties and other functions related to path additions:

$$\begin{aligned}
&N' :\equiv \mathsf{N}_G + [\![n]\!].\\
&E' : N' \to N' \to \mathcal{U}.\\
&E'(\mathsf{inl}(x), \mathsf{inl}(y)) \ :\equiv \mathsf{E}_G(x, y).\\
&E'(\mathsf{inl}(x), \mathsf{inr}(y)) \ :\equiv (x = u) \times (y = (0, r)).\\
&E'(\mathsf{inr}(x), \mathsf{inl}(y)) \ :\equiv (x = \mathsf{pred}((0, r))) \times (y = v).\\
&E'(\mathsf{inr}(x), \mathsf{inr}(y)) \ :\equiv \mathsf{E}_{P_n}(x, y).
\end{aligned}$$

Remember that the path graph P_n with n nodes can be defined as follows:

$$P_n :\equiv ([\![n]\!], \lambda\, u\, v.\mathsf{toNat}(u) + 1 = \mathsf{toNat}(v)),$$

where toNat is defined as follows:

$$\begin{aligned}
&\mathsf{toNat} : [\![n]\!] \to \mathbb{N}.\\
&\mathsf{toNat}(k, !) :\equiv k.
\end{aligned}$$

We also conveniently define the non-simple path addition of P_n to G at nodes u and v in G. This operation mirrors the symmetrisation of a simple path addition. This construction of non-simple path addition is needed for subsequent sections, as it is used to establish the planar graphs, which involve the symmetrisation of the given graph.

Definition 6.10. *Let G be a graph with nodes u, v. The* non-simple path addition *of P_n to G at these nodes yields a new graph. This graph is constructed in a similar fashion as the simple path addition, by linking G and the graph $\mathsf{Sym}(P_n)$ using four edges. Two of these edges go from node u to 0 in $\mathsf{Sym}(P_n)$ and back. The other two edges link v to n in $\mathsf{Sym}(P_n)$ and back.*

To ease the upcoming discussion, we must introduce the following conventions:

- G is a locally connected finite graph with decidable equality on its nodes.
- n is a positive natural number.
- In the graph $G \bullet_{u,v} P_n$, we denote the walk from u to v via the addition of P_n to G as p. This is illustrated in Fig. 10(a). By an abuse of notation, we may also refer to this walk as $e_0 \cdot P_n \cdot e_n$. Here, e_0 and e_n are the edges connecting nodes u to 0 and nodes $n-1$ to v, respectively. The remaining edges, denoted as e_i, connect nodes $i-1$ and i and represent the new additions from the path addition.
- For brevity, we denote $G \bullet_{u,v} P_n$ by $G \bullet p$. This notation is often used below when the specifics of n and u, v are not crucial to the discussion.
- We denote $G \bullet_{u,v} \mathsf{Sym}(P_n)$ by $G \bullet \overline{p}$. Here, $\overline{p}$ represents the subgraph added to G through the non-simple path addition of P_n at nodes u and v. This is illustrated in Fig. 10(b).

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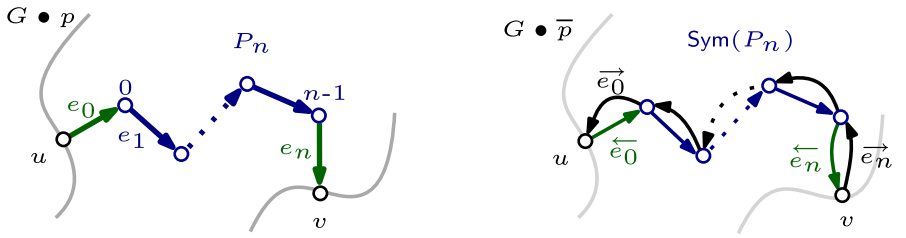

Figure 10. Path graph additions of P_n to G. The left figure illustrates the path addition $G \bullet_{u,v} P_n$, achieved by adding path graph P_n to graph G at nodes u and v. This process introduces two new edges, e_0 and e_n, along with n new nodes from path P_n. We define p as the walk $e_0 \cdot P_n \cdot e_n$ from u to v in $G \bullet_{u,v} P_n$, simplifying notation. Similarly, the right figure depicts the non-simple path addition of P_n to G at nodes u and v, extending graph G with P_n's symmetrisation and four additional edges.

- ▹ In $G \bullet \overline{p}$, we adopt similar notation regarding edges in the symmetrisation of a graph, as introduced in Fig. 4. The walk $\overleftarrow{p}$ signifies the walk in $\overline{p}$ induced by the sequence $\overleftarrow{e_0} \cdot \overleftarrow{e_1} \cdot \cdots \cdot \overleftarrow{e_n}$. Conversely, $\overrightarrow{p}$ denotes the opposite direction walk, induced by the sequence $\overrightarrow{e_n} \cdot \overrightarrow{e_{n-1}} \cdot \cdots \cdot \overrightarrow{e_0}$. See Fig. 10(b) for an illustration.
- ▹ Both $G \bullet p$ and $G \bullet \overline{p}$ are referred to as *graph extensions*.
- ▹ The operator $(\bullet)$ is left associative.
- ▹ The variables p_i denote finite path graphs of positive length, with respective endpoints u_i and v_i, adhering to the same considerations as for p in the previous items.
- ▹ A *simple cyclic addition* to G is the path addition $G \bullet_{u,u} p$ for some p, where u is a node in G.

Theorem 6.11. *If a graph G is connected, then both $G \bullet p$ and $G \bullet \overline{p}$ remain connected.*

Proof. To show the connectedness of $G \bullet_{u,v} P_n$, it suffices to consider connectivity between all node pairs in the augmented graph. The case for $G \bullet_{u,v} \mathsf{Sym}(P_n)$ is analogous. Additionally, we assume a walk can always be constructed to connect any two nodes in G. This is justified by eliminating the propositional truncation in the definition of connectedness since we want to prove connectedness for a graph, which is a proposition itself. The proof is followed by cases, depending on the location of the nodes in the augmented graph.

Let x and y be distinct nodes in $G \bullet_{u,v} P_n$; for identical nodes, a trivial walk suffices. If both are in G, their connectivity is inherent. If x is in G and y is in P_n, their connectivity is established via a concatenated walk from x to u within G, followed by the subwalk of $e_0 \cdot P_n \cdot e_n$ that connects 0 to y. If x and y lie in P_n, say they correspond to i and j, we can use as the walk to connect them, $e_{i+1} \cdot \cdots \cdot e_j$ if $i < j$. Otherwise, the walk is $e_i \cdot \cdots \cdot e_n \cdot w \cdot e_0 \cdot \cdots \cdot e_j$, where w denotes a given walk from v to u in G. □

We establish that graph symmetrisation is functorial with respect to path addition.

Theorem 6.12. $\mathsf{Sym}(G \bullet p) \cong \mathsf{Sym}(G) \bullet \overline{p}$.

Proof. To show these graphs are isomorphic, we compare their node and edge sets for equivalence. By definitions of Sym and path-addition, the node sets are identical:

$$\mathsf{N}_{\mathsf{Sym}(G \bullet p)} \equiv \mathsf{N}_G + [\![n]\!] \equiv \mathsf{N}_{\mathsf{Sym}(G)} + [\![n]\!] \equiv \mathsf{N}_{\mathsf{Sym}(G) \bullet \overline{p}}.$$

For the edge sets, we want to show that for given nodes x and y,

$$\mathsf{E}_{\mathsf{Sym}(G \bullet p)}(x, y) \simeq \mathsf{E}_{\mathsf{Sym}(G) \bullet \overline{p}}(x, y).$$

To address this equivalence, we notice how the path addition operation affects the edge sets of the original graph. This operation affects the edges differently based on the location of x and y, but

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within G or P_n, and because symmetrisation does not alter the edge sets:

$$\mathsf{E}_{\mathsf{Sym}(G \bullet p)}(x, y) \equiv \mathsf{E}_{\mathsf{Sym}(G)}(x, y) \equiv \mathsf{E}_{\mathsf{Sym}(G) \bullet \overline{p}}(x, y).$$

When x is in G and y in P_n, or vice versa, symmetry allows us to consider two cases: $x \equiv u$ and $y \equiv 0$, or $x \equiv v$ and $y \equiv n$. In both scenarios, the new edges introduced by path addition result in equivalent edge sets:

$$\mathsf{E}_{\mathsf{Sym}(G \bullet_{u,v} P_n)}(u, 0) \simeq [\![2]\!] \simeq \mathsf{E}_{\mathsf{Sym}(G) \bullet \overline{p}}(u, 0).$$

The first part of this chain, the equivalence, $\mathsf{E}_{\mathsf{Sym}(G \bullet_{u,v} P_n)}(u, 0) \simeq [\![2]\!]$ which is due to the fact that u and 0 are adjacent in $\mathsf{Sym}(G \bullet_{u,v} P_n)$. These two edges are the one induced in $G \bullet_{u,v} P_n$ and the other one from the symmetrisation process. On the other hand, the equivalence $\mathsf{E}_{\mathsf{Sym}(G) \bullet \overline{p}}(u, 0) \simeq [\![2]\!]$ follows by applying $(\bullet \overline{p})$ to $\mathsf{Sym}(G)$. The case for $x \equiv v$ and $y \equiv n$ is analogous. Consequently, the edge sets coincide, confirming the expected isomorphism. □

Theorem 6.13. *Let $\mathscr{M}$ represent a planar map of G, $\mathscr{F}$ a specific face, and u and v two nodes on the boundary walk of $\mathscr{F}$. An extended planar map of $G \bullet p$ can be constructed from $\mathscr{M}$, where p is situated onto $\mathscr{F}$, splitting it into two faces.*

The proof of Theorem 6.13 unfolds in several steps. We first define a map that extends $\mathscr{M}$ to a proper map of $G \bullet p$ with defined values for the nodes in p. Next, as illustrated in Fig. 12, we establish two faces resulting from placing p onto $\mathscr{F}$. The final step involves demonstrating that the candidate map for $G \bullet p$ is planar. That is, per Definition 6.6, that all pairs of walks in the symmetrisation of $G \bullet p$ are walk-homotopic with respect to the given map.

Proof of Theorem 6.13. Let $\mathscr{M}$ be a planar map of G, $\mathscr{F}$ a specific face, and u and v two nodes on the boundary walk of $\mathscr{F}$. We denote the graph $G \bullet p$ as H and the prospective planar map for this graph as $\mathscr{M}'$. In the context of Definition 5.3, within the face walk boundary $\partial\mathscr{F}$ of the given face $\mathscr{F}$, we identify an edge preceding u, represented as $a : \mathsf{E}_G(\mathsf{pred}(u), u)$, and its succeeding edge $a^+ : \mathsf{E}_G(u, \mathsf{suc}(u))$. Analogously for v, we have $b : \mathsf{E}_G(\mathsf{pred}(v), v)$ and $b^+ : \mathsf{E}_G(v, \mathsf{suc}(v))$, as depicted in Fig. 11(a).

We define the map $\mathscr{M}'$ at each node x in H. We begin with the endpoints of p, that is, $x = u$ and $x = v$. For $x = u$, we alter the cycle $\mathscr{M}(u)$ by introducing e_0 between the edges a and a^+, resulting in the cycle $\mathscr{M}'(u) = (\cdots a\, e_0\, a^+ \cdots)$. Similarly, for $x = v$, the modified cycle $\mathscr{M}'(v)$ is $(\cdots b\, e_n\, b^+ \cdots)$. For internal nodes of p, that is, nodes x in P_n, the map $\mathscr{M}'$ is defined directly. At each of these nodes, we encounter only two edges, denoted as e_i and e_{i+1}, where i ranges from 0 to $n-1$. Remember that e_0 connects nodes u and 0, e_n links nodes $n-1$ and v, and for the remaining, e_i bridges nodes $i-1$ and i.

Assume the face $\mathscr{F}$, induced by (A, h) of degree m according to Definition 5.3. Here, h is an edge-injective graph homomorphism from A to $\mathsf{Sym}(H)$, satisfying the map-compatibility condition. Let $\partial\mathscr{F}$ be the boundary walk of $\mathscr{F}$ of length k and define n_1, n_2 as $k + (n+1)$ and $(m-k) + (n+1)$, respectively.

Let us denote F_1, F_2 as faces induced by (C_{n_1}, h_1) and (C_{n_2}, h_2), respectively, where $h_1 = (\alpha_1, \beta_1)$ and $h_2 = (\alpha_2, \beta_2)$ are morphisms of type $\mathsf{Hom}(C_{n_i}, \mathsf{Sym}(H))$ for $i = 1, 2$. The boundary walks of these faces, ∂F_1 and ∂F_2, are defined as $\mathsf{cw}_{\mathscr{F}}(u, v) \cdot \overrightarrow{p}$ and $\mathsf{ccw}_{\mathscr{F}}(u, v) \cdot \overrightarrow{p}$, respectively. The subsequent image provides a visual representation of this concept.

To establish the planarity of $\mathscr{M}'$, we must first demonstrate that for each face F_1 and F_2 of $\mathscr{M}'$, h_1 and h_2 satisfy the map-compatibility condition and uphold the edge-injectivity property. Beginning with h_1, consider the nodes in C_{n_1}, namely $0, 1, \ldots, n_1 - 1$. Each node $i : \mathsf{N}_{C_{n_1}}$ maps to a node defined by α from $\mathscr{F}$. Specifically, $\alpha_1(i)$ equals $\alpha(i)$ for $i < k$, while $\alpha_(i)$ positions the node in $\mathsf{cw}_{\mathscr{F}}(u, v)$. For the corresponding edges, $e : \mathsf{E}_{C_{n_1}}(i, i+1)$, we employ the function β from $\mathscr{F}$ to define β_1, such that $\beta_1(i, i+1, e)$ corresponds to $\beta(i, i+1, e)$.

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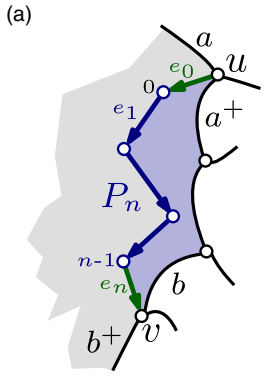
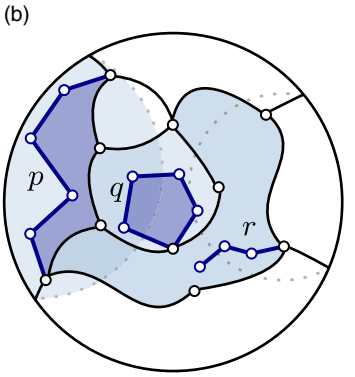


Path addition used in Theorem 6.13.



The embedded graph $\mathsf{Sym}(G \bullet p \bullet q \bullet r)$.

Figure 11. Figure (a) in the caption illustrates the path addition $G \bullet p$ as detailed in Theorem 6.13. Figure (b) presents the planar map for G from Fig. 5(b), showcasing three graph extensions: the path addition of p, cyclic addition of q and spike addition of r. Though it is feasible to define the construction of r, it is not necessary for this discussion. The additions of p and q split faces F_2 and F_3 from Fig. 5, generating two new faces each. The spike addition of r substitutes F_4 with a face of higher degree.

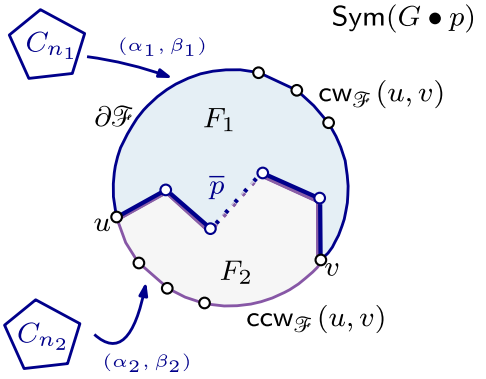


Figure 12. The figure demonstrates the partitioning of face $\mathscr{F}$ into two, F_1 and F_2, via $G \bullet p$ when p resides on face $\mathscr{F}$.

However, if $k \leq i \leq n_1$, node i must be placed in $\overline{p}$, then $\alpha_1(i)$ is $n - i$. Correspondingly, for edges, we set $\beta_1(i, i+1, e)$ as the edge $\mathsf{inl}(e_i)$ in $\mathsf{Sym}(H)$. It is clear by construction that h_1 is an edge-injective, map-compatible graph homomorphism with the map $\mathscr{M}'$, properties naturally inherited from h. In a similar vein, it can be proven that h_2 is well defined and fulfils the map-compatibility condition and the edge-injectivity property.

To prove that $\mathscr{M}'$ is planar, we must first show that it is spherical. To see this, we rely on Theorem 6.4, which allows us to apply the elimination of the propositional truncation to the evidence that $\mathscr{M}$ is spherical. This enables us to obtain a walk homotopy for any pair of walks in $\mathsf{Sym}(G)$ sharing endpoints, which is perhaps used henceforth without explicit mention. This entails that homotopic walks in $\mathsf{Sym}(G)$, deforming along faces other than $\mathscr{F}$, maintain their homotopy in $\mathsf{Sym}(H)$. Therefore, our focus narrows down to:

(i) the set of walks in $\mathsf{Sym}(G)$ deforming along $\mathscr{F}$, and
(ii) the set of walks resulting from possible compositions of $\overline{p}$ with existing walks in $\mathsf{Sym}(G)$.

For both walks originating from set (i), their homotopy is defined by the vertical composition of homotopies along F_1 and F_2, as referenced in Theorem 6.2, (Prieto-Cubides 2022*b*, Section 5).

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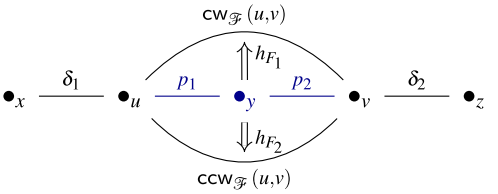

Figure 13. The figure shows a part of the graph $\mathsf{Sym}(G \bullet p)$ embedded in the 2-sphere. As constructed in the proof of Theorem 6.13, the faces, F_1 and F_2, of the map $\mathscr{M}'$ are given by a face division of $\mathscr{F}$ by the path p. Such gives rise to new walk homotopies, as h_{F_1} and h_{F_2} in the picture. The walk $\overleftarrow{p}$ from u to v is the walk composition of p_1, a walk from u to y, and p_2, a walk from y to v. The walks δ_1 and δ_2 are walks in $\mathsf{Sym}(G)$ from x to z.

In case (ii), we consider walks without inner loops, following Lemma 5.8 in Prieto-Cubides (2022*b*). We examine three subcases without loss of generality, where the walk $\overline{p}$ from u to v decomposes into p_1 and p_2. Here, p_1 is a walk from u to node y in $\mathsf{Sym}(G \bullet p)$, and p_2 from y to v, as shown in Fig. 13. Recall that a walk homotopy for any pair of walks in $\mathsf{Sym}(G)$ sharing endpoints is always accessible by hypothesis.

(a) Either w_1, w_2, or both, include $\overleftarrow{p}$ as a subwalk from x to z. If w_1 composes as $\delta_1 \cdot \overleftarrow{p} \cdot \delta_2$, and $\overleftarrow{p}$ is not a subwalk of w_2, with δ_1 and δ_2 being walks in $\mathsf{Sym}(G)$ from x to u and v to z, a homotopy of walks can be obtained as in the calculation below. The remaining cases are demonstrated similarly:

$$
\begin{aligned}
w_1 &\equiv \delta_1 \cdot \overleftarrow{p} \cdot \delta_2 \\
&\equiv \delta_1 \cdot \mathsf{ccw}_{F_1}(u,v) \cdot \delta_2 && \text{(By construction of } F_1\text{)} \\
&\sim_{\mathscr{M}'} \delta_1 \cdot \mathsf{cw}_{F_1}(u,v) \cdot \delta_2 && \text{(By hcollapse constructor applied to } F_1, \delta_1 \text{ and } \delta_2\text{)} \\
&\equiv \delta_1 \cdot \mathsf{cw}_{\mathscr{F}}(u,v) \cdot \delta_2 && \text{(By construction of } F_1\text{)} \\
&\sim'_{\mathscr{M}} w_2 && \text{(By hypothesis: walks in } \mathsf{Sym}(G) \text{ are homotopic).}
\end{aligned}
$$

(b) The walks w_1 and w_2 from x to y share a suffix (p_1) or a prefix (p_2). Without loss of generality, let $w_1 = \delta_1 \cdot p_1$ and $w_2 = \delta \cdot p_1$, where δ is a walk from x to u. These walks are homotopic in $\mathsf{Sym}(G)$ via the spherical map $\mathscr{M}$, that is, $\delta_1 \sim_{\mathscr{M}} \delta$. The construction of $\mathscr{M}'$ ensures $\delta_1 \sim_{\mathscr{M}'} \delta$. Utilising right whiskering, we deduce $\delta_1 \cdot p_1 \sim_{\mathscr{M}'} \delta \cdot p_1$, thereby reaching our desired conclusion. Similarly, if w_1 is $p_2 \cdot \delta_2$ and w_2 is $p_2 \cdot \delta$, where δ is a walk from v to z, one can show that $\delta_2 \sim_{\mathscr{M}} \delta$, and hence $\delta_2 \cdot p_2 \sim_{\mathscr{M}'} \delta \cdot p_2$ by left whiskering.

(c) The walks w_1 and w_2 from x to y can be expressed as composites of $\delta \cdot p_1$ and $\delta' \cdot \overrightarrow{p_2}$, respectively. Here, δ and δ' are walks from x to u and x to v, without sharing a common prefix or suffix subwalk. We aim to show $w_1 \sim_{\mathscr{M}'} w_2$ via F_2 deformation:

$$
\begin{aligned}
w_1 &\equiv \delta \cdot \overleftarrow{p_1} \\
&\equiv \delta \cdot \mathsf{cw}_{F_2}(u,y) && \text{(By construction of } F_2\text{)} \\
&\sim_{\mathscr{M}'} \delta \cdot \mathsf{ccw}_{F_2}(u,y) && \text{(By constructor hcollapse applied to } F_2, \delta_1 \text{ and } \langle y \rangle\text{)} \\
&\equiv \delta \cdot (\mathsf{ccw}_{\mathscr{F}}(u,v) \cdot \overrightarrow{p_2}) && \text{(By construction of } F_2\text{)} \\
&\equiv (\delta \cdot \mathsf{ccw}_{\mathscr{F}}(u,v)) \cdot \overrightarrow{p_2} && \text{(By assoc. of walk concat.)} \\
&\sim_{\mathscr{M}'} \delta' \cdot \overrightarrow{p_2} && \text{(By whiskering applied to the walk htpy. by hyp.)} \\
&\equiv w_2.
\end{aligned}
$$

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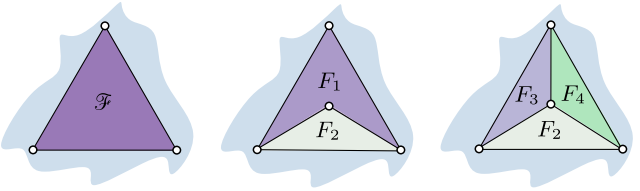

Figure 14. The figure illustrates a planar synthesis for constructing a K_4 planar map using a C_3 planar map. Initially, face $\mathscr{F}$ is divided into F_1 and F_2. Subsequently, F_1 is split into F_3 and F_4. The resulting map ends up with four faces, including the outer face.

Concluding our proof of Theorem 6.13, we have shown that $\mathscr{M}$ extends to a spherical map $\mathscr{M}'$ of $G \bullet p$. By identifying F_1 as the outer face, we further establish that $\mathscr{M}'$ is a planar map. □

Given that $\mathscr{M}$ is a planar map, we denote its planar extension derived from Theorem 6.13 by $E(\mathscr{M}, \mathscr{F}, u, v, P_n)$. To shorten the notation, this planar extension is denoted by $E(\mathscr{M}, \mathscr{F}, p)$, when the specifics of u, v and P_n are not really crucial to the discussion. We refer to this as the *face division* of $\mathscr{F}$ by p, since this construction results in the placement of p in $\mathscr{F}$, dividing it into two new faces.

Definition 6.14. *For a finite graph G with map $\mathscr{M}$, the Euler characteristic $\chi_{\mathscr{M}}$ is defined as the number $v - e + f$, where v, e and f denote the cardinalities of the sets of nodes, edges and faces, respectively:*

$$\chi_{\mathscr{M}} \coloneqq v - e + f. \tag{23}$$

Theorem 6.15. *For a graph G with planar map $\mathscr{M}$, any planar extension of $\mathscr{M}$ maintains Euler's characteristic. That is, for any face $\mathscr{F}$ of $\mathscr{M}$ and nodes u, v within $\mathscr{F}$ connected by a path P_n, we have $\chi_{\mathscr{M}}$ equals $\chi_{E(\mathscr{M},\mathscr{F},u,v,P_n)}$.*

Proof. The lemma follows from the construction detailed in the proof of Theorem 6.13. The path addition of P_n between nodes u and v on face $\mathscr{F}$ increases the node count by $n + 1$, edges by $n + 2$ and faces by 1, preserving the Euler characteristic. □

Euler characteristic serves as a planarity criterion for connected finite graphs. Specifically, according to Euler's formula, a graph G is planar under the map $\mathscr{M}$ if and only if $\chi_{\mathscr{M}}$ equals 2. The constructions detailed in this section facilitate the verification of Euler's formula for graphs constructed via path additions, an approach also employed later for biconnected planar graphs.

However, for arbitrary graphs not derived from graph extensions, validating Euler's formula remains challenging, primarily due to the non-trivial task of determining the cardinality of the set of faces for an arbitrary map $\mathscr{M}$ of a given graph G, that is, computing the set of elements of type $\mathsf{Face}(G, \mathscr{M})$ (see Definition 5.3). Progress was made by establishing that the type of faces forms a finite set. This suggests the feasibility of extracting this number in practice, possibly utilising the employed proof-assistant. We leave this to future work.

6.5 Planar synthesis of graphs

Inductive graph construction methods abound, such as Whitney–Robbins synthesis, ear decomposition of a graph and the K_4 construction depicted in Fig. 14. Drawing inspiration from these methods and face divisions as in Theorem 6.13, we propose a method to build larger planar graphs using graph extensions, ensuring that we remain within the type of planar graphs.

Definition 6.16. *A Whitney* synthesis *(synthesis for short) of graph G from graph H is defined as a sequence of graphs $G_0, G_1, \cdots, G_n$, where G_0 is H, G_n is G, and each G_i results from the path*

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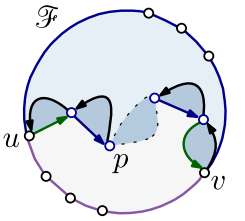

Figure 15. The figure illustrates the face division of $\mathscr{F}$ by a non-simple path addition.

addition of p_i to G_{i-1} for i in the range 1 *to n. Consequently, G can be viewed as the result of adding paths $p_1, p_2, \cdots, p_n$ to H:*

$$G \equiv H \bullet p_1 \bullet p_2 \bullet \cdots \bullet p_n.$$

The length *of this synthesis is n. A* simple synthesis *refers to a sequence containing only simple additions. Conversely, a sequence composed solely of non-simple additions is termed a* non-simple synthesis.

Theorem 6.17. *Syntheses preserve graph-connectedness. Specifically, if a graph H is connected and G is synthesised from H, then each intermediate graph G_i in the synthesis sequence is also connected.*

Proof. We prove this by induction on the length of the synthesis and the fact that path additions preserve connectedness, Theorem 6.11. □

Definition 6.18. *Given a planar map $\mathscr{M}$ of the graph H with outer face $\mathscr{F}$, we define a* planar synthesis *of G from H of length n as a sequence:*

$$(G_0, \mathscr{M}_0, \mathscr{F}_0), (G_1, \mathscr{M}_1, \mathscr{F}_1) \cdots, (G_n, \mathscr{M}_n, \mathscr{F}_n),$$

where

- ▹ *$(G_0, \mathscr{M}_0, \mathscr{F}_0)$ is equivalent to $(H, \mathscr{M}, \mathscr{F})$, and*
- ▹ *$(G_n, \mathscr{M}_n)$ corresponds to $(G, E(\mathscr{M}_{n-1}, \mathscr{F}_{n-1}, p_{n-1}))$.*

For each i in the range 1 *to n, the graph G_i is $G_{i-1} \bullet p_i$, and the map $\mathscr{M}_i$ is $E(\mathscr{M}_{i-1}, \mathscr{F}_{i-1}, p_{i-1})$, where $\mathscr{F}_{i-1}$ is a face of $\mathscr{M}_{i-1}$.*

Theorem 6.19. *If a graph G is synthesised from a planar graph H via planar synthesis, then G and every graph in the corresponding sequence are planar.*

Proof. Through planar synthesis, each G_i is derived from G_{i-1} via path addition, ensuring planarity by Theorem 6.13. □

While we have not yet employed non-simple additions, they have become relevant when we characterise planar biconnected graphs in the next section. It is possible to extend the face division theorem and its construction to utilise non-simple additions, Theorem 6.13, allowing us to adapt not only the planar synthesis in Definition 6.18 to *non-simple planar syntheses* but also Theorem 6.19 to accommodate non-simple additions. Hence, given a map $\mathscr{M}$ for G with a face $\mathscr{F}$, the corresponding planar map for $G \bullet \overline{p}$ is denoted as $E(\mathscr{M}, \mathscr{F}, \overline{p})$, maintaining a similar notation as before. As with path additions, extending the map with non-simple additions introduces new faces.

Taking into account $G \bullet_{u,v} \mathsf{Sym}(P_n)$ and the map $E(\mathscr{M}, \mathscr{F}, \overline{p})$, the number of faces increases to $n+3$, the number of nodes increases to $v+n+1$ and the number of edges increases to $2 \cdot (n+2)$, as illustrated in Figure 15. Consequently, the Euler characteristic of $E(\mathscr{M}, \mathscr{F}, \overline{p})$ is equal to the Euler characteristic of $\mathscr{M}$.

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Consequently, the Euler characteristic of $E(\mathcal{M}, \mathcal{F}, \overline{p})$ is equal to the Euler characteristic of $\mathcal{M}$:

$$\chi_{E(\mathcal{M},\overline{p})} :\equiv (v + (n+1)) - (e + 2 \cdot (n+2)) + (f + (n+3)) \equiv \chi_{\mathcal{M}}.$$

Fig. 13(b) demonstrates the construction of larger planar graphs using various path, cycle and spike additions. A *spike addition* to G, although not precisely defined here, as it is not extensively used for further constructions, can be essentially described as a path addition sharing only one node with G. With a given map for G, a simple addition of a spike creates a new face of a higher degree than the face where the spike is inserted. Consequently, non-simple spike additions also increase the number of faces for the extended map due to the emergence of new faces between edge pairs that share endpoints.

The remainder of this section aims to characterise the construction of all 2-connected planar graphs. In general, a graph is *k-connected* if it cannot be disconnected by removing less than k nodes. Depending on k, there are various methods to construct the set of k-connected graphs. For instance, any undirected 2-connected graph can be constructed by applying path additions to an appropriate cyclic graph (Diestel 2012, Section 3). Our focus in the following will be on the construction of 2-connected planar graphs.

Definition 6.20. *A graph G is defined as* 2-connected, *or biconnected, when the proposition* $\mathsf{Biconnected}(G)$ *holds. This is, when the resulting graph $G - x$, formed by removing a node x from G, remains connected.*

Specifically, $G - x$ is the graph made up of the set of nodes, $\Sigma_{y:\mathsf{N}_G}(x \neq y)$, and their corresponding edges in G:

$$\mathsf{Biconnected}(G) :\equiv \prod_{(x\,:\,\mathsf{N}_G)} \mathsf{Connected}(G - x).$$

Theorem 6.21. *If G is a cyclic graph, then* $\mathsf{Sym}(G)$ *is 2-connected.*

Proof. The cyclic nature of G ensures that in $\mathsf{Sym}(G)$, there are two inner loop-free walks between any pair of nodes: a direct walk following the cycle of G and a reverse walk counter to the cycle. These walks are edge-disjoint and thus preserve the graph's connectivity despite the removal of any single node-only one of the walks might be affected, leaving the other intact to sustain connectedness. □

The property of 2-connectedness in a graph does not remain invariant under simple path additions. Clearly, removing a node from the added path p disconnects $G \bullet p$. Yet, through non-simple path additions, it is possible to maintain and even augment 2-connected graphs.

Theorem 6.22. *Let G denote a 2-connected graph. The graph extensions, $G \bullet \overline{p}$ and* $\mathsf{Sym}(G) \bullet \overline{p}$, *preserve the 2-connected graph property.*

Proof. To show that $G \bullet_{u,v} \mathsf{Sym}(P_n) - x$ remains connected for any node x in $G \bullet \overline{p}$, we consider the location of x. If x is within G, then $G - x$ is connected by hypothesis. Applying Theorem 6.11, it follows that $G \bullet \overline{p} - x$ is also connected, showing the 2-connectivity of $G \bullet \overline{p}$. Otherwise, if x lies on $\overline{p}$, its removal divides $\overline{p}$ into two parts, p_1 and p_2. For any two nodes in $G \bullet \overline{p} - x$, we show they are connected. If both nodes are in G or the same part p_i, they are connected by prior arguments or direct traversal, respectively. If they are located in distinct subgraphs, say x is in p_1 and y is in p_2. We can construct a walk from x to u, another walk across G from u to v (since G is connected), and then to the second node, Hence, $G \bullet \overline{p}$ maintains 2-connectivity. □

Inspired by Yamamoto's work (Yamamoto et al. 1995), our focus is on the construction of 2-connected planar graphs. Within a different theoretical setting (HOL) and using a different graph definition, Yamamoto shows that any undirected 2-connected planar graph can be inductively

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built by adding diverse *paths* to *circuits* (their term for *cyclic graphs*). In our context, we initiate constructions with any 2-connected graph $\mathsf{Sym}(C_n)$ and subsequently extend these graphs by non-simple planar additions.

Theorem 6.23. *In a non-simple Whitney synthesis of G originating from a 2-connected graph H, with planarity ensured by a map $\mathscr{M}$, each graph in the synthesis maintains 2-connectivity and planarity via planar extension of $\mathscr{M}$ using non-simple additions.*

Proof. Assuming a non-simple Whitney synthesis of G from H of length n is given, we proceed by induction on n.

- ▹ Base case ($n = 0$). The graph G is H, and by hypothesis, H is a 2-connected planar graph. Thus, the conclusion follows.
- ▹ Inductive step. For the inductive step, we assume that the claim holds for a sequence of length n, thus establishing G_n as a 2-connected planar graph via map $\mathscr{M}_n$. We then aim to demonstrate that G, defined as $G_n \bullet \overline{p_i}$ for some path p_i, also qualifies as a 2-connected planar graph.
- ▹ Given that G_n is 2-connected, it follows from Theorem 6.22 that $G_n \bullet \overline{p_i}$ also retains this property. We then extend the planar map $\mathscr{M}_n$ of G_n to a planar map $\mathscr{M}$ for G, preserving the outer face or selecting a new outer face from the additions. This construction of $\mathscr{M}$ follows the method in Theorem 6.13, where we expand planar maps using simple additions, as required here. □

Theorem 6.24. *Any graph synthesised from $\mathsf{Sym}(C_n)$ through non-simple Whitney syntheses is a 2-connected planar graph.*

Proof. Given that C_n is planar by Example 6.8 and consequently connected, $\mathsf{Sym}(C_n)$ is 2-connected by Theorem 6.21. By repeatedly applying Theorem 6.23 to each step in the given synthesis sequence, we ensure the resulting graph's 2-connectivity and planarity. □

Lemma 3 and Proposition 4 in Yamamoto et al. (1995) discuss undirected 2-connected planar graphs similar to the converse of Theorem 6.24. It is possible to follow Yamamoto's argument closely, even though it was presented in an informal way. However, this requires preliminary formalisation of several technicalities, such as maximal subgraphs, adjacent faces and edge sequence deletion. Subsequently, one can assert that non-simple Whitney syntheses entirely determine 2-connected planar graphs, as expressed similarly in Diestel (2012, Section 3). In essence, any graph defined as planar in Definition 6.6 and 2-connected in Definition 6.20 can be inductively generated from $\mathsf{Sym}(C_n)$ via iterative non-simple path additions and proper map extensions.

Further exploration of graph extensions, such as amalgamations, appendages, deletions, contractions and subdivisions, should be considered to generate planar graphs (Gross et al. 2018, Section 7.3).

7. Related Work

The study of planar graphs and more general graph-theoretic topics can be found in relevant projects and large libraries formalised in Coq (Doczkal and Pous 2020), Isabelle/HOL (Noschinski 2015), and most recently, in Agda (Rijke et al. 2023), and Lean (The Mathlib Community 2020). Examples of notable projects on the subject include the formal proof of the Four-Colour Theorem (FCT) in Coq (Gonthier 2008), the proof of the discrete form of the Jordan Curve Theorem in Coq (Dufourd and Puitg 2000), and the verification of Kepler's Conjecture in HOL (Hales et al. 2017).

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Different approaches have been proposed to address the planarity of graphs in formal systems. These works use different mathematical objects depending on the system. We use combinatorial maps in this work, but other related constructions are, for example, root maps defined in terms of permutations (Dubois et al. 2016) and hypermaps (Dufourd 2009; Dufourd and Puitg 2000; Gonthier 2008), among others. In particular, one can see that the notion of a *hypermap* is a generalisation of a combinatorial map for undirected finite graphs. Such a concept is one fundamental construction to formalise mathematics of graph maps amongst in theorem provers, along with the computer-checked proof of FCT. Additionally, Dufourd states and proves the Euler's polyhedral formula and the Jordan Curve Theorem using an inductive characterisation of hypermaps (Dufourd 2009; Dufourd and Puitg 2000). Recently, for a more standard representation of finite graphs, Doczkal proved that, according to his notion of a plane map based on hypermaps, every $K_{3,3}$-free graph and K_5-free graph without isolating nodes is planar, a direction of Wagner's theorem (Doczkal 2021).

An alternate strategy for planar graphs is to build them iteratively. For example, in Yamamoto et al. (1995), the authors showed that any biconnected and finite planar graph can be decomposed as a finite set of cycle graphs, where every face is the region bounded by a closed walk (Gross et al. 2018, Sections 5.2, 7.3). This construction defines an inductive data type that begins with a cycle graph C_n serving as the base case, and by repeatedly merging new instances of cycle graphs, one gets the final planar graph. Bauer formalises in Isabelle/HOL a similar construction of planar graphs from a set of faces (Bauer and Nipkow 2002; Bauer 2005). A related approach in our setting is of the treatment of planar graph extensions, as described in Section 6.4.

To our knowledge, previous work in type theory related to graph planarity has been conducted within different formal systems and for distinct classes and notions of graphs. These studies predominantly define planarity for undirected finite graphs. In contrast, our definition encompasses the broader class of connected and locally finite directed multigraphs. Our research aligns more closely with the foundations of mathematics, particularly the formalisation of mathematics in HoTT, rather than the practical aspects of graph theory. This necessitates proposing new constructs, occasionally even for the most fundamental concepts within the theory.

8. Concluding Remarks

This document is a case study of graph-theoretic concepts in constructive mathematics using HoTT. An elementary characterisation of planarity of connected and locally finite directed multigraphs is presented in Section 6.3. We collected all the maps of a graph in the two-dimensional plane – identified up to isotopy – in a homotopy set, see Theorem 6.7. The type of these planar maps displays some of our main contributions, for example, the type of spherical maps stated in Definition 6.3 and the type of faces for a given map in Definition 5.3. As far as we know, the presentation of these types in a dependent type theory like HoTT is novel. For example, besides its rather technical definition, we believe the type of faces encodes in a better combinatorial way the essence of the topological intuition behind it, rather than, being defined as simply cyclic lists of nodes, as by other authors (Bauer 2005; Gonthier 2008; Yamamoto et al. 1995), see Section 5. We did not include a proof that the type of faces of a map of a finite graph is finite. We omitted it here due to its technical complexity, involving multiple applications of finiteness results and identification of convenient equivalences. This detailed proof is included in an upcoming thesis by the first author, an extended version of this work and related formalisations in Agda.

Additionally, as a way to construct planar graphs inductively, we presented extensions for planar maps. We demonstrated that any cycle graph is planar, and by means of planar extensions such as path additions, one can construct larger planar graphs; for example, to illustrate this approach, a planar map for K_4 using simply path additions from a planar map of C_3 is illustrated in Fig. 14. Other relevant notions for this work are cyclic types, cyclic graphs, homotopy for walks (Prieto-Cubides 2022*b*) and spherical maps.

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We chose HoTT as the reasoning framework to directly study the symmetry of our mathematical constructions. Many of the proofs supporting our development could only be constructed by adopting the UA, a main principle in HoTT. A primary example of using Univalence in this paper is the structural identity principle for graphs, as stated in Theorem 3.7.

Another contribution of this work includes the (computer-checked) proofs. Not all but the relevant concepts in this document have been formalised in the proof assistant Agda, in a fully self-contained development (Prieto-Cubides 2022*a*). However, for technical reasons, the formalisation of Example 6.8 and further in-depth studies on the main results in Section 6.4 like Theorems 6.13 and 6.24 are left for future work.

This work can serve as a starting point for further developments of graph theory in HoTT or related dependently typed theories. We expect further research to provide other interesting results, such as the equivalences between different characterisations of planarity for graphs, for example, the Kuratowski's and Wagner's characterisations for planar graphs, and the realisation of planar graphs defined via HITs.

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