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GENERATING FUNCTIONS FOR THE QUOTIENTS OF NUMERICAL SEMIGROUPS

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Abstract

We propose generating functions, $\mathrm{RGF}_p(x)$, for the quotients of numerical semigroups which are related to the Sylvester denumerant. Using MacMahon's partition analysis, we can obtain $\mathrm{RGF}_p(x)$ by extracting the constant term of a rational function. We use $\mathrm{RGF}_p(x)$ to give a system of generators for the quotient of the numerical semigroup $\langle a_1, a_2, a_3\rangle$ by p for a small positive integer p, and we characterise the generators of $\langle A\rangle/p$ for a general numerical semigroup A and any positive integer p.

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1. Introduction

Throughout this paper, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}^+$ denote the set of all integers, nonnegative integers and positive integers, respectively.

A subset S of $\mathbb{N}$ is a *numerical semigroup* if $0 \in S$, $\mathbb{N} \setminus S$ is finite and S is closed under the addition in $\mathbb{N}$. Given a positive integer sequence $A = (a_1, a_2, \ldots, a_k)$, if $\gcd(A) = 1$, then

$$\langle A\rangle = \{x_1a_1 + x_2a_2 + \cdots + x_ka_k \mid k \geq 2, x_i \in \mathbb{N}, 1 \leq i \leq k\}$$

is a numerical semigroup (see [13]) and A is a *system of generators* of $S = \langle A\rangle$. If no proper subset of A generates S, then we say that A is a *minimal system of generators* of S. Sylvester [16] defined the denumerant $d(a_0; a_1, a_2, \ldots, a_k)$ as

$$d(a_0; a_1, a_2, \ldots, a_k) = \#\{(x_1, \ldots, x_k) \mid x_1a_1 + x_2a_2 + \cdots + x_ka_k = a_0, x_i \in \mathbb{N}\}.$$

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If $\gcd(A) = 1$, then there exists a positive integer N such that $d(a_0; a_1, \ldots, a_k) > 0$ for any integer $a_0 \geq N$ (see, for example, [11, Theorem 1.0.1]). The greatest integer not belonging to $\langle A \rangle$ is the *Frobenius number* of A defined by

$$F(A) = \max\{a_0 \in \mathbb{Z}^+ \mid d(a_0; a_1, a_2, \ldots, a_k) = 0\}.$$

For more descriptions and results about numerical semigroups, see [4, 11, 13].

Suppose $\langle A \rangle$ is a numerical semigroup and $p \in \mathbb{Z}^+$. The *quotient of* $\langle A \rangle$ *by* p,

$$\frac{\langle A \rangle}{p} = \{n \in \mathbb{N} \mid pn \in \langle A \rangle\} = \{n \mid pn = x_1 a_1 + x_2 a_2 + \cdots + x_k a_k, x_i \in \mathbb{N}, 1 \leq i \leq k\},$$

was introduced in [14]. It is easy to verify that $\langle A \rangle / p$ is a numerical semigroup, that $\langle A \rangle \subseteq \langle A \rangle / p$, and that $\langle A \rangle / p = \mathbb{N}$ if and only if $p \in \langle A \rangle$. For example, let $p = 3$ and $\langle A \rangle = \langle 5, 6 \rangle = \{0, 5, 6, 10, 11, 12, 15, 16, 17, 18, 20 \rightarrow\}$, where the symbol $\rightarrow$ means that all subsequent integers are included. Then $\langle A \rangle / 3 = \{0, 2, 4 \rightarrow\} = \langle 2, 5 \rangle$.

Let a_1, a_2, p be pairwise relatively prime positive integers. Rosales [12] obtained a system of generators for $\langle a_1, a_2 \rangle / 2$ and Rosales and Urbano-Blanco [15] gave a characterisation of a system of generators for $\langle a_1, a_2 \rangle / p$ by means of modular permutations and certain congruence equations. In [6], Cabanillas discussed the minimal generators of $\langle a_1, a_2 \rangle / p$. In [10], Moscariello also gave a characterisation of the generating system of $\langle A \rangle / p$ by defining a class of partitions. There are many open problems related to $\langle A \rangle / p$ (see, for example, [7]).

The *representation generating function* of $\langle A \rangle / p$ is the generating function

$$\mathrm{RGF}_p(x) = \sum_{n \geq 0} d(pn; a_1, a_2, \ldots, a_k) x^n.$$

The function $\mathrm{RGF}_p(x)$ is easily seen to be rational (see Section 2.1) and we can use it to obtain a system of generators for $\langle A \rangle / p$. For example, let a_1 and a_2 be relatively prime odd positive integers. Then

$$\sum_{n \geq 0} d(n; a_1, a_2) x^n = \frac{1}{(1 - x^{a_1})(1 - x^{a_2})}$$

and the representation generating function of $\langle a_1, a_2 \rangle / 2$ is determined by

$$\mathrm{RGF}_2(x^2) = \frac{1}{2}\Big(\frac{1}{(1 - x^{a_1})(1 - x^{a_2})} + \frac{1}{(1 - (-x)^{a_1})(1 - (-x)^{a_2})}\Big) = \frac{1 + x^{a_1 + a_2}}{(1 - x^{2a_1})(1 - x^{2a_2})}.$$

Therefore, $\langle a_1, a_2 \rangle / 2 = \langle a_1, a_2, (a_1 + a_2)/2 \rangle$.

We use *MacMahon's partition analysis* [9] to represent $\mathrm{RGF}_p(x)$ as the constant term of a rational function in a new variable λ. For small $p \in \mathbb{Z}^+$ and $A = (a_1, a_2, a_3)$ with $\gcd(A) = 1$, we can calculate $\mathrm{RGF}_p(x)$ and obtain a system of generators of the quotient of the numerical semigroup $\langle A \rangle$ by p. We give the results for $p = 2$ and 3 in Table 1. We write $a_i = pk_i + t_i$, where $0 \leq t_i \leq p - 1$ and $p, k_i \in \mathbb{Z}^+$ for $1 \leq i \leq 3$.

We can extend this idea to give the following simple characterisations for $\langle A \rangle / p$.

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TABLE 1. A system of generators of $\langle a_1, a_2, a_3\rangle/p$ for $p = 2, 3$.

p	t_1	t_2	t_3	A system of generators of $\langle a_1, a_2, a_3\rangle/p$
2	0	0	1	$\langle a_1/2, a_2/2, a_3\rangle$
2	0	1	1	$\langle a_1/2, a_2, a_3, (a_2 + a_3)/2\rangle$
2	1	1	1	$\langle a_1, a_2, a_3, (a_1 + a_2)/2, (a_1 + a_3)/2, (a_2 + a_3)/2\rangle$
3	0	0	1	$\langle a_1/3, a_2/3, a_3\rangle$
3	0	0	2	$\langle a_1/3, a_2/3, a_3\rangle$
3	0	1	1	$\langle a_1/3, a_2, a_3, (2a_3 + a_2)/3, (2a_2 + a_3)/3\rangle$
3	0	1	2	$\langle a_1/3, a_2, a_3, (a_2 + a_3)/3\rangle$
3	0	2	2	$\langle a_1/3, a_2, a_3, (2a_2 + a_3)/3, (a_2 + 2a_3)/3\rangle$
3	1	1	1	$\langle a_1, a_2, a_3, (2a_1 + a_2)/3, (2a_1 + a_3)/3, (2a_2 + a_1)/3, (2a_2 + a_3)/3,$ $(2a_3 + a_1)/3, (2a_3 + a_2)/3, (a_1 + a_2 + a_3)/3\rangle$
3	1	1	2	$\langle a_1, a_2, a_3, (a_1 + a_3)/3, (a_2 + a_3)/3, (2a_1 + a_2)/3, (2a_2 + a_1)/3\rangle$
3	1	2	2	$\langle a_1, a_2, a_3, (a_1 + a_2)/3, (a_1 + a_3)/3, (2a_2 + a_3)/3, (2a_3 + a_2)/3\rangle$
3	2	2	2	$\langle a_1, a_2, a_3, (2a_1 + a_2)/3, (2a_1 + a_3)/3, (2a_2 + a_1)/3, (2a_2 + a_3)/3,$ $(2a_3 + a_1)/3, (2a_3 + a_2)/3, (a_1 + a_2 + a_3)/3\rangle$

THEOREM 1.1. *Suppose* $A = (a_1, a_2, \ldots, a_n) = (pk_1 + t_1, pk_2 + t_2, \ldots, pk_n + t_n)$ *with* $\gcd(A) = 1$, $p \in \mathbb{Z}^+$, $k_i \in \mathbb{N}$, $1 \le t_i \le p - 1$ *for* $1 \le i \le n$ *and* $n \ge 2$. *Let*

$$\mathcal{T}_p = \Big\{(x_1, x_2, \ldots, x_n) \mid 0 \le x_1, x_2, \ldots, x_n \le p - 1, p \,\Big|\, \sum_{i=1}^{n} x_i t_i \ (\neq 0)\Big\}.$$

Then a system of generators of the quotient of the numerical semigroup $\langle A\rangle$ *by* p *is given by*

$$\frac{\langle A\rangle}{p} = \Big\langle a_1, a_2, \ldots, a_n, \frac{1}{p}\sum_{i=1}^{n} x_i a_i \,\Big|\, (x_1, x_2, \ldots, x_n) \in \mathcal{T}_p\Big\rangle.$$

The paper is organised as follows. In Section 2, we introduce MacMahon's partition analysis and the constant term method following [17, 18]. We calculate $\mathrm{RGF}_p(x)$ and obtain a system of generators for $\langle 3k_1 + 1, 3k_2 + 2, 3k_3 + 2\rangle/3$ and $\langle a, a + 1\rangle/(a - 1)$ to illustrate how to use the method. In Section 3, we give the proof of Theorem 1.1.

2. MacMahon's partition analysis

In algebraic combinatorics, MacMahon's partition analysis [9] is one of the tools for solving counting problems connected to linear Diophantine equations and inequalities. Such problems can be transformed into finding the constant term of an Elliott rational function, that is, a rational function whose denominator is a product of binomials. This process has been studied by Andrews *et al.* using computer algebra [1–3]. Algorithms

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have been developed, such as the Omega package [2], the Ell package [17] and the CTEuclid package [18]. We will work with symbolic data.

We introduce some basic definitions and results from [17, 18]. We work in the field $K = \mathbb{Q}((\lambda))((x))$ of double Laurent series. In this field, every rational function has a unique Laurent series expansion, so that the following definition makes sense.

DEFINITION 2.1 [17]. Suppose an element in $K = \mathbb{Q}((\lambda))((x))$ is written as a formal Laurent series $\sum_{i=-\infty}^{\infty} a_i\lambda^i$ in λ, where a_i are elements in $\mathbb{Q}((x))$. Then the constant term operator CT_λ acts by

$$\mathrm{CT}_\lambda \sum_{i=-\infty}^{\infty} a_i\lambda^i = a_0.$$

This definition is extended to CT_Λ for a set of variables $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ in [17]. Here we only need the case $m = 1$.

To work with rational functions in K, we need to clarify their series expansions. A monomial $M = x^k\lambda^\ell \neq 1$ is said to be *small*, denoted $M < 1$, if $k > 0$ or if $k = 0$ and $\ell > 0$, and is said to be *large*, denoted $M > 1$, otherwise. The series expansion for $1/(1 - M)$ in K is

$$\frac{1}{1-M} = \begin{cases} \displaystyle\sum_{k\geq 0} M^k & \text{if } M < 1; \\ \displaystyle\frac{1}{-M(1-1/M)} = -\sum_{k\geq 0}\frac{1}{M^{k+1}} & \text{if } M > 1. \end{cases}$$

To obtain the series expansion of an Elliott rational function E, we write E in its proper form,

$$E = \frac{L}{\prod_{j=1}^{n}(1-M_j)} = L\prod_{j=1}^{n}\Big(\sum_{k\geq 0}(M_j)^k\Big),$$

where L is a Laurent polynomial and each monomial M_j is small. Note that the proper form of E is not unique. For instance, $1/(1 - x) = (1 + x)/(1 - x^2)$ are both proper forms.

2.1. Extracting the constant term. Consider the $\mathrm{RGF}_p(x)$ of a numerical semigroup $\langle A\rangle/p$, where $\langle A\rangle = \langle a_1, a_2, \ldots, a_k\rangle$, $\gcd(A) = 1$ and $p \in \mathbb{Z}^+$. We introduce a new variable λ to replace the linear constraint $pn = c_1a_1 + c_2a_2 + \cdots + c_ka_k$, so that

$$\begin{aligned} \mathrm{RGF}_p(x) &= \sum_{n\geq 0} d(pn; a_1, a_2, \ldots, a_k)x^n \\ &= \sum_{n\geq 0, c_i\geq 0} \mathrm{CT}_\lambda\, \lambda^{c_1a_1+c_2a_2+\cdots+c_ka_k-pn}x^n \\ &= \mathrm{CT}_\lambda \frac{1}{(1-x/\lambda^p)(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})}. \end{aligned} \tag{2.1}$$

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In the third line, we used the sum of a geometric series and the linearity of the CT operator. This gives a power series in x with the powers of λ ranging from $-\infty$ to ∞. Thus, we have represented $\mathrm{RGF}_p(x)$ as the constant term of an Elliott rational function. It follows that $\mathrm{RGF}_p(x)$ is also an Elliott rational function, since by [17, Theorem 3.2], the constant term of an Elliott rational function is still Elliott rational.

REMARK 2.2. By definition, the Frobenius number of $\langle A\rangle/p$ is the greatest integer m with $\mathrm{RGF}_p^{(m)}(0) = 0$, that is,

$$F\Big(\frac{\langle A\rangle}{p}\Big) = \max\{n \in \mathbb{N} \mid d(pn; a_1, a_2, \ldots, a_k) = 0\} = \max\{m \in \mathbb{N} \mid \mathrm{RGF}_p^{(m)}(0) = 0\}.$$

To extract the constant term, we use partial fraction decompositions of univariate rational functions, from which the constant term can be read off. To this end, we write

$$E = \frac{L(\lambda)}{\prod_{i=1}^{n}(1 - u_i\lambda^{a_i})}, \tag{2.2}$$

where $L(\lambda)$ is a Laurent polynomial, the u_i are free of λ and the a_i are positive integers for all i. Note that we might have $u_i\lambda^{a_i} = x^{-1}\lambda^2 > 1$, so that (2.2) is not a proper form.

PROPOSITION 2.3 [18]. *Suppose that the partial fraction decomposition of E is given by*

$$E = P(\lambda) + \frac{p(\lambda)}{\lambda^k} + \sum_{i=1}^{n} \frac{A_i(\lambda)}{1 - u_i\lambda^{a_i}}, \tag{2.3}$$

where the u_i are free of λ, $P(\lambda)$, $p(\lambda)$ and $A_i(\lambda)$ are all polynomials, $\deg_p(\lambda) < k$, *and* $\deg A_i(\lambda) < a_i$ *for all i. Then*

$$\mathrm{CT}_\lambda E = P(0) + \sum_{u_i\lambda^{a_i}<1} A_i(0),$$

where the sum ranges over all i such that $u_i\lambda^{a_i}$ is small in $\mathbb{Q}((\lambda))((x))$.

We can see that the proposition holds by direct series expansion:

$$\frac{A_i(\lambda)}{1 - u_i\lambda^{a_i}} = \begin{cases} \dfrac{A_i(\lambda)}{1 - u_i\lambda^{a_i}} \xrightarrow{\mathrm{CT}_\lambda} A_i(0) & \text{if } u_i\lambda^{a_i} < 1; \\ \dfrac{A_i(\lambda)}{-u_i\lambda^{a_i}(1 - 1/u_i\lambda^{a_i})} = \dfrac{\lambda^{-a_i}A_i(\lambda)}{-u_i(1 - 1/u_i\lambda^{a_i})} \xrightarrow{\mathrm{CT}_\lambda} 0 & \text{if } u_i\lambda^{a_i} > 1. \end{cases}$$

For clarity, we have written the rational function in its proper form before applying the operator CT_λ.

THEOREM 2.4 [18]. *Let E be as in (2.3). Then $A_s(\lambda)$ is uniquely characterised by*

$$A_s(\lambda) \equiv E \cdot (1 - u_s\lambda^{a_s}) \bmod \langle 1 - u_s\lambda^{a_s}\rangle, \quad deg_\lambda A_s < a_s, \tag{2.4}$$

where $\langle 1 - u_s\lambda^{a_s}\rangle$ denotes the ideal generated by $1 - u_s\lambda^{a_s}$.

To compute $\mathrm{CT}_\lambda E$ for E as in (2.2) in K, we need to compute

$$A_s(0) := \mathcal{A}_{1-u_s\lambda^{a_s}} E = \mathcal{A}_{1-(u_s\lambda^{a_s})^{-1}} E,$$

where $A_s(\lambda)$ is characterised by (2.4). In this new notation, Proposition 2.3 reads

$$\mathrm{CT}_\lambda E = P(0) + \sum_i \chi(u_i\lambda^{a_i} < 1)\mathcal{A}_{1-u_i\lambda^{a_i}} E,$$

where $\chi(\varepsilon) = 1$ if the proposition ε is true and $\chi(\varepsilon) = 0$ if ε is false.

THEOREM 2.5 [18]. *Let E be as in (2.2). If E is proper in λ, that is, the degree in the numerator is less than the degree in the denominator, then*

$$\mathrm{CT}_\lambda E = \sum_{i=1}^{n} \chi(u_i\lambda^{a_i} < 1)\mathcal{A}_{1-u_i\lambda^{a_i}} E. \tag{2.5}$$

If $E|_{\lambda=0} = \lim_{\lambda\to 0} E$ exists, then

$$\mathrm{CT}_\lambda E = E|_{\lambda=0} - \sum_{i=1}^{n} \chi(u_i\lambda^{a_i} > 1)\mathcal{A}_{1-u_i\lambda^{a_i}} E. \tag{2.6}$$

Equation (2.6) is a kind of dual of (2.5). Because of these two formulae, it is convenient to call the denominator factor $1 - u_i\lambda^{a_i}$ *contributing* if $u_i\lambda^{a_i}$ is small and *dually contributing* if $u_i\lambda^{a_i}$ is large. We also write

$$\mathrm{CT}_\lambda \frac{1}{\underline{1 - u_s\lambda^{a_s}}} E(1 - u_s\lambda^{a_s}) = \mathcal{A}_{1-u_s\lambda^{a_s}} E = A_s(0).$$

For this notation, we allow $a_s < 0$. One can think that only the single underlined factor of the denominator contributes when taking the constant term in λ.

LEMMA 2.6. *If E given by (2.2) is proper in λ, that is, the degree in the numerator is less than the degree in the denominator, and $E|_{\lambda=0} = 0$, then*

$$\sum_{s=1}^{n} \mathcal{A}_{1-u_s\lambda^{a_s}} E = 0.$$

2.2. Two examples. In this section, we obtain systems of generators for the numerical semigroups $\langle 3k_1 + 1, 3k_2 + 2, 3k_3 + 2\rangle/3$ and $\langle a, a + 1\rangle/(a - 1)$ by calculating their representation generating functions $\mathrm{RGF}_p(x)$.

PROPOSITION 2.7. *Let* $A = (a_1, a_2, a_3) = (3k_1 + 1, 3k_2 + 2, 3k_3 + 2)$, $k_1, k_2, k_3 \in \mathbb{N}$, *with* $\gcd(A) = 1$. *A system of generators of $\langle A\rangle/3$ is given by*

$$\frac{\langle A\rangle}{3} = \Big\langle a_1, a_2, a_3, \frac{a_1 + a_2}{3}, \frac{a_1 + a_3}{3}, \frac{2a_2 + a_3}{3}, \frac{2a_3 + a_2}{3}\Big\rangle. \tag{2.7}$$

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PROOF. The right-hand side of (2.7) is easily seen to be contained in the left-hand side. To show that the left-hand side is contained in the right-hand side, we compute as follows. By (2.1),

$$\begin{aligned}
\mathrm{RGF}_3(x) &= \mathrm{CT}_\lambda \frac{1}{(1-x/\lambda^3)(1-\lambda^{3k_1+1})(1-\lambda^{3k_2+2})(1-\lambda^{3k_3+2})} \\
&= \mathrm{CT}_\lambda \frac{-1}{\underline{(1-x/\lambda^3)}(1-\lambda^{3k_1+1})(1-\lambda^{3k_2+2})(1-\lambda^{3k_3+2})} \quad \text{(by Theorem 2.5)} \\
&= \mathrm{CT}_\lambda \frac{-1}{\underline{(1-x/\lambda^3)}(1-x^{k_1}\lambda)(1-x^{k_2}\lambda^2)(1-x^{k_3}\lambda^2)} \quad \text{(by Theorem 2.4)} \\
&= \mathrm{CT}_\lambda \frac{1}{(1-x/\lambda^3)\underline{(1-x^{k_1}\lambda)(1-x^{k_2}\lambda^2)(1-x^{k_3}\lambda^2)}} \quad \text{(by Lemma 2.6)} \\
&= \frac{1}{(1-x^{3k_1+1})(1-x^{k_2-2k_1})(1-x^{k_3-2k_1})} \\
&\quad + \mathrm{CT}_\lambda \frac{1}{(1-x/\lambda^3)(1-x^{k_1}\lambda)\underline{(1-x^{k_2}\lambda^2)}(1-x^{k_3}\lambda^2)} \\
&\quad + \mathrm{CT}_\lambda \frac{1}{(1-x/\lambda^3)(1-x^{k_1}\lambda)(1-x^{k_2}\lambda^2)\underline{(1-x^{k_3}\lambda^2)}}.
\end{aligned} \tag{2.8}$$

The second term of (2.8) is

$$\begin{aligned}
&\mathrm{CT}_\lambda \frac{1}{(1-x/\lambda^3)(1-x^{k_1}\lambda)\underline{(1-x^{k_2}\lambda^2)}(1-x^{k_3}\lambda^2)} \\
&= \mathrm{CT}_\lambda \frac{1}{(1-x^{k_2+1}/\lambda)(1-x^{k_1}\lambda)\underline{(1-x^{k_2}\lambda^2)}(1-x^{k_3-k_2})} \quad \text{(by Theorem 2.4)} \\
&= \mathrm{CT}_\lambda \frac{-\lambda x^{-k_2-1}}{(1-\lambda/x^{k_2+1})(1-x^{k_1}\lambda)\underline{(1-x^{k_2}\lambda^2)}(1-x^{k_3-k_2})} \\
&= \mathrm{CT}_\lambda \frac{\lambda x^{-k_2-1}}{\underline{(1-\lambda/x^{k_2+1})(1-x^{k_1}\lambda)}(1-x^{k_2}\lambda^2)(1-x^{k_3-k_2})} \quad \text{(by Lemma 2.6)} \\
&= \frac{1}{(1-x^{k_1+k_2+1})(1-x^{3k_2+2})(1-x^{k_3-k_2})} + \frac{x^{-k_1-k_2-1}}{(1-x^{-k_1-k_2-1})(1-x^{k_2-2k_1})(1-x^{k_3-k_2})}.
\end{aligned}$$

Similarly, the third term of (2.8) is

$$\begin{aligned}
&\mathrm{CT}_\lambda \frac{1}{(1-x^{k_3+1}/\lambda)(1-x^{k_1}\lambda)(1-x^{k_2-k_3})\underline{(1-x^{k_3}\lambda^2)}} \quad \text{(by Theorem 2.4)} \\
&= \mathrm{CT}_\lambda \frac{\lambda x^{-k_3-1}}{\underline{(1-\lambda/x^{k_3+1})(1-x^{k_1}\lambda)}(1-x^{k_2-k_3})(1-x^{k_3}\lambda^2)} \quad \text{(by Lemma 2.6)} \\
&= \frac{1}{(1-x^{k_1+k_2+1})(1-x^{k_3-k_2})(1-x^{3k_3+2})} + \frac{x^{-k_1-k_3-1}}{(1-x^{-k_1-k_3-1})(1-x^{k_3-k_2})(1-x^{k_3-2k_1})}.
\end{aligned}$$

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Therefore, we obtain the representation generating function in the form

$$\begin{aligned}
&\mathrm{RGF}_3(x)\\
&= \frac{1 + (x^{k_1+1} + x^{k_2+k_3+2})(x^{k_2} + x^{k_3}) + x^{2k_1+2}(x^{2k_2} + x^{2k_3}) + x^{k_1+k_2+k_3}(x^{k_1+2} + x^{k_2+k_3+3})}{(1 - x^{3k_1+1})(1 - x^{3k_2+2})(1 - x^{3k_3+2})}\\
&= \frac{1 + x^{(a_1+a_2)/3} + x^{(a_1+a_3)/3} + x^{(2a_2+a_3)/3} + x^{(2a_3+a_2)/3} + x^{2(a_1+a_2)/3} + x^{2(a_1+a_3)/3} + x^{(2a_1+a_2+a_3)/3} + x^{(a_1+2a_2+2a_3)/3}}{(1 - x^{a_1})(1 - x^{a_2})(1 - x^{a_3})}.
\end{aligned}$$

Since $2a_1 + a_2 + a_3 = (a_1 + a_2) + (a_1 + a_3)$ and $a_1 + 2a_2 + 2a_3 = (a_1 + a_2) + (2a_3 + a_2)$, the power of each term in the series expansion of $\mathrm{RGF}_3(x)$ is contained in the right-hand side. This completes the proof. □

PROPOSITION 2.8. *Let $A = (a, a + 1)$, $a \in \mathbb{Z}^+$, $a \geq 3$. A system of generators of $\langle a, a + 1\rangle/(a - 1)$ is given by*

$$\frac{\langle a, a+1\rangle}{a-1} = \begin{cases} \left\langle \dfrac{a+1}{2}, \dfrac{a+3}{2}, \dots, a-1, a \right\rangle & \text{if } a \text{ is odd;}\\[2ex] \left\langle \dfrac{a}{2}+1, \dfrac{a}{2}+2, \dots, a, a+1 \right\rangle & \text{if } a \text{ is even.} \end{cases}$$

PROOF. As in the previous proof, we only need to show that the left-hand side is contained in the right-hand side. By (2.1),

$$\begin{aligned}
\mathrm{RGF}_{a-1}(x) &= \mathrm{CT}_\lambda \frac{1}{(1 - x/\lambda^p)(1 - \lambda^a)(1 - \lambda^{a+1})}\\
&= \mathrm{CT}_\lambda \frac{-1}{\underline{(1 - x/\lambda^{a-1})}(1 - \lambda^a)(1 - \lambda^{a+1})} \quad \text{(by Theorem 2.5)}\\
&= \mathrm{CT}_\lambda \frac{-1}{\underline{(1 - x/\lambda^{a-1})}(1 - x\lambda)(1 - x\lambda^2)} \quad \text{(by Theorem 2.4)}\\
&= \mathrm{CT}_\lambda \frac{1}{(1 - x/\lambda^{a-1})\underline{(1 - x\lambda)(1 - x\lambda^2)}} \quad \text{(by Lemma 2.6)}\\
&= \frac{1}{(1 - x^a)(1 - x^{-1})} + \mathrm{CT}_\lambda \frac{1}{(1 - x/\lambda^{a-1})(1 - x\lambda)\underline{(1 - x\lambda^2)}}.
\end{aligned}$$

The computation of the second term depends on the parity of a. If a is odd, it is

$$\begin{aligned}
&\mathrm{CT}_\lambda \frac{1}{(1 - x^{(a+1)/2})(1 - x\lambda)\underline{(1 - x\lambda^2)}} = \frac{1}{(1 - x^{(a+1)/2})}\Big(1 - \mathrm{CT}_\lambda \frac{1}{\underline{(1 - x\lambda)}(1 - x\lambda^2)}\Big)\\
&\quad = \frac{1}{(1 - x^{(a+1)/2})}\Big(1 - \frac{1}{1 - x^{-1}}\Big) = \frac{1}{(1 - x)(1 - x^{(a+1)/2})}.
\end{aligned}$$

Thus, we obtain

$$\mathrm{RGF}_{a-1}(x) = \frac{1 - x - x^a + x^{(a+3)/2}}{(1 - x)(1 - x^a)(1 - x^{(a+1)/2})} = \frac{1 + x^{(a+3)/2} + x^{(a+5)/2} + \cdots + x^{a-1}}{(1 - x^a)(1 - x^{(a+1)/2})}.$$

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If instead a is even, the second term is

$$\begin{aligned}\mathrm{CT}_\lambda \frac{1}{(1-x^{a/2}/\lambda)(1-x\lambda)\underline{(1-x\lambda^2)}} &= \mathrm{CT}_\lambda \frac{-(\lambda/x^{a/2})}{(1-\lambda/x^{a/2})(1-x\lambda)\underline{(1-x\lambda^2)}} \\ &= \mathrm{CT}_\lambda \frac{\lambda/x^{a/2}}{\underline{(1-\lambda/x^{a/2})(1-x\lambda)}(1-x\lambda^2)} \\ &= \frac{1}{(1-x^{(a+2)/2})(1-x^{a+1})} + \frac{x}{(1-x^{(a+2)/2})(1-x)}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\mathrm{RGF}_{a-1}(x) &= \frac{-x}{(1-x)(1-x^a)} + \frac{1}{(1-x^{(a+2)/2})(1-x^{a+1})} + \frac{x}{(1-x^{(a+2)/2})(1-x)} \\ &= \frac{1+(x^{(a+2)/2}+x^{a+2})(1+x+x^2+\cdots+x^{a/2-2})}{(1-x^a)(1-x^{a+1})}.\end{aligned}$$

The proposition then follows. □

Note that $\langle a, a+1\rangle/(a-1)$ is a half-line numerical semigroup (see [5]). Therefore, its Frobenius number is given by

$$F\Big(\frac{\langle a, a+1\rangle}{a-1}\Big) = \begin{cases} \dfrac{a-1}{2} & \text{if } a \text{ is odd,} \\ \dfrac{a}{2} & \text{if } a \text{ is even.}\end{cases}$$

3. A system of generators of $\langle A\rangle/p$

Let $A = (a_1, a_2) = (pk_1 + t_1, pk_2 + t_2)$, $0 \le t_i \le p-1$, $p, k_1, k_2 \in \mathbb{Z}^+$ and $\gcd(A) = 1$. We can compute $\mathrm{RGF}_p(x)$ for $p = 2, 3, 4, 5$ as in the proof of Proposition 2.7 and obtain a system of generators of $\langle a_1, a_2\rangle/p$. The results agree with those in [15, Proposition 17].

Similarly, for $A = (a_1, a_2, a_3)$, we can obtain $\mathrm{RGF}_p(x)$ for $p = 2, 3$. The corresponding systems of generators are given in Table 1. This table illustrates a pattern summarised in Theorem 1.1. The theorem has a simple direct proof.

Proof of Theorem 1.1. Let $\overline{B} := \langle a_1, a_2, \ldots, a_n, \sum_{i=1}^n x_i a_i/p \mid (x_1, x_2, \ldots, x_n) \in \mathcal{T}_p\rangle$. This is well defined since $(x_1a_1 + x_2a_2 + \cdots + x_na_n)/p \in \mathbb{Z}^+$ by definition of $\mathcal{T}_p$. Let $\overline{A} := \langle A\rangle/p$. The containment $\overline{A} \supseteq \overline{B}$ is obvious and we need to show that $\overline{A} \subseteq \overline{B}$.

If $x \in \overline{A} = \langle A\rangle/p$, then $xp = y_1a_1 + y_2a_2 + \cdots + y_na_n$ for some $y_1, y_2, \ldots, y_n \in \mathbb{N}$. Each y_i is uniquely written as $y_i = m_ip + r_i$ for some $m_i \ge 0$ and $0 \le r_i \le p-1$. Then we have $x = m_1a_1 + m_2a_2 + \cdots + m_na_n + (r_1a_1 + r_2a_2 + \cdots + r_na_n)/p$ and we have $p \mid (r_1a_1 + r_2a_2 + \cdots + r_na_n)$. By the definition of $\mathcal{T}_p$, $(r_1a_1 + r_2a_2 + \cdots + r_na_n)/p$ is either 0 or an element in $\{(x_1a_1 + x_2a_2 + \cdots + x_na_n)/p \mid (x_1, x_2, \ldots, x_n) \in \mathcal{T}_p\}$. In either case, $x \in \overline{B}$. Therefore, $\overline{A} \subseteq \overline{B}$. □

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COROLLARY 3.1. *Suppose that* $A = (a_1, a_2, a_3) = (pk_1 + t_1, pk_2 + t_2, pk_3 + t_3)$ *with* $p \in \mathbb{Z}^+$, $k_i \in \mathbb{N}$ *and* $1 \le t_i \le p - 1$ *for* $1 \le i \le 3$. *If* $\gcd(A) = 1$, *then a system of generators of the quotient of the numerical semigroup* $\langle A\rangle$ *by* p *is given by*

$$\frac{\langle A\rangle}{p} = \left\langle a_1, a_2, a_3, \frac{1}{p}(x_1a_1 + x_2a_2 + x_3a_3) \;\middle|\; (x_1, x_2, x_3) \in \mathcal{T}_p \right\rangle,$$

where

$$\mathcal{T}_p = \{(x_1, x_2, x_3) \mid 0 \le x_1, x_2, x_3 \le p - 1, p \mid (t_1x_1 + t_2x_2 + t_3x_3)(\neq 0)\}.$$

We observe that Theorem 1.1 can be strengthened in the following sense. Suppose that $A = (a_1, \ldots, a_e, a_{e+1}, \ldots, a_n)$, $p \mid a_i$ for $1 \le i \le e$ and $p \nmid a_j$ for $e + 1 \le j \le n$. Then

$$\frac{\langle A\rangle}{p} = \left\langle \frac{a_1}{p}, \frac{a_2}{p}, \ldots, \frac{a_e}{p}, \mathcal{L}_p \right\rangle, \tag{3.1}$$

where $\mathcal{L}_p$ is a system of generators of $\langle a_{e+1}, \ldots, a_n\rangle/p$. We only explain why the left-hand side is contained in the right-hand side because the other containment is trivial. For any $x \in \langle A\rangle/p$, there exists $xp = y_1a_1 + y_2a_2 + \cdots + y_na_n$ for some $y_1, y_2, \ldots, y_n \in \mathbb{N}$ and

$$x = y_1\frac{a_1}{p} + \cdots + y_e\frac{a_e}{p} + \frac{1}{p}(y_{e+1}a_{e+1} + \cdots + y_na_n).$$

Therefore, $x \in \langle a_1/p, a_2/p, \ldots, a_e/p, \mathcal{L}_p\rangle$.

REMARK 3.2. Theorem 1.1 only gives a system of generators of $\langle A\rangle/p$, rather than a minimal system of generators. A minimal system of generators of $\langle a_1, a_2\rangle/p$ is given in [6].

Combining (3.1) and Theorem 1.1, we reobtain the following result.

COROLLARY 3.3 [15, Corollary 18]. *Let* $a_1, a_2, k_1, k_2 \in \mathbb{Z}^+$ *and* $\gcd(a_1, a_2) = 1$. *Then*

$$\frac{\langle a_1, a_2\rangle}{2} = \begin{cases} \left\langle \frac{a_1}{2}, a_2 \right\rangle & \text{if } a_1 = 2k_1, a_2 = 2k_2 + 1, \\ \left\langle a_1, a_2, \frac{a_1 + a_2}{2} \right\rangle & \text{if } a_1 = 2k_1 + 1, a_2 = 2k_2 + 1. \end{cases}$$

Another consequence of Theorem 1.1 is the following result.

COROLLARY 3.4 [15, Corollary 19]. *Let* $a_1, a_2, k_1, k_2 \in \mathbb{Z}^+$ *and* $\gcd(a_1, a_2) = 1$. *If* $a_1 = 3k_1 + 1$, $a_2 = 3k_2 + 1$, *or* $a_1 = 3k_1 + 2$, $a_2 = 3k_2 + 2$, *then*

$$\frac{\langle a_1, a_2\rangle}{3} = \left\langle a_1, a_2, \frac{2a_1 + a_2}{3}, \frac{2a_2 + a_1}{3} \right\rangle.$$

If $a_1 = 3k_1 + 1$, $a_2 = 3k_2 + 2$, *then*

$$\frac{\langle a_1, a_2\rangle}{3} = \left\langle a_1, a_2, \frac{a_1 + a_2}{3} \right\rangle.$$

PROOF. If $(t_1, t_2) = (1, 1)$ or $(t_1, t_2) = (2, 2)$, then $\mathcal{T}_p = \{(1, 2), (2, 1)\}$. If $(t_1, t_2) = (1, 2)$, then $\mathcal{T}_p = \{(1, 1), (2, 2)\}$. This completes the proof. □

4. Future work

Let $s \in \mathbb{N}$, $A = (a_1, a_2, \ldots, a_k)$ and $\gcd(A) = 1$. Komatsu [8] introduced the *s-numerical semigroup* defined by $\langle A; s\rangle = \{n \in \mathbb{N} \mid d(n; a_1, a_2, \ldots, a_k) \geq s + 1\} \cup \{0\}$ and considered its Frobenius number, called the *s-Frobenius number* $F_s(A)$ of A. In other words, $F_s(A)$ is the largest number N satisfying $d(N; a_1, \ldots, a_k) \leq s$. These concepts reduce to the classical one when $s = 0$. It would be of interest to see if our methods can be used to compute these more general Frobenius numbers.

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