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# EQUIVALENCE OF CABLES OF MUTANTS OF KNOTS

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**0. Introduction.** There is the nice formula which links the Alexander polynomial of $(m, k)$-cable of a link with the Alexander polynomial of the link [**5**] [**36**] [**38**]. H. Morton and H. Short investigated whether a similar formula holds for the Jones-Conway (Homfly) polynomial and they found that it is very unlikely. Morton and Short made many calculations of the Jones-Conway polynomial of $(2, q)$-cables along knots (2 was chosen because of limited possibility of computers) and they get very interesting experimental material [**24**], [**25**]. In particular they found that using their method they were able to distinguish some Birman [**4**] and Lozano-Morton [**22**] examples (all which they tried) and the $9_{42}$ knot (in the Rolfsen [**37**] notation) from its mirror image. On the other hand they were unable to distinguish the Conway knot and the Kinoshita-Terasaka knot. Other pairs of mutants were tried with similar results. The above finding of Morton and Short (which the author learned of at Oberwolfach Conference, September 1985) was the motivation for the author to prove (in January 1986) that generally $(2, q)$ cables of mutants of knots are skein equivalent so in particular they have the same Jones-Conway polynomial [**33**]. Independently, and about the same time, the above result was proven by R. Lickorish and A. Lipson [**19**] and generalized by them to the Kauffman polynomial. In April 1986 the similar method was used by H. Morton and P. Traczyk [**26**] to prove a more general result for the Jones polynomial, in particular Morton and Traczyk proved that $(p, q)$-cables of mutually mutant knots have the same Jones polynomial, the fact which does not hold for the Jones-Conway polynomial [**26**] and is expected to be false for the Kauffman polynomial.

There is however one case in which we show that the Morton-Traczyk result still holds for the Jones-Conway and Kauffman polynomials. Namely no new polynomial invariant of links can distinguish $(p, q)$ cables of $K_1 \# K_2$ and $K_1 \# -K_2$ where $-K_2$ denotes the knot $K_2$ with reversed orientation; see Corollary 6.3.

The paper is organized as follows:

The first part contains the preliminary material about cables, satellites and mutants of links. We also include into this part the related result about the Alexander polynomial.

In the second part we review new invariants of links including the Jones, Jones-Conway and Kauffman polynomials.

In the third part we develop general tools which allow us to study equivalence of cables (or satellites) of mutants.

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In the fourth part we prove the Morton-Traczyk result on the Jones polynomial of cables of mutants of knots.

In the fifth part we deal with the Jones-Conway and Kauffman polynomials.

In the sixth part we show that no new invariant of links can distinguish satellites (e.g. cables) of $K_1 \# K_2$ and $K_1 \# -K_2$.

Finally we consider, in the seventh part, some other situations to which the tools of Part 3 can be applied and we discuss problems which arise.

**1. Preliminaries. Cables, satellites, mutants and the Alexander polynomial.** Let $L$ be a link in $S^3$. We denote by $V_L$ a regular neighbourhood of $L$ in $S^3$ and by $M_L$ the link space $S^3$-int $V_L$. If $L$ is a knot then $V_L$ is a solid torus. There is a simple closed curve $m$ on $\partial V_L$ which is not trivial on $\partial V_1$ but bounds a 2-disk in $V_L$. We call $m$ a meridian of the knot $L$. There is a simple closed curve $l$ on $\partial V_L$ which is not trivial on $\partial V_L$ but is nullhomologous in $M_L$. We call $l$ a *longitude* of $L$. If $L$ is oriented, we may assign orientations to $m$ and $l$: a longitude of $l$ is isotopic to $L$ in $V_L$ and we orient it as $L$. A meridian will be oriented in such a way that its linking number $\operatorname{lk}(m,L)$ with $L$ in $S^3$ is +1.

*Definition* 1.1. Let $L$ be a link in a 3-sphere $S^3$ such that $L$ lies in an unknotted solid torus $V \subset S^3$ and $L$ intersects any meridian disk of $V$. Let $K$ be a nontrivial knot in $S^3$, $V_K$ its tubular neighbourhood and $h : V \longrightarrow V_K$ a homeomorphism which maps a longitude of $V$ onto a longitude of $K$. $L_s = h(L)$ is a link called a *satellite* of $K$, and $K$ is its *companion*. The pair $(V, L)$ is the pattern of $L_s$. If $L$ is a link ambient isotopic in $V$ to a $(p,q)$-torus link (on $\partial V$) then $L_s$ is called a $(p,q)$-cable link on $K$. Let $D^2$ be a meridian disk of $V$. The algebraic crossing number of $D^2$ and $L$ is called the winding number of the satellite $L_s$ and the geometric crossing number of $D^2$ and $L$ is called the wrapping number of the satellite $L_s$.

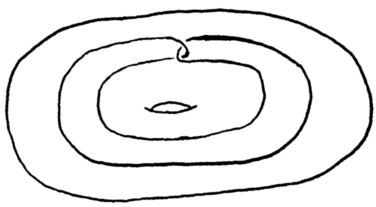


Figure 1.1.

Now let $\Delta_L(t)$ be the (normalized) reduced Alexander polynomial of a link $L$. For our purposes, we use the characterization of $\Delta_L(t)$ found by Alexander [**1**] and extensively used by Conway [**7**]. Namely $\Delta_L(t)$ is uniquely determined by the following conditions:

(1) $\quad \Delta_{T_1}(t) = 1$

where $T_1$ is a trivial knot,

(2) $\quad \Delta_{L_+}(t) - \Delta_{L_-}(t) = (\sqrt{t} - 1/\sqrt{t})\Delta_{L_0}(t)$

where $L_+$, $L_-$, and $L_0$ are diagrams of oriented links which are identical except near one crossing point, where they look like in Fig. 1.2.

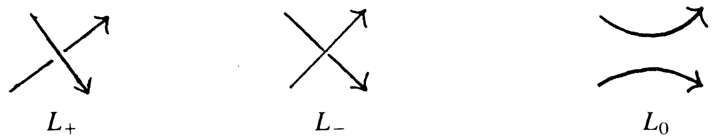


Figure 1.2.

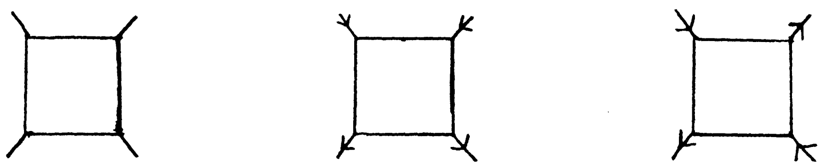
The Alexander polynomial of a satellite $L_s$ can be computed from the Alexander polynomial of the companion $K$, the pattern link $L$ and the linking number $n = \mathrm{lk}(m, L)$ where $m$ is the meridian of $V$ (for pattern $(V, L)$),

THEOREM 1.2. (Burau, Seifert, Fox).

$$\Delta_{L_s}(t) \equiv \Delta_L(t) \cdot \Delta_K(t^n)$$

*where* $\equiv$ *means an equality up to multiplication by* $\pm t^i$.

*Definition* 1.3 [**20**]. (a) A *tangle* is a part of a diagram of a link with two inputs and two outputs (Fig. 1.3(a)). It depends on an orientation of the diagram which arcs are inputs and which ones are outputs. For oriented diagrams we distinguish tangles with neighbouring inputs (Fig. 1.3(b)) and alternated tangles (Fig. 1.3(c)).



(a) (b) tangle with neighbouring inputs (c) alternated tangle

Figure 1.3.

(b) Let $L_1$ and $L_2$ be oriented diagrams of links. Then $L_2$ is a mutation of $L_1$ if $L_2$ can be obtained from $L_1$ by the following process (Fig. 3.2):

(i) remove from $L_1$ an inhabitant $T$ of a tangle $B$,

(ii) rotate $T$ through angle $\pi$ about the central axis (perpendicular to the plane of the diagram) ($\tau$-mutation) or about the horizontal ($\rho$-mutation) or vertical ($\delta$-mutation) of the tangle and if necessary change the orientation of $T$ (so that inputs and outputs are preserved).

(iii) place the new inhabitant into the tangle $B$ to get $L_2$.

It follows immediately from the properties (1) and (2) of the Alexander polynomial that if $L_2$ is a mutant of $L_1$ then

$$\Delta_{L_2}(T) = \Delta_{L_1}(t)$$

therefore one gets from Theorem 1.2,

COROLLARY 1.4. *If a knot $L_2$ is a mutant of a knot $L_1$ and $L_{s_1}$ and $L_{s_2}$ are satellites of $L_1$ and $L_2$ respectively with the same pattern $(V, K)$ then*

$$\Delta_{L_{s_2}}(t) \equiv \Delta_{L_{s_1}}(t).$$

In Part 4, we will show (following [**26**]) that the same holds for the Jones polynomial.

**2. New invariants of links;** $(0, \infty)$**, skein and Kauffman equivalences.** If $p$ is a crossing of oriented components (component) of a diagram of a link, we associate with $p$ the sign according to the convention of Fig. 2.1.


Figure 2.1.

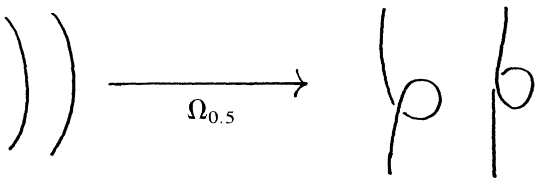
The twist (or write) number of an oriented diagram, tw( ), is defined to be the sum of the signs over all crossings. For selfcrossings the sign does not depend on orientation so the twist number is well defined even for an unoriented knot diagram.

Consider four diagrams of links, $L_+, L_-, L_0$, and $L_\infty$ which are identical except near one crossing point where they look like in Fig. 2.2.


Figure 2.2.

We distinguish between $L_+$ and $L_-$, and $L_0$ and $L_\infty$ whenever the sign of the crossing is well defined. We will work mostly with diagrams of links which are either oriented (all components) or nonoriented (no component is oriented), but to allow more general approach we will allow also partially oriented diagrams (i.e., some components are oriented and some are not).

Two diagrams of links are (ambient) isotopic if and only if (due to Reidemeister theorem) one can go from one to another using Reidemeister moves of type $\Omega_1^{\mp}$, $\Omega_2^{\mp}$, $\Omega_3^{\mp}$ (see Fig. 2.3). To define his polynomial, *L*. Kauffman used weaker relation than (ambient) isotopy; namely regular isotopy which allows only Reidemeister moves of type $\Omega_2^{\mp}$, $\Omega_3^{\mp}$. In fact the most natural is equivalent relation on link diagrams which is somewhere between (ambient) isotopy and regular isotopy, we call it weak regular isotopy. We say that two link diagrams are *weakly regular isotopic* if one can go from one to another using Reidemeister moves of type $Q_2^{\mp 1}$, $Q_3^{\mp 1}$ and the weak first Reidemeister moves of type $Q_{0.5}^{\mp 1}$ (Fig. 2.3). The twist number, tw( ), and the (global) linking number are the invariants of weak regular isotopy of oriented diagrams and it is easy to verify [**39**], [**31**] that two oriented diagrams are weakly regular isotopic if and only if they are (ambient) isotopic and have the same twist number. Similarly the selftwist number st( ) = tw( ) −2lk( ) is the invariant of weak regular isotopy of unoriented (and partially oriented) diagrams and two partially oriented diagrams are isotopic if and only if they are (ambient) isotopic and have the same selftwist number.




($\Omega_{0.5}$ creates the pair of circles of the opposite sign)

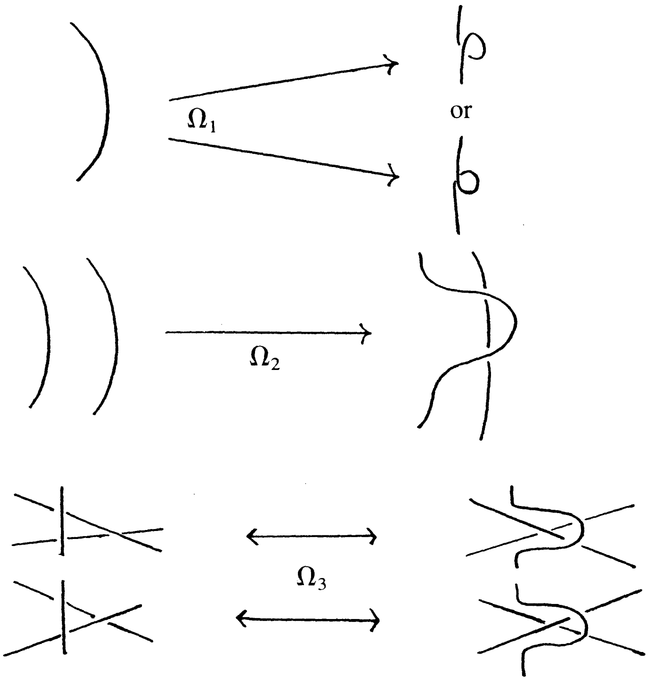



Figure 2.3.

Most generally we will consider invariants of weak regular isotopy classes of partially oriented link diagrams which have the property that the value of the invariant for $L_+$ is uniquely determined by the value of the invariant for $L_-$, $L_0$ and $L_\infty$ and the value of the invariant for $L_-$ is uniquely determined by the value of the invariant for $L_+$, $L_0$ and $L_\infty$. We call such invariants the *invariants of Kauffman type* and if two partially oriented link diagrams cannot be distinguished by any invariant of Kauffman type we say that they are *Kauffman equivalent* (see **[31]** for more details). The Kauffman polynomial, $\Lambda_L(a,x)$, is an example of a Kauffman type invariant. It is defined uniquely by (see Fig. 2.5)

(i) $\Lambda_{T_1}(a,x) = 1$

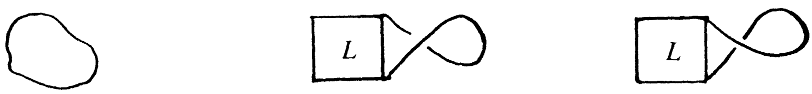
(ii) $\Lambda_{Lp.\mathrm{tw}}(a,x) = a\Lambda_L(a,x), \quad \Lambda_{Ln.\mathrm{tw}}(a,x) = a^{-1}\Lambda_L(a,x)$,

(iii) $\Lambda_{L+}(a,x) + \Lambda_{L-}(a,x) = x\Lambda_{L_0}(a,x) + x\Lambda_{L_\infty}(a,x)$.

The polynomial $\Lambda_L(a,x)$ can be easily modified to the Kauffman polynomial of oriented links, $F_L(a,x)$ :

$$F_L(a,x) = a^{-\mathrm{tw}(L)}\Lambda_L(a,x) \quad \textbf{[16]}.$$

We can consider invariants of weak regular isotopy classes of unoriented link diagrams which have the property that the values of the invariant for $L_{\mathrm{cr}}$ is uniquely determined by the value of the invariant for $L_{\mathrm{hor}}$ and $L_{\mathrm{ver}}$ ($L_{\mathrm{cr}}, L_{\mathrm{hor}}$ and $L_{\mathrm{ver}}$ denote the link diagrams which are identical except near one crossing point where they look like in Fig. 4.3.)




$T_1$ $L_{\mathrm{p.tw}}$ $L_{\mathrm{n.tw}}$

Figure 2.5.

We call such invariants the invariants of $(0,\infty)$ *type* and if two unoriented link diagrams cannot be distinguished by any invariant of $(0,\infty)$ type we say that they are $(0,\infty)$ *equivalent*. The Kauffman bracket polynomial, $\langle\ \rangle$, (i.e., the unoriented version of the Jones polynomial) is an example of a $(0,\infty)$-invariant (see **[13]**, **[17]**). It is defined uniquely by

(i) $\langle T_1 \rangle = 1$

(ii) $\langle L_{p.\mathrm{tw}} \rangle = -A^3\langle L \rangle, \quad \langle L_{n.\mathrm{tw}} \rangle = -A^{-3}\langle L \rangle$,

(iii) $\langle L_{\mathrm{cr}} \rangle = A\langle L_{\mathrm{hor}} \rangle + A^{-1}\langle L_{\mathrm{ver}} \rangle$.

It can be easly modified to the invariant of oriented links

$$f_L(A) = (-A^3)^{-\mathrm{tw}(L)}\langle L \rangle.$$

After substitution $A = -t^{-1/4}$ one gets the Jones polynomial; i.e.,

$$V_L(t) = f_L(-t^{-1/4}).$$

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Finally if we consider invariant of (ambient) isotopy of oriented links which have the property that the value of the invariant for $L_+$ (resp. $L_-$) is uniquely determined by the value of the invariant for $L_-$ and $L_0$ (resp. $L_+$ and $L_0$) we get skein type invariants. If two oriented links cannot be distinguished by any skein type invariant we say that they are skein equivalent (see [7], [**11**], [**20**], [**34**] or [**31**]). The Jones-Conway polynomial $P_L(a, z)$ (named also skein, twisted Alexander, generalized Jones, 2-variable Jones, Homfly or Flypmoth) is an example of a skein type invariant. It is defined uniquely by:

(i) $P_{T_1}(a, z) = 1$
(ii) $aP_{L_+}(a, z) + a^{-1}P_{L_-}(a, z) = zP_{L_0}(a, z)$

(see [**10**], [**20**], [**12**], [**30**], [**34**], [**9**]).

We will mean by new knot invariants, skein type invariants, $(0, \infty)$-invariants or invariants of Kauffman type. For a given link we will consider binary or trinary trees which can be used to compute an invariant of the link from the invariants of links at leaves of the tree. Depending on the invariant we will consider binary $(0, \infty)$-tree, binary skein-tree and trinary Kauffman tree. For example, the situation at each node (except leaves) of $(0, \infty)$-tree looks like in Fig. 2.4.

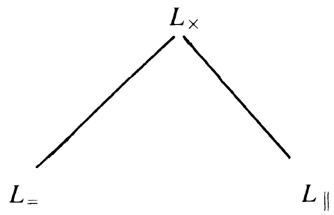



Figure 2.4.

### 3.1. Parallel and symmetric $n$-rooms.

*Definition* 3.1. An $n$-room with $(p_1, p_2, p_3, p_4)$-corners (where $p_1 + p_2 + p_3 + p_4 = 2n, p_i \geqq 0$) is a part of a diagram of a link with $n$ inputs and $n$ outputs distributed in such a way that $p_1$ of them lie in the north-west corner of the $n$-room, $p_2$ in the north-east corner, $p_3$ in the south-east corner and $p_4$ in the south-west corner of the $n$-room (see Fig. 3.1).

*Definition* 3.2. (a) Let $A$ be an $n$-room. We denote by $\rho(A), \delta(A), \tau(A)$ the $n$-room obtained from $A$ by rotating it through angle 180° about one of the three axes as shown in Fig. 3.2 (compare Definition 1.3).

(b) We say that an $n$-room is of $\rho$ (resp. $\delta$ or $\tau$) type if $p_1 = p_2$, $p_3 = p_4$ (resp. $p_1 = p_4$, $p_2 = p_3$, or $p_1 = p_3$, $p_2 = p_4$) and in the case of oriented room $\rho$ (resp. $\delta$ or $\tau$) sends all inputs to inputs or all inputs to outputs.

(c) Let $L$ be a diagram of a link and $A$ its $n$-room of type $\rho$ (resp. $\delta$ or $\tau$). Then we denote by $\rho(L)$ (resp. $\delta(L)$ or $\tau(L)$) the link obtained from $L$ by $\rho$ (resp.

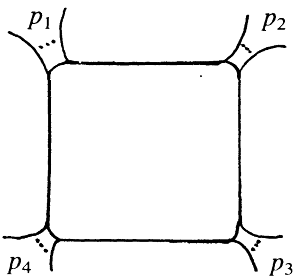



Figure 3.1.

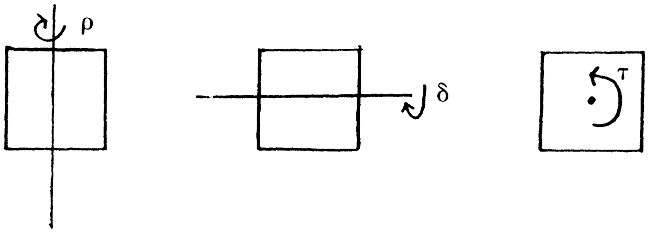



Figure 3.2.

$\delta$ or $\tau$) operation on $A$. If necessary we change the orientation of $A$ so that inputs and outputs are preserved.

*Definition* 3.3. If $A$ is a $n$-room of a link diagram $L$ then $A'$— the complement of $A$ is also an $n$-room. We say that $A'$ is a *$\rho$-parallel* $n$-room if strings of N-W corner of $A$ are joined in $A'$ with strings of N-E corner of $A$ by $p_1 = p_2$ parallel (possibly twisted) strings (there are possibly other components which do no interact with parallel strings) and S-W corner is joined with S-E corner by $p_3 = p_4$ parallel strings. In particular, $a_i$ is joined with $\rho(a_i)$ for each $i \leqq 2n$ (see Fig. 3.3). If $A'$ is a $\rho$-parallel $n$-room then $A$ is an $n$-room of type $\rho$. Similarly we define $\delta$-parallel $n$-room (we join N-W corner with S-W corner and N-E corner with S-W corner) and $\tau$-parallel $n$-room (we join N-W corner with S-E corner and N-E with S-W), see Fig. 3.3.

Now we will consider a link diagram $L$ composed of two $n$-rooms $A$ and $A'$ where $A'$ is a $\rho$-parallel $n$-room (resp. $\delta$-parallel or $\tau$-parallel) and we will compare $L$ with $\rho(L)$ (resp. $\delta(L)$ or $\tau(L)$).

Now for simplicity (but without a loss of generality) we will concentrate on $\rho$. Our strategy is the following: we build a binary (or trinary, depending on which invariant is considered) tree for $L$ modifying $L$ only in $A$. Simultaneously we can build the tree for $\rho(L)$. Of course the leaves of trees for $L$ and $\rho(L)$ are still related by $\rho$. Now we try to choose leaves so that either they are unchanged by $\rho$ or one can reduce the number of strings involved and proceed by induction.

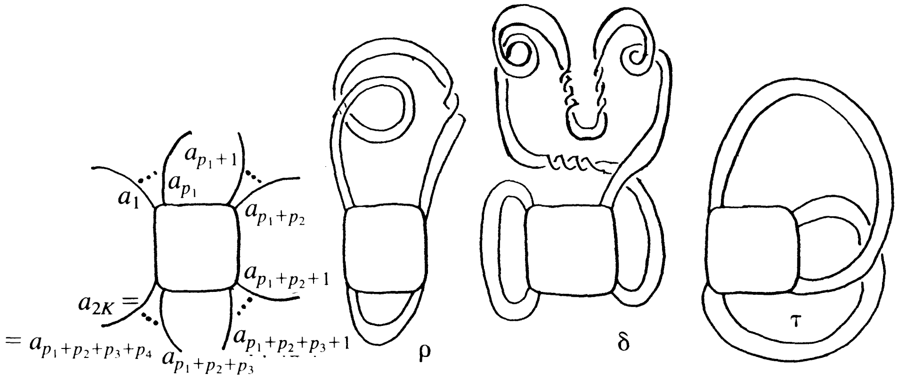



Figure 3.3.

We need more definitions to make the above idea precise.

We have the natural partial ordering of 4-tuples $(p_1, p_2, p_3, p_4)$ where $p_i \geqq 0$. Namely $(p'_1, p'_2, p'_3, p'_4) \leqq (p_1, p_2, p_3, p_4)$ if and only if $p'_i \leqq p_i$ for each $i$. Furthermore our set of 4-tuples has a unique minimal element (0, 0, 0, 0).

*Definition* 3.4. (a) Let $A$ be an $n$-room of type $\rho$. We say that $A$ is *$\rho$-symmetric* if $\rho(A) = A$.

(b) If a link diagram $L$ is composed of an $n$-room $A$ of type $\rho$ and its complement $A'$ then we say that $A$ is *almost $\rho$-symmetric modulo $A'$* if $L$ and $\rho(L)$ are (ambient) isotopic (in fact they are then weakly regular isotopic becauuse $\rho$ does not change the twist number).

(c) If a link diagram $L$ is composed of an $n$-room $A$ of type $\rho$ with $(p_1, p_2, p_3, p_4)$-corners, and its complement $\rho$-parallel $n$-room $A'$, then $A$ is called *reducible* (more precisely *reducible with respect to $A'$*) if $L$ is weakly regular isotopic to a link diagram $L^*$ composed of an $m$-room $A^*$ of type $\rho$ with $(p^*_1, p^*_2, p^*_3, p^*_4)$ corners, and $\rho$-parallel $m$-room $A^{*\prime}$ and $(p^*_1, p^*_2, p^*_3, p^*_4) < (p_1, p_2, p_3, p_4)$, and furthermore $\rho(L)$ is weakly regular isotopic to $\rho(L^*)$.

THEOREM 3.5. *Let $L$ be a link diagram composed of an $n$-room $A$ of type $\rho$ with $(p_1, p_2, p_3, p_4)$-corners and its complement $\rho$-parallel $n$-room $A'$. Assume that $L$ has binary skein (resp. binary $(0, \infty)$ or trinary Kauffman) tree whose leaves are composed of $A'$ and an $n$-room $B$ of type $\rho$ with $(p_1, p_2, p_3, p_4)$ corners which is either almost symmetric modulo $A'$ or is reducible with respect to $A'$. Additionally assume that for each link diagram $L^*$ composed of an $n^*$-room $A^*$ of type $\rho$ with $(p^*_1, p^*_2, p^*_3, p^*_4)$-corners where*

$$(p^*_1, p^*_2, p^*_3, p^*_4) < (p_1, p_2, p_3, p_4)$$

*and $\rho$-parallel $n^*$-room $A^{*\prime}$ one has $L^*$ skein equivalent (resp. $(0, \infty)$ equivalent or Kauffman equivalent) to $\rho(L^*)$. Then $L$ is skein equivalent (resp. $(0, \infty)$ equivalent or Kauffman equivalent) to $\rho(L)$. The same holds when we change $\rho$ to $\delta$ or $\tau$.*

*Proof.* If $(p_1, p_2, p_3, p_4) = 0$ then obviously $L$ is isotopic to $\rho(L)$. Now assumptions of Theorem 3.5 are formulated in such a way that the inductive step can be performed (if $L^*$ is equivalent to $\rho(L^*)$ for

$$(p_1^*, p_2^*, p_3^*, p_4^*) < (p_1, p_2, p_3, p_4)$$

then $L$ is equivalent to $\rho(L)$).

So Theorem 3.5 is obvious but very important. In specific cases we will have to check whether the hypothesis of Theorem 3.5 is satisfied.

In the case of $(0, \infty)$-equivalence we will show, after [**26**], that no assumptions on $(p_1, p_2, p_3, p_4)$ are needed. For other equivalences we need some restrictions (and they are essential).

## 4. $(0, \infty)$ equivalence and the Jones polynomial.

THEOREM 4.1 ([**26**]). *Let $L$ be a link diagram composed of an $n$-room $A$ of type $\rho$ (resp. $\delta$ or $\tau$) with $(p_1, p_2, p_3, p_4)$-corners and its complement a $\rho$-parallel (resp. $\delta$ or $\tau$-parallel) $n$-room $A'$. Then $L$ and $\rho(L)$ (resp. $\delta(L)$ or $\tau(L)$) are* $(0, \infty)$ *equivalent.*

*Proof.* Without loss of generality we can limit ourselves to the case of $\rho$. We use Theorem 3.5 for $(0, \infty)$-equivalence. Therefore we have to verify the hypothesis of Theorem 3.5 that $L$ has a binary $(0, \infty)$ tree whose leaves are the same as $L$ outside $A$ and which are either almost symmetric modulo $A'$ or reducible with respect to $A'$. We can assume that any leaf (say $L^* = A^* \cup A'$) has no crossings in $A^*$. We can ignore trivial components in $A^*$ because they also appear in $\rho(A^*)$.

Now assume that $A^*$ is not reducible. Then no string of $A^*$ starts and ends in the same corner. Then $A^*$ looks as in Fig. 4.1 so is symmetric, therefore almost symmetric.

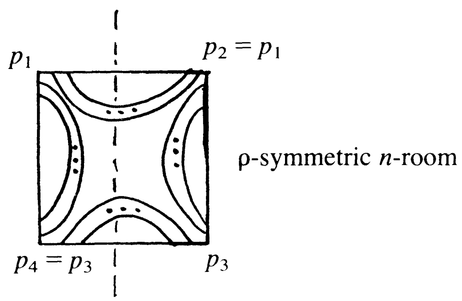



Figure 4.1.

COROLLARY 4.2. *Let $L$ be a link diagram and $A$ its tangle with complementary tangle $A'$. Assume that the mutation $\rho$ (resp. $\delta$ or $\tau$) preserves components of $L$ that is if $a_0$ is a point of $A \cap A' \cap L$ then $a_0$ and $\rho(a_0)$ (resp. $\delta(a_0)$ or $\tau(a_0)$) lie on* *the same component of L. Then for any satellite $L_s$ of L (i.e., we take a satellite for each component of L) and the corresponding satellite $(\rho(L))_s$ (resp. $(\delta(L))_s$ or $(\tau(L))_s$), L and $(\rho(L))_s$ (resp. $(\delta(L))_s$ or $(\tau(L))_s$) are $(0,\infty)$-equivalent.*

*Proof.* If $a_0$ and $\rho(a_0)$ are on the same component of $A'$ then one can find a diagram of $L_s$ which can be divided naturally (reflecting division $L = A \cup A'$) into the $n$-room $A_s$ and $\rho$-parallel room $A'_s$. Furthermore $(\rho(L))_s = \rho(L_s)$. Then we apply Theorem 4.1. If $a_0$ and $\rho(a_0)$ are on the same component of $A$ then we use the same argument exchanging the roles of $A$ and $A'$. A similar argument holds for $\delta$ and $\tau$. Finally consider the case when $a_0$ and $\rho(a_0)$ are on the same component of $L$ but they are not on the same component neither of $A$ nor of $A'$ (see Fig. 4.2).

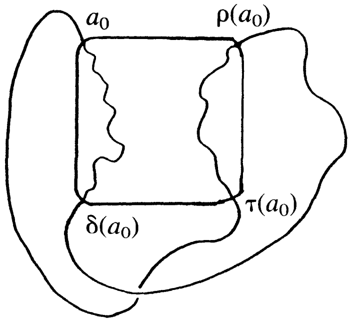



Figure 4.2.

Then either (i) $a_0$ and $\delta(a_0)$ are on the same component of $A$ and $a_0$ and $\tau(a_0)$ are on the same component of $A'$, or (ii) $a_0$ and $\tau(a_0)$ are on the same component of $A$ and $a_0$ and $\delta(a_0)$ are on the same component of $A'$. We will consider the case (i) ((ii) is analogous). Now by the first part of the proof $L_s$ is $(0,\infty)$-equivalent to $(\delta(L))_s$ and $(\delta(L))_s$ is $(0,\infty)$-equivalent to $(\tau \cdot \delta(L))_s$. We perform $\delta$ on $A$ and $\tau$ on $A'$ but up to weak regular isotopy we get the same diagram as performing $\rho = \tau\delta$ on $A$ so $L_s$ is $(0,\infty)$-equivalent to $(\rho(L))_s$. A similar argument holds for $\delta$ and $\tau$.

COROLLARY 4.3. [**26**] *Let $K_1$ and $K_2$ be satellites (e.g. $(p,q)$-cables) of oriented knot K and its mutant (constructed in the same way). Then for the Jones polynomial $V(t)$: $V_{K_1}(t) = V_{K_2}(t)$.*

*Proof.* By Corollary 4.2 $K_1$ and $K_2$ are $(0,\infty)$-equivalent as unoriented knots and additionally they have the same twist numbers so they have the same Jones polynomial.

COROLLARY 4.4. *Let knots $K_1$ and $K_2$ be satellites of oriented knot K and its mutant (with the same pattern). Then they have the same signature ($\sigma(K_1) = \sigma(K_2)$).*

*Proof.* To prove Corollary 4.4 and its extension analogous to Theorem 4.1 it is enough to show that the signature of knots is an invariant of $(0,\infty)$-equivalent classes. In fact the following, more general theorem holds.

THEOREM 4.5. *Let L be an unoriented link diagram with the Kauffman bracket* $\langle L\rangle \neq 0$ *for* $A^2 = i$, *then* $\hat{\sigma}(L) = \sigma(L) + \mathrm{lk}(L)$ *is a* $(0,\infty)$*-equivalence classes invariant (it was shown by Murasugi* [**29**] *that* $\hat{\sigma}(L)$ *does not depend on an orientation of L).*

*Proof.* It is useful to consider one more variation of the Jones polynomial. Namely $\hat{V}_L(t) \in Z[t^{\mp 1/2}]$ is defined by

$$\hat{V}_L(t) = t^{-\frac{3}{2}\mathrm{lk}(L)} V_L(t)$$

or equivalently for $A = -t^{-1/4}$ it is equal to

$$(-A^3)^{-\mathrm{st}(L)}\langle L\rangle \quad \text{where } \mathrm{st}(L) = \mathrm{tw}(L) - 2\mathrm{lk}(L).$$

$\hat{V}_L(t)$ is an invariant of (global) isotopy of unoriented links. Further let us define the unoriented determinant of $L$ by $\hat{D}_L = \hat{V}_L(t)$ for $\sqrt{t} = i (t = -1)$. Of course for knots $\hat{D}_L \neq 0$. The following fact easily follows from [7] and [**29**]

$$\frac{\hat{D}_L}{|\hat{D}_L|} = i^{\hat{\sigma}(L)} \quad \text{for } \hat{D}_L \neq 0, \tag{4.6}$$

in particular $\hat{D}_L \in R \cup iR$. Now to prove Theorem 4.5 we need only the following lemma.

LEMMA 4.7. *Let*

$$\check{\sigma}(L) = \begin{cases} \hat{\sigma}(L) & \text{if } \hat{D}_L \neq 0 \\ \infty & \text{if } \hat{D}_L = 0 \end{cases}$$

*and* $c(L)$ *denote the number of components of L, then if one knows,* $\hat{D}_{L_{\mathrm{hor}}}$, $\check{\sigma}(L_{\mathrm{hor}})$, $\mathrm{st}(L_{\mathrm{hor}})$, $c(L_{\mathrm{hor}})$, $\hat{D}_{L_{\mathrm{ver}}}$, $\check{\sigma}(L_{\mathrm{ver}})$, $\mathrm{st}(L_{\mathrm{ver}})$, *and* $c(L_{\mathrm{ver}})$ *then one can find* $\hat{D}_{L_{\mathrm{cr}}}$, $\check{\sigma}(L_{cr})$, $\mathrm{st}(L_{\mathrm{cr}})$ *and* $c(L_{\mathrm{cr}})$.

*Proof.* For $c(L_{\mathrm{cr}})$, $\mathrm{st}(L_{\mathrm{cr}})$ and $\hat{D}_{L_{\mathrm{cr}}}$ this is an easy task. Namely:

$$c(L_{\mathrm{cr}}) = \begin{cases} c(L_{\mathrm{hor}}) + 1 & \text{if } c(L_{\mathrm{hor}}) = c(L_{\mathrm{ver}}) \\ \min(c(L_{\mathrm{hor}}), c(L_{\mathrm{ver}})) & \text{if } c(L_{\mathrm{hor}}) \neq c(L_{\mathrm{ver}}) \end{cases}$$

$$\mathrm{st}(L_{\mathrm{cr}}) = \begin{cases} \frac{\mathrm{st}\, L_{\mathrm{hor}} + \mathrm{st} L_{\mathrm{ver}}}{2} & \text{if } c(L_{\mathrm{cr}}) > c(L_{\mathrm{hor}}) \\ 2\mathrm{st}\, L_{\mathrm{hor}} - \mathrm{st} L_{\mathrm{ver}} + 1 & \text{if } c(L_{\mathrm{hor}}) > c(L_{\mathrm{ver}}) \\ 2\mathrm{st}\, L_{\mathrm{ver}} - \mathrm{st} L_{\mathrm{hor}} - 1 & \text{if } c(L_{\mathrm{ver}}) > c(L_{\mathrm{hor}}) \end{cases}$$

$$\hat{D}_{L_{\mathrm{cr}}} = i^{\frac{1}{2}(\operatorname{st} L_{\mathrm{hor}} - \operatorname{st} L_{\mathrm{cr}} - 1)} \hat{D}_{L_{\mathrm{hor}}} + i^{\frac{1}{2}(\operatorname{st} L_{\mathrm{ver}} - \operatorname{st} L_{\mathrm{cr}} + 1)} \hat{D}_{L_{\mathrm{ver}}}.$$

Now we will work with the signature. If $\hat{D}_{L_{\mathrm{cr}}} = 0$ then

$$\check{\sigma}(L_{\mathrm{cr}}) = \infty,$$

so assume $\hat{D}_{L_{\mathrm{cr}}} \neq 0$. Then, formula 4.6 allows us to find $\check{\sigma}(L_{\mathrm{cr}})(= \hat{\sigma}(L_{\mathrm{cr}}))$ modulo 4. We will use the following known formulas (see [**7**], [**29**], [**35**] or [**31**]):

$$(4.8) \qquad \begin{aligned} &(a) \quad |\sigma(L_{\mp}) - \sigma(L_0)| \leqq 1 \\ &(b) \quad \text{if } \hat{D}_{L_{\mp}} \neq 0 \text{ but } \hat{D}_{L_0} = 0 \text{ then } \sigma(L_{\mp}) = \sigma(L_0). \end{aligned}$$

Now we have to consider two main cases:

I. $c(L_{\mathrm{hor}}) = c(L_{\mathrm{ver}})$, that is, two different components are involved in the crossing of $L_{\mathrm{cr}}$. From 4.8(a) it follows easily that:

$$(4.9) \qquad \begin{cases} |\hat{\sigma}(L_{\mathrm{cr}}) - \hat{\sigma}(L_{\mathrm{hor}}) + \frac{1}{2}(\operatorname{st} L_{\mathrm{cr}} - \operatorname{st} L_{\mathrm{hor}} - 1)| \leqq 1 \\ |\hat{\sigma}(L_{\mathrm{cr}}) - \hat{\sigma}(L_{\mathrm{ver}}) + \frac{1}{2}(\operatorname{st} L_{\mathrm{cr}} - \operatorname{st} L_{\mathrm{ver}} + 1)| \leqq 1. \end{cases}$$

Now because either $\hat{D}_{L_{\mathrm{hor}}}$ or $\hat{D}_{L_{\mathrm{ver}}}$ is different from zero so either

$$\check{\sigma}(L_{\mathrm{hor}}) = \hat{\sigma}(L_{\mathrm{hor}}) \quad \text{or} \quad \check{\sigma}(L_{\mathrm{ver}}) = \hat{\sigma}(L_{\mathrm{ver}})$$

and using 4.9 one can determine $\check{\sigma}(L_{\mathrm{cr}})$.

II. $c(L_{\mathrm{hor}}) \neq c(L_{\mathrm{ver}})$; that is $L_{\mathrm{cr}}$ has the self-crossing. Now one has two possibilities

$$c(L_{\mathrm{hor}}) < c(L_{\mathrm{ver}}) \quad \text{and} \quad c(L_{\mathrm{hor}}) > c(L_{\mathrm{ver}}).$$

Let us concentrate on the latter (the first being analogous). Now either

$$\text{(i)} \quad \hat{D}_{L_{\mathrm{hor}}} \neq 0 \quad \text{so } \check{\sigma}(L_{\mathrm{hor}}) = \hat{\sigma}(L_{\mathrm{hor}}).$$

Because 4.8(a) gives us

$$|\hat{\sigma}(L_{\mathrm{cr}}) - \hat{\sigma}(L_{\mathrm{hor}}) + \frac{1}{2}(\operatorname{st} L_{\mathrm{cr}} - \operatorname{st} L_{\mathrm{hor}} - 1)| \leqq 1$$

therefore we can determine $\check{\sigma}(L_{\mathrm{cr}})$, or

$$\text{(ii)} \quad \hat{D}_{L_{\mathrm{hor}}} = 0 \quad (\text{so } \hat{D}_{L_{\mathrm{ver}}} \neq 0).$$

then 4.8(b) gives us

$$(4.10) \qquad \hat{\sigma}(L_{\mathrm{cr}}) = \hat{\sigma}(L_{\mathrm{hor}}) + \frac{1}{2}(\operatorname{st} L_{\mathrm{hor}} - \operatorname{st} L_{\mathrm{cr}} + 1).$$

On the other hand we can write $L_{\text{ver}}$ as $L_{\text{dver}}$ (see Fig. 4.3) and then the smoothing of a new crossing of $L_{\text{dver}}$ (with any orientation of $L_{\text{dver}}$) gives us again $L_{\text{hor}}$. We can use 4.8(b) to get (see Fig. 4.3):

$$\text{(4.11)} \quad \begin{aligned} \hat{\sigma}(L_{\text{ver}}) = \hat{\sigma}(L_{\text{dver}} &= \hat{\sigma}(L_{\text{htw}}) + \frac{1}{2}(\operatorname{st} L_{\text{htw}} - \operatorname{st} L_{\text{dver}} + 1) \\ &= \hat{\sigma}(L_{\text{hor}}) + \frac{1}{2}(\operatorname{st} L_{\text{hor}} - \operatorname{st} L_{\text{ver}}). \end{aligned}$$

Combining 4.10 and 4.11, we get

$$\text{(4.12)} \quad \hat{\sigma}(L_{\text{cr}}) = \hat{\sigma}(L_{\text{ver}}) + \frac{1}{2}(\operatorname{st} L_{\text{ver}} - \operatorname{st} L_{\text{cr}} + 1)$$

so we can determine $\check{\sigma}(L_{\text{cr}})$. This completes the proof of Lemma 4.7 and Theorem 4.5.

Theorem 4.5, Lemma 4.7 and Corollary 4.4 likely can be extended by the same method to any supersignature associated with the Jones polynomial (see [**35**] and [**31**, Theorem 4.20]).

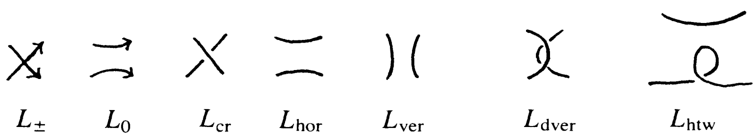



Figure 4.3.

## 5. Skein equivalence and the Jones-Conway polynomial, Kauffman equivalence and the Kauffman polynomial of satellites of mutants.

THEOREM 5.1. *Let $L$ be a link diagram composed of an $n$-room $A$ of type $\rho$ (resp. $\delta$ or $\tau$) with $(p_1, p_2, p_3, p_4)$-corners*

$$((p_1, p_2, p_3, p_4) \leqq (2, 2, 2, 2))$$

*and its complement a $\rho$-parallel (resp. $\delta$ or $\tau$-parallel) $n$-room $A$. Then $L$ and $\rho(L)$ (resp. $\delta(L)$ or $\tau(L)$) are Kauffman and skein equivalent.*

The following example of Lickorish and Lipson [**19**] shows that the assumptions of Theorem 5.1 are essential i.e., if $A$ and $A'$ are of type $\delta$ then $L$ and $\rho(L)$ are not necessarily Kauffman or skein equivalent.

*Example* 5.2 [**19**]. Consider the link $L_1$ composed of three trivial components (Fig. 5.1). If we apply $\rho$-rotation to the room $A$ we get the link $L_2$ composed of a trivial component and the Hopf link. Therefore $L_1$ and $L_2$ have different

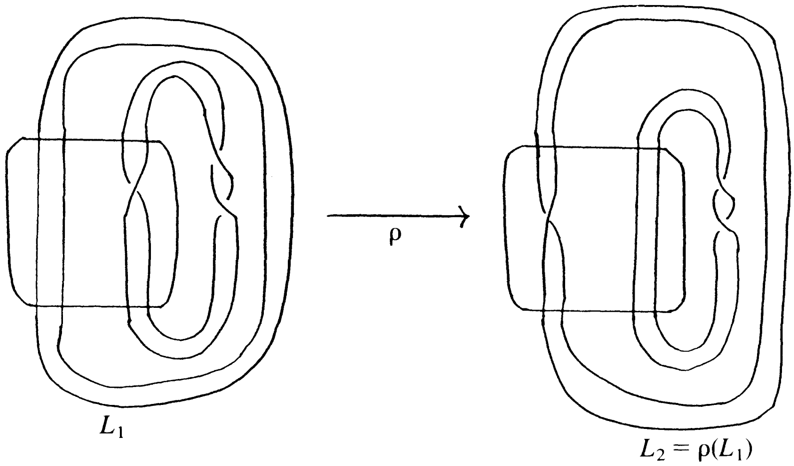
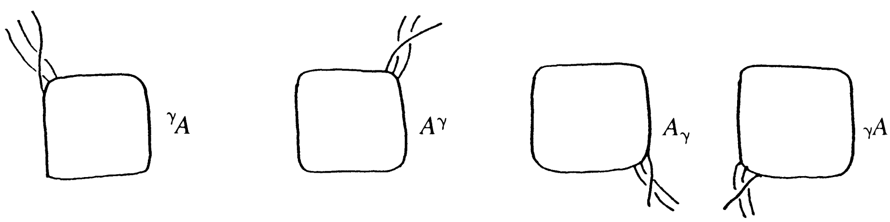



Figure 5.1.

Jones, Jones-Conway and Kauffman polynomials (so they are neither $(0, \infty)$ nor skein, nor Kauffman equivalent). To prove Theorem 5.1 we use Theorem 3.5 and we need a workable criterion to recognize almost symmetric $n$-rooms.

*Definition* 5.3. Let $A$ be an $n$-room with $(p_1, p_2, p_3, p_4)$-corners and $\gamma$ a $p$-string braid. We denote by ${}^{\gamma}A$ the $n$-room obtained from $A$ by adding $\gamma$ at the NW corner of $A$ (we need $p = p_1$). Similarly we define $A^{\gamma}, A_{\gamma}$ and ${}_{\gamma}A$. We always orient $\gamma$ from outside to inside; see Fig. 5.2. We say that $A$ and $B$ are *corner equivalent* if and only if $B$ can be got from $A$ using finite numbers of moves of type $A \longrightarrow {}^{\gamma}A$, $A \longrightarrow A^{\gamma}$, $A \longrightarrow A_{\gamma}$ and $A \longrightarrow {}_{\gamma}A$.




*the case of* $(p_1 = p_2 = p_3 = p_4)$ $\gamma = \delta_1\delta_2 \in B_3$

Figure 5.2.

Lemma 5.4. (a) *Let L be a link diagram composed of a* 4*-room A of type* $\rho$ *(resp.* $\delta$ *or* $\tau$*) with* $(2,2,2,2)$*-corners and its complement a* $\rho$*-parallel (resp.* $\delta$ *or* $\tau$*-parallel)* 4*-room* $A'$*. Then if A is corner equivalent to a* $\rho$ *(resp.* $\delta$ *or*

*$\tau$)-symmetric 4-room then $A$ is an almost $\rho$ (resp. $\delta$ or $\tau$)-symmetric 4-room with respect to $A'$.*

(b) *Consider all 4-rooms $A$ with $(2,2,2,2)$-corners. There is $7 \cdot 5 \cdot 3 = 105$ pairings of 8 inputs and outputs of $A$. 60 of them do not connect strings of the same corner. Then for each of these 60 connections there is a 4-room realizing the connection which is corner equivalent to a $\rho, \delta$ and $\tau$-symmetric 4-room.*

*Proof.* (a) If $A$ is corner equivalent to $\rho$-symmetric 4-room then

$$A = {}^{\gamma_4}_{\gamma_3} B^{\gamma_1}_{\gamma_2}$$

where $B$ is $\rho$-symmetric 4-room and $\gamma_1, \gamma_2, \gamma_3$ and $\gamma_4$ are 2-braids. Then

$$\rho(A) = {}^{\gamma_1}_{\gamma_2} B^{\gamma_4}_{\gamma_3}$$

and because $A'$ is a $\rho$-parallel 4-room

$$\rho(L) = {}^{\gamma_1}_{\gamma_2} B^{\gamma_4}_{\gamma_3} \cup A'$$

is regular isotopic to

$${}^{\gamma_4}_{\gamma_3} B^{\gamma_1}_{\gamma_2} \cup A' = L$$

(we just transport twists from one corner to the other), so $A$ is almost $\rho$-symmetric with respect to $A'$. A similar proof works for $\delta$ and $\tau$.

(b) Consider $\rho, \delta$ and $\tau$-symmetric rooms in Fig. 5.3.

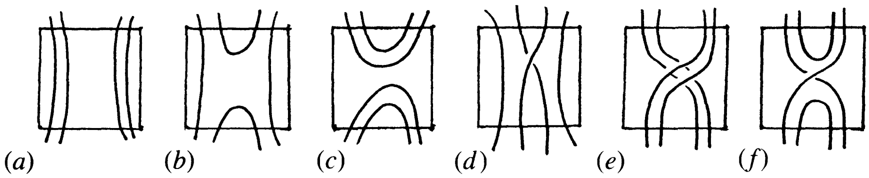



Figure 5.3.

Among 60 pairings of inputs and outputs which we have to consider, 4 are realized by 4-rooms corner equivalent to Fig. 5.3(a), 16 – Fig. 5.3(b), 4 – Fig. 5.3(c), 16 – Fig. 5.3(d), 4 – Fig. 5.3(e) and the last 16 are realized by 4-rooms corner equivalent to Fig. 5.3(f).

Now to prove Theorem 5.1 we use Theorem 3.5. For each pairing of inputs and outputs of $A$ there is one possible leaf of binary skein (or trinary Kauffman) tree for $L$ so it is sufficient to show that each pairing is represented by a reducible or almost symmetric 4-room. If we connect strings from the same corner then we get a room reducible with respect to $A'$. If we do not connect strings from the same corner then by Lemma 5.4 we can always realize our pairing by a

4-room almost $\rho$ (resp. $\delta$ or $\tau$)-symmetric modulo $A'$. So we get the hypothesis of Theorem 3.5 for $(p_1,p_2,p_3,p_4) = (2,2,2,2)$. For $(p_1,p_2,p_3,p_4) < (2,2,2,2)$ the same proof works (in fact the shorter proof because there are less cases to check). Therefore all hypotheses of Theorem 3.5 are satisfied so the proof of Theorem 5.1 is completed.

COROLLARY 5.5. *Let $L$ be a link and $A$ its tangle with complementary tangle $A'$. Assume that the mutation $\rho$ (resp. $\delta$ or $\tau$) preserves components of $L$, that is if $a_0$ is any input or output of $A$ then $a_0$ and $\rho(a_0)$ (resp. $\delta(a_0)$ or $\tau(a_0)$) lie on the same component of $L$. Then for any satellite $L_s$ of $L$ such that for each component $K$ of $L$ which cuts the boundary of $A$, the wrapping number of the satellite of $K$ is less or equal to 2 (see Definition 1.1) and the corresponding satellite $(\rho(L))_s$ (resp. $(\delta(L))_s$ or $(\tau(L))_s$) one has that $L_s$ and $(\rho(L))_s$ (resp. $(\delta(L))_s$ or $(\tau(L))_s$) are skein and Kauffman equivalent.*

*Proof.* This is analogous to that of Corollary 4.2.

COROLLARY 5.6. **[19]**, **[33]** *Let $K_1$ and $K_2$ be satellites of wrapping number 2 (e.g. $(2,q)$-cables or doubles) of an oriented knot $K$ and its mutant (constructed in the same way). Then $K_1$ and $K_2$ are skein and Kauffman equivalent. In particular they have the same Jones-Conway and Kauffman polynomials and the same signature. They have also the same Tristram-Levine signature $\sigma(\zeta)$ provided*

$$P_{K_1}(i, 2-\zeta-\bar{\zeta}) \neq 0$$

*where $P$ denotes the Jones-Conway polynomial.*

*Proof.* It is analogous to that of Corollaries 4.3 and 4.4. We use the fact that under assumption $P(i,2-\zeta-\bar{\zeta}) \neq 0$ the Tristram-Levine signature $(\sigma(\zeta))$ is an invariant of skein equivalence classes of links (see **[35]** or **[31]**).

## 6. New knot invariants of satellites of $K_1 \# K_2$ and $K_1 \# -K_2$.

THEOREM 6.1. *Let $L$ be a link diagram composed of an $n$-room $A$ of type $\rho$ with $(n,n,0,0)$-corners and its complement $\rho$-parallel $n$-room $A'$. Then $L$ and $\rho(L)$ are skein and Kauffman equivalent. Similar results hold for $\delta$ and $\tau$.*

*Proof.* We use Theorem 3.5. We proceed as in the proofs of Theorems 4.1 and 5.1. We have to show that for each pairing of $2n$ inputs and outputs there is always $n$-room which represents it and which is either reducible with respect to $A'$ or is almost symmetric modulo $A'$. If a pairing joins two points from the same corner then one can easily find an $n$-room representing this pairing which is reducible with respect to $A'$. Now consider only pairings which join points from different corners. There are $n!$ of such pairings and they are in natural bijection with a group $S_n$ of all permutations of $n$ elements. Now consider an $n$ braid $\gamma$ going from NW corner to NE corner of an $n$-room. It represents an $n$-room

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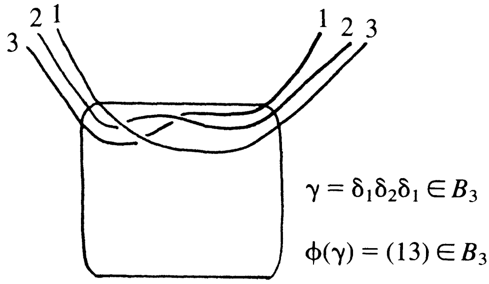


Figure 6.1.

with pairing given by $\phi(\gamma)$ where $\phi$ is the standard homomorphism $\phi : B_n \longrightarrow S_n$ given by $\phi(\delta_i) = (i, i+1)$; compare Fig. 6.1.

Now observe that if an $n$-room $B$ is filled by an $n$-braid

$$\gamma = \delta_{i_1}^{p_1}\delta_{i_2}^{p_2}, \dots, \delta_{i_k}^{p_k}$$

then the $n$-room $\rho(B)$ is filled by the braid

$$\gamma^* = \delta_{i_k}^{p_k}, \dots, \delta_{i_1}^{p_1}.$$

So if $\gamma$ is conjugated to $\gamma^*$ in $B_n$ then the diagrams $B \cup A'$ and $\rho(B) \cup A'$ are regularly isotopic so $B$ is almost $\rho$-symmetric. Therefore to complete the proof of Theorem 6.1 we need only the following lemma.

LEMMA 6.2. *Consider the natural homomorphis $\phi : B_n \longrightarrow S_n$, then for each $s \in S_n$ there exists $\alpha_s \in \phi^{-1}(s)$ such that $\alpha_s$ and $\alpha_s^*$ are conjugated in $B_n$ where for*

$$\alpha_s = \delta_{i_1}^{p_1}, \dots, \delta_{i_k}^{p_k},$$
$$\alpha_s^* = \delta_{i_k}^{p_k}, \dots, \delta_{i_1}^{p_1}.$$

*Proof.* Step 1. If for a given $s \in S_n$ there exists $\alpha_s$ from Lemma 6.2. Then for $s'$ conjugated to $s$ there exists $\alpha_s'$. To see this let

$$s' = tst^{-1} \quad \text{and} \quad \alpha_s = \beta\alpha_s^*\beta^{-1}$$

where $t \in S_n$ and $\beta \in B_n$. Let $T \in \phi^{-1}(t)$ then

$$\phi(T\alpha_s T^{-1}) = \phi(T)\phi(\alpha_s)\phi(T^{-1}) = tst^{-1} = s'.$$

Now $\alpha_{s'} = T\alpha_s T^{-1}$ is conjugate to

$$\alpha_{s'}^* = T^{*-1}\alpha_s^* T^*.$$

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Namely

$$(T\beta T^*)\alpha_s^*, (T\beta T^*)^{-1} = T\beta(T^*\alpha_s^*, T^{*-1})\beta^{-1}T^{-1}$$
$$= T(\beta\alpha_s^* B^{-1})T^{-1} = T\alpha_s T^{-1} = \alpha_{s'}.$$

Step 2. Now we proceed by induction on $n$. For $n = 1$ Lemma 6.2 obviously holds. Assume it holds for $1, 2, \ldots, n-1$ and consider $s \in S_n$. Now either

(i) $s$ is a composition of disjoint cycles of length less than $n$ and then $s$ is conjugated to

$$s' = (1, 2, \ldots, i_1)(i_1 + 1, \ldots, i_2)\ldots(i_{k-1} + 1, \ldots, i_k).$$

Now Lemma 6.2 holds for $s'$ by inductive assumption and for $s$ by Step 1.

(ii) $s$ is an $n$-cycle so $s$ is conjugated to $s' = (1, 2, \ldots, n)$. Now consider

$$\alpha_{s'} = \delta_1\delta_2\ldots\delta_{n-1} \in B_n.$$

We have $\phi(\alpha_{s'}) = s'$ and $\alpha_{s'}$ is conjugated to $\alpha_{s'}^* = \delta_{n-1}\ldots\delta_1$. Namely

$$\alpha_{s'}^* = \Delta\,\alpha_{s'}\Delta^{-1} \quad \text{where } \Delta = (\delta_1\ldots\delta_{n-1})(\delta_1\ldots\delta_{n-2})\ldots(\delta_1\delta_2)\delta_1$$

(see [**3**, Lemma 2.5.2]). Therefore Lemma 6.2 holds for $s'$ and by Step 1 for $s$ too. This completes the proof of Lemma 6.2.

COROLLARY 6.3. *Consider an oriented composed link $L_1 \# L_2$ and let $-L_2$ denote the link obtained from $L_2$ by reversing orientations of all components of $L_2$. Let $L_1 \# -L_2$ denote the connected sum of links which join the same components as $L_1 \# L_2$. Let $L_s$ and $L'_s$ be satellites of $L_1 \# L_2$ and $L_1 \# -L_2$ respectively (constructed in the same way). Then no new invariants of links can distinguish $L_s$ from $L'_s$.*

*Proof.* Consider the diagram of $L_1 \# L_2$ consisting of two 2-rooms (Fig. 6.2)

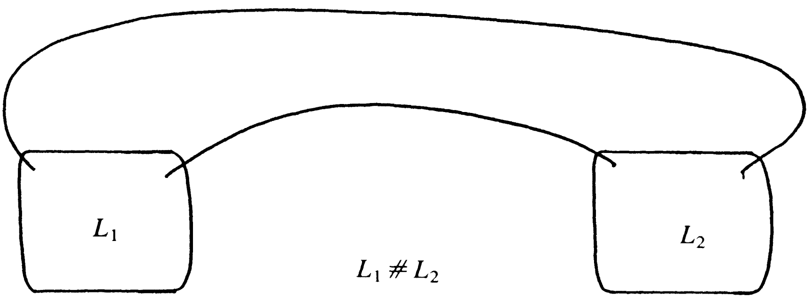


Figure 6.2.

Now consider a diagram $D$ of the satellite $L_s$ of $L_1 \# L_2$ such that it is composed of an $n$-room $A$ of type $\rho$ with $(n, n, 0, 0)$ corners (this room "covers"

$L_2$), and $\rho$-parallel $n$-room $A'$ (it "covers" $L_1$). Now $\rho(D) = \rho(A) \cup A'$ is a diagram of $L'_s$ and $D$ and $\rho(D)$ are skein and Kauffman equivalent by Theorem 6.1. Therefore Corollary 6.3 follows.

**7. Further applications of Theorem 3.5; examples and open questions.** We cannot expect that the hypothesis of Theorem 3.5 is satisfied for each 4-tuple $(p_1, p_2, p_3, p_4)$, that is we do not expect that for each pairing of $2n$ inputs and outputs there is an $n$-room with $(p_1, p_2, p_3, p_4)$ corners which realizes the pairing and which is either reducible or almost symmetric. Consider for example the 5-room of type $\rho$ with (3, 3, 2, 2)-corners and 6-rooms of type $\rho$ with (3, 3, 3, 3)-corners shown in Fig. 7.1.

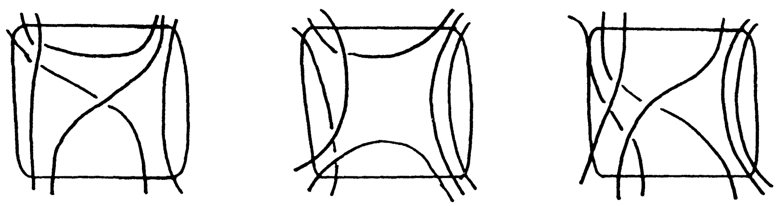

Figure 7.1.

However we can still slightly generalize Theorem 5.1.

THEOREM 7.1. *Let $L$ be a link diagram composed of an $n$-room $A$ of type $\rho$ (resp. $\delta$ or $\tau$) with $(p_1, p_2, p_3, p_4)$-corners and its complement a $\rho$ (resp. $\delta$ or $\tau$)-parallel $n$-room $A$. Then for $n \leqq 4$ or $n = 5$ and $(p_1, p_2, p_3, p_4) = (4, 4, 1, 1)$ $L$ and $\rho(L)$ (resp. $\delta(L)$ or $\tau(L)$) are skein and Kauffman equivalent.*

*Proof.* For $(p_1, p_2, p_3, p_4) = (2, 2, 2, 2)$ this is Theorem 5.1. For $p_i = 0$, for some $i$ this is a special case of Theorem 6.1. Therefore if we limit ourselves to $n$-rooms; of type $\rho$ we have to check additionally the cases $(p_1, p_2, p_3, p_4) = (3, 3, 1, 1)$ and $(4, 4, 1, 1)$.

We use Theorem 3.5 and follow the proof of Theorem 5.1. We start from the case $(p_1, p_2, p_3, p_4) = (3, 3, 1, 1)$. There are $7 \cdot 5 \cdot 3 = 105$ pairings of 8 inputs and outputs of 4-rooms. 63 of them join two strings from the same corner so they have reducible 4-rooms which represent them. So we have to show that the remaining 42 pairings have almost $\rho$-symmetric representatives. Each of these pairings is represented by a 4-room corner equivalent to one of the following three $\rho$-symmetric 4-rooms (Fig. 7.2).

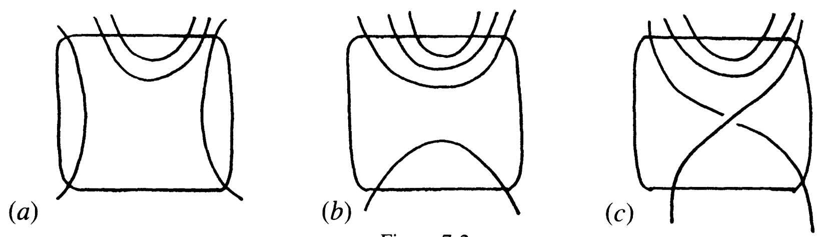

Figure 7.2.

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There is no reason to expect that each 4-room corner equivalent to a $\rho$-symmetric one is almost $\rho$-symmetric but it is not difficult to check that one can find 18 4-rooms corner equivalent to that of Fig. 7.2(a) which are almost $\rho$-symmetric and which represent 18 different pairings on inputs and outputs (similarly 6 4-rooms are related to Fig. 7.2(b) and 18 to Fig. 7.2(c)). One example is shown on Fig. 7.3.

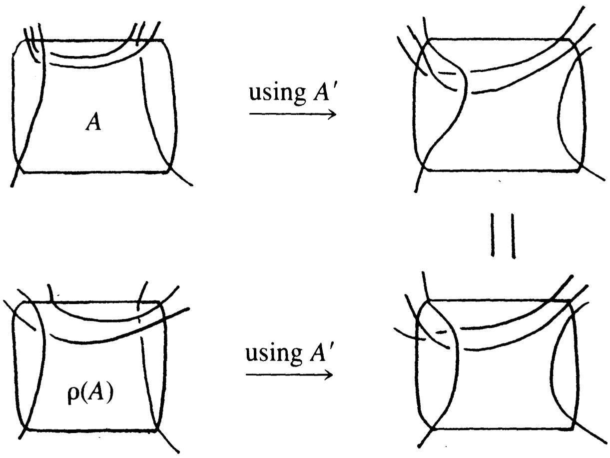


Figure 7.3.

Now we have to check the case of $(p_1, p_2, p_3, p_4) = (4,4,1,1)$. There are $9 \cdot 7 \cdot 5 \cdot 3 = 945$ possible pairings of 10 inputs and outputs of 5-rooms with $(4,4,1,1)$-corners and 729 of them contain a pair from the same corner so they have reducible 5-rooms which represent them. Therefore we have to show that the remaining 216 pairings have almost $\rho$-symmetric representatives. Each of these 216 pairings is realized by a 5-room corner equivalent to one of the following three $\rho$-symmetric 5-rooms of Fig. 7.5 (96 are corner equivalent to Fig. 7.5(a), 96 to Fig. 7.5(b) and 24 to Fig. 7.5(c)).

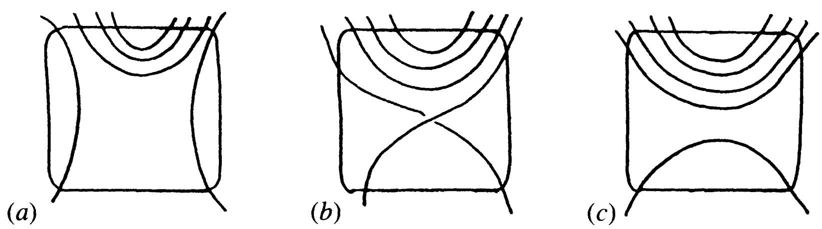


Figure 7.5.

24 pairings related to Fig. 7.5(c) are the special cases of pairings considered in Theorem 6.1 and pairings related to Fig. 7.5(a) and (b) are the same when the

proof is considered so we can limit ourselves to 96 pairings realized by 5-rooms corner equivalent to Fig. 7.5(a). It is an easy but tedious task; one can shorten the calculation using a simple algebraic lemma similar to Lemma 6.2. We leave it to the reader.

THEOREM 7.2. *Let $L$ be an oriented link diagram composed of $n$-room $A$ of type $\rho$ (resp. $\delta$ or $\tau$) with $(p_1, p_2, p_3, p_4)$-corners and its complement a $\rho$ (resp. $\delta$ or $\tau$)-parallel $n$-room $A$. Assume additionally that at each corner all strings have the same orientation. Then for $n \leqq 5, L$ and $\rho(L)$ (resp. $\delta(L)$ or $\tau(L)$) are skein equivalent.*

*Proof.* By Theorems 6.1 and 7.1 if we limit ourselves to $n$- rooms of type $\rho$ we have to check additionally the case (3, 3, 2, 2).

There are 228 allowed pairings of 10 inputs and outputs of considered 5-rooms. We will show that each such pairing is realized by an almost $\rho$-symmetric 5-room. Each of these 228 pairings is realized by a 5-room corner equivalent to one of the following five $\rho$-symmetric 5-rooms of Fig. 7.4 (36 is corner equivalent to Fig. 7.4(a), 72 to Fig. 7.4(b), 12 to Fig. 7.4(c), 36 to Fig. 7.4(d) and 72 to Fig. 7.4(e).

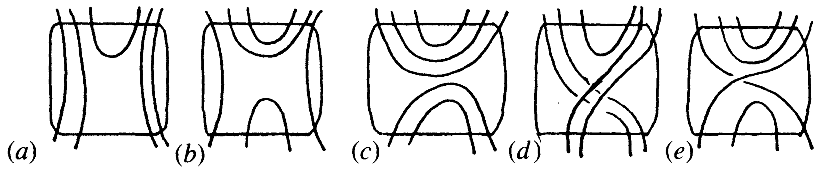


Figure 7.4.

For 156 pairings realized by 5-rooms corner equivalent to Fig. 7.4(b), (c) or (e) the situation reduces essentially to that of $(p_1, p_2, p_3, p_4) = (3, 3, 1, 1)$. The situation described by Fig. 7.4(a) and (d) are essentially the same so we can limit ourselves to 36 pairings realized by 5-rooms corner equivalent to that of Fig. 7.4(a). It is an easy but tedious task and we omit it.

COROLLARY 7.3. *Let $L$ be a link diagram and $A$ its tangle with complementary tangle $A'$. Assume that $L$ has exactly two components $L_1$ and $L_2$ which cut the boundary of $A$, and that the mutation $\rho$ (resp. $\delta$ or $\tau$) preserves $L_1$ and $L_2$. Then*

(i) *For any satellite $L_s$ of $L$ such that the satellite for $L_1$ has the wrapping number at most 4 and for $L_2$ the wrapping number is one and the corresponding satellite $(\rho(L))_s$ (resp. $(\delta(L))_s$ or $(\tau(L))_s$) of $\rho(L)$ (resp. $\delta(L)$ or $\tau(L)$), one has that $L_s$ and $(\rho(L))_s$ (resp. $(\delta(L))_s$ or $(\tau(L))_s$) are skein and Kauffman equivalent*

(ii) *for any satellite $L_s$ of $L$ such that the satellites for $L_1$ and $L_2$ have the wrapping numbers 3 for $L_1$ and 2 for $L_2$, and the absolute values of winding numbers equal to 3 for $L_1$ and 2 for $L_2$ (see Definition* 1.1*), and the corresponding satellite $(\rho(L))$ (resp. $(\delta(L))_s$ or $(\tau(L))_s$) one has that $L_s$ and $(\rho(L)_s$ (resp. $(\delta(L))_s$ or $(\tau(L))_s$) are skein equivalent.*

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*Proof.* This follows from Theorems 7.1 and 7.2 similarly as Corollaries 4.2, 5.3 and 6.3 have followed from Theorems 4.1, 5.1 and 6.1.

The starting points of all theorems considered previously were tangle and mutation. It has very nice generalization based on the Tutte's idea of rotors. It will be discussed in the forthcoming paper of R. Anstee, D. Rolfsen and the author [**2**].

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