PERFECT CATEGORIES

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Introduction

This note extends to categories Fountain's theorem (2) that for a perfect monoid S, every flat S-set is projective. (The converse is known (4).)

Fountain used the theorem for monoids to prove that perfection is also equivalent to the pair of properties:

A. Every locally cyclic S-set is cyclic.

M_R. The principal right ideals of S satisfy the minimum condition.

A similar result for categories $\mathscr{C}$ is a corollary, since it was known (4) that $\mathscr{C}$ is perfect if and only if it has property A and every monoid $\mathscr{C}(X, X)$ is perfect.

Theorem. *The following three conditions on a category* $\mathscr{C}$ *are equivalent*:

(*a*) $\mathscr{C}$ *is perfect*;

(*b*) *Every flat* (*set-valued*) *functor on* $\mathscr{C}$ *is projective*;

(*c*) *Every weakly flat* (*set-valued*) *functor on* $\mathscr{C}$ *is projective*.

The implication $(c)\Rightarrow(b)$ is trivial; $(b)\Rightarrow(a)$ is known (4); so we need to prove $(a)\Rightarrow(c)$.

Definitions and background

The basic reference for flat functors is the seminar notes of Grothendieck and Verdier (3), which treat them as generalised representable functors and call them " ind-objets " (" objet " meaning a representable functor). They are defined (3) as the direct limits of directed systems of representable functors. The standard term is now *flat functor*, although as far as I know it has not made its way from lectures into print. As with (flat) modules, so with functors $F: \mathscr{C}\rightarrow\mathscr{S}$, the property is equivalent to this: the set-valued functor $(\ )\otimes F$, on cat$(\mathscr{C}^{\mathrm{op}}, \mathscr{S})$ to $\mathscr{S}$, is left exact. There are results like that in (3), from which this result became clear when the tensor product $G\otimes F$ of contravariant G and covariant F was defined. F. W. Lawvere tells me that this was well known in Zurich certainly by 1966. The explicit definition of $\otimes$ is simple enough (using the notation of (4); Latin letters denote points of F, which are ordered pairs consisting of an object X of $\mathscr{C}$ and an element e of $F(X)$, Greek letters denote morphisms of $\mathscr{C}$, and of course a juxtaposition αp is meaningful only when the object of p is the domain of α); form $G\otimes F$ from the coproduct

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of all $G(X)\otimes F(X)$ by identifying $(q\alpha, p)$ with $(q, \alpha p)$, for every meaningful instance of these expressions.

The definition of $\otimes$ was published by B. Mitchell (**5**), whose results do not touch on flatness but include these fundamentals: for representable $G = h_Y$, $G\otimes F$ is $F(Y)$; for representable $F = h^X$, $G\otimes F$ is $G(X)$; in general, $\otimes$ is co-continuous in each variable.

Weakly flat functors (as in Stenström (**5**)) are the direct limits of projective functors. Recall that projective functors are the retracts of free functors; in turn, free functors are coproducts of representable ones. One verifies by inspection that each of these constructions preserves the following property which the representable functors plainly have:

(*) *In F there are no relations $\alpha p = \beta q$ except those given by relations $\alpha v = \beta\delta$ in $\mathscr{C}$ and points o, by means of $p = vo$, $q = \delta o$; and there are no relations $\alpha p = \beta p$ except those given by $\alpha v = \beta v$, $p = vo$.*

Thus (*) *is true of weakly flat functors F.* (Actually it characterises them. Grothendieck-Verdier showed (**3**, Theorem 8.3.3) that this and indecomposability characterise flat functors.)

Proof of $(a)\Rightarrow(c)$

We can use Fountain's lemmas to prove:

Theorem. *If $\mathscr{C}$ is perfect then every weakly flat functor F: $\mathscr{C}\rightarrow S$ is projective.*

Proof. Perfection implies (**4**) property A: every locally cyclic functor is cyclic, or in other words every ascending chain of cyclic subfunctors is finite. Then (following Fountain's numbering of lemmas):

(2) Every generating set of points of F contains an irredundant generating set.

For let M be the set of all maximal cyclic subfunctors $\mathscr{C}p$ of F. A generating set S must include, for each $\mathscr{C}p$ in M, some s such that $p \in \mathscr{C}s$; then

$$\mathscr{C}p \subset \mathscr{C}s,\ \mathscr{C}p = \mathscr{C}s \in M.$$

(3) The indecomposable summands of F are cyclic.

For let S be an irredundant generating set. The claim is that $\mathscr{C}p$, $\mathscr{C}q$ are disjoint for $p \neq q$ in S. Indeed, if not, then (*) gives $p = vo$, $q = \delta o$, with $\mathscr{C}o$ containing $\mathscr{C}p$ and $\mathscr{C}q$, contrary to maximality.

The indecomposable summands are, of course, also flat (but we need only (*), which they evidently inherit).

(5) The indecomposable summands of F are projective.

For this we must recall that perfection implies (**4**) property D: every isotropy set has a minimal left ideal generated by an idempotent. (An isotropy set, meaning a coset of an identity 1_X in a left congruence, is (4) simply an isotropy set in $\mathscr{C}(X, X)$, or left unitary subsemigroup (2).)

Let E be an indecomposable summand of F. E being flat cyclic, of the form h^X/ρ for some left congruence ρ, we shall construct an isomorphism with a projective $h^X\alpha$, where $\alpha: X \to X$ is idempotent. ($h^X\alpha(Y)$ is the subset of $h^X(Y)$ consisting of all elements $\xi\alpha$; h^X retracts onto $h^X\alpha$ by ϕ, where $\phi_Y(\xi) = \xi\alpha$.) We need to note that when $\beta\rho = \nu\rho$ in $E(Y)$, there exists $\theta: X \to X$ in $\mathscr{C}$, ρ-equivalent to 1_X, such that $\beta\theta = \nu\theta$. For $\beta(1_X\rho) = \nu(1_X\rho)$ and (*) yield $\beta\delta = \nu\delta$ and $1_X\rho = \delta o$ for some $\delta: W \to X$ and $o \in E(W)$. But $1_X\rho$ generates E, so o is $\eta(1_X\rho)$ and we may put $\theta = \delta\eta$.

Let $B = 1_X\rho$ and let $B\alpha$ be a minimal left ideal of B generated by idempotent α. By Lemma 8.12 of (1), αB is a minimal right ideal of B. By the remark above, for each β in B (since $\beta\rho = \alpha\rho$) there is θ in B with $\beta\theta = \alpha\theta$. So $\beta\theta B \subset \alpha B$, $\alpha B = \beta\theta B$ by minimality, and therefore $\alpha \in \beta B$.

Define f: $h^X/\rho \to h^X\alpha$ by $f_Y(\xi\rho) = \xi\alpha$. If $\xi\rho$ is equal to $\zeta\rho$, then $\xi\beta = \zeta\beta$ for some β in B (by the remark above), so $\xi\alpha = \zeta\alpha$ since $\alpha \in \beta B$. Evidently f is natural, and surjective. Since $(\xi\alpha)\rho = \xi(\alpha\rho) = \xi(1_X\rho) = \xi\rho$, f is injective and (5) is proved.

The theorem follows since F is a coproduct of projectives.

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