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An asymptotic result for a subcritical branching process with immigration

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Under weak hypotheses, the geometric ergodicity of the states of a subcritical branching process with immigration is demonstrated by obtaining an asymptotic representation of the transition probabilities.

Consider a branching process in which each individual reproduces independently of all others and has probability a_j $(j = 0, 1, \ldots)$ of giving rise to j progeny in the following generation, and in which there is an independent immigration component where, with probability b_j $(j = 0, 1, \ldots)$, j individuals enter the population at each generation. Letting X_n $(n = 0, 1, \ldots)$ be the population size of the n-th generation, it is well known [1], [2] that $\{X_n\}$ defines a Markov chain on the non-negative integers. Moreover, if $p_{ij}^{(n)}$ $(i, j, n = 0, 1, \ldots)$ represents the n-step transition probabilities of this Markov chain, then

$$P_i^{(n)}(x) = \sum_{j=0}^{\infty} p_{ij}^{(n)} x^j = \left[B\left(A_n(x)\right)\right]^i \prod_{m=0}^{n-1} B\left(A_m(x)\right) \tag{1}$$

where, if $|x| \le 1$, $A(x) = \sum_{j=0}^{\infty} a_j x^j$, $B(x) = \sum_{j=0}^{\infty} b_j x^j$, $A_0(x) = x$ and $A_{n+1}(x) = A\left(A_n(x)\right)$, $(n = 0, 1, \ldots)$.

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If $0 < a_0$, $b_0 < 1$, $0 < a_0 + a_1 < 1$ and $\alpha = A'(1-) < 1$, then the condition $\sum_{j=1}^{\infty} b_j \log j < \infty$ is necessary and sufficient for the existence of a limiting distribution, $\{\pi_j\}$, of the Markov chain and

$$\Pi(x) = \sum_{j=0}^{\infty} \pi_j x^j = \prod_{m=0}^{\infty} B\left(A_m(x)\right) \qquad (|x| \le 1) , \tag{2}$$

see [1], [2] and [6].

In [5] the author studied the rate of convergence of $p_{ij}^{(n)}$ to its limit π_j and it was shown that if $\beta = B'(1-) < \infty$, $p_{ij}^{(n)} - \pi_j = O\left(\alpha^n\right)$, showing that all the states of the Markov chain are geometrically ergodic. More precisely, if also $\sum_{j=1}^{\infty} a_j j \log j < \infty$ then $p_{ij}^{(n)} - \pi_j = \alpha^n\left(M_{ij}+o(1)\right)$ and the exact form of M_{ij} was found, but if $\sum_{j=1}^{\infty} a_j j \log j = \infty$ then $p_{ij}^{(n)} - \pi_j = o\left(\alpha^n\right)$.

The purpose of this note is to show that qualitatively similar results obtain if we relax the requirement that the immigration distribution have a finite mean, but we shall assume that $B(x)$ is given in terms of a slowly varying function. We now prove the following theorem.

THEOREM. *Let* $0 < a_0$, $b_0 < 1$, $0 < a_0 + a_1 < 1$, $\alpha < 1$ *and* $\sum_{j=1}^{\infty} a_j j \log j < \infty$ *for a branching process with immigration. If*

$$B(1-x) = 1 - x^{\delta} L(x) \qquad (0 < \delta \le 1) \tag{3}$$

where $L(x)$ *is slowly varying at the origin, then*

$$P_i^{(n)}(x) = \Pi(x) + \alpha^{n\delta} L\left(\alpha^n\right)\left\{H(s)+o(1)\right\} \qquad (i = 0 , \quad |x| < 1) \tag{4}$$

where $H(x) = \Pi(x)\left(1-G(x)\right)^{\delta}/c^{\delta}\left(1-\alpha^{\delta}\right)$, $c = G'(1-)$ *and* $G(x)$ *is a*

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probability generating function satisfying the functional equation

$$1 - G\big(A(x)\big) = \alpha\big(1-G(x)\big) . \tag{5}$$

The n-step transition probabilities have the form

$$p_{ij}^{(n)} = \pi_j + \alpha^{n\delta}L(\alpha^n)\big(h_j + o(1)\big) \qquad (i = 0 , \quad j = 0, 1, \ldots) \tag{6}$$

where $H(x) = \sum_{j=0}^{\infty} h_j x^j$. *If* $0 < \delta < 1$ *or* $\delta = 1$ *and* $L(x) \to \infty$, $(x \downarrow 0)$ *then equations* (4) *and* (5) *also hold for* $i = 1, 2, \ldots$.

Proof. The condition for the existence of a limiting distribution, $\sum_{j=1}^{\infty} b_j \log j < \infty$, is equivalent to the condition $\int_0^{\gamma} \big(1-B(1-x)\big)/x dx < \infty$, $(0 < \gamma \le 1)$, and it follows from equation (3) that this latter condition is indeed satisfied on recalling that for a slowly varying function

$$x^{\varepsilon}L(x) \to 0 \qquad (x \downarrow 0 ; \quad \varepsilon > 0) . \tag{7}$$

Considering the case $i = 0$, it follows from equations (1) and (2) that

$$P_0^{(n)}(x) - \Pi(x) = P_0^{(n)}(x)\Big[1 - \prod_{m=n}^{\infty} B\big(A_m(x)\big)\Big] .$$

The first member on the right converges to $\Pi(x)$ and it is not hard to show that for the second member,

$$1 - \prod_{m=n}^{\infty} B\big(A_m(x)\big) = \sum_{m=n}^{\infty} \Big[1-B\big(A_m(x)\big)\Big] + R^{(n)}(x) \tag{8}$$

where

$$-\Big[\sum_{m=n}^{\infty} \Big[1-B\big(A_m(0)\big)\Big]\Big]^2/2 \le R^{(n)}(x) \le \sum_{m=n}^{\infty} \Big[1-B\big(A_m(0)\big)\Big]^2/B\big(A_m(0)\big) \tag{9}$$

and we have assumed that $0 \le x < 1$. We shall now obtain an asymptotic estimate of the series in (8). Using equation (3) we have

$$\sum_{m=n}^{\infty} \frac{1-B\big(A_m(x)\big)}{\alpha^{n\delta}L(\alpha^n)} = \sum_{m=0}^{\infty} \left[\frac{1-A_{m+n}(x)}{\alpha^{n+m}}\right]^{\delta} \cdot \frac{L\big(1-A_{m+n}(x)\big)}{L\big(\alpha^{n+m}\big)} \cdot \frac{L\big(\alpha^{n+m}\big)}{L\big(\alpha^{n}\big)}\alpha^{m\delta} . \tag{10}$$

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Since $\sum_{j=1}^{\infty} a_j j \log j < \infty$, we can combine the refined forms of Yaglom's and Kolmogorov's limit theorems (see [3]) to obtain

$$(11) \qquad 1 - A_n(x) = \alpha^n[(1 - G(x))/c + o(1)]$$

where $G(x)$ and c are as given in the statement of the theorem. Thus, for each $x \in [0, 1)$, the first factor of the terms of the series on the right hand side of (10) is bounded, and the boundedness of the second factor follows from (11) and the fact that $L(ty)/L(y) \to 0$, $(y \downarrow 0)$, uniformly for $0 < u < t < U < \infty$; see [4]. Finally use of (7) makes it clear that the dominated convergence theorem is applicable to equation (10) thus yielding

$$\sum_{m=n}^{\infty} \left[1 - B\left(A_m(x)\right)\right] = \alpha^{n\delta} L\left(\alpha^n\right)\left[\left(1 - G(x)\right)^{\delta}/c^{\delta}\left(1-\alpha^{\delta}\right) + o(1)\right] \qquad (0 \le x < 1) .$$

It is apparent from the working above and Kolmogorov's limit theorem that

$$1 - B\left(A_n(0)\right) = \alpha^{n\delta} L\left(\alpha^n\right)\left(1/c + o(1)\right) .$$

Using this result and the well known fact that $A_n(0) \uparrow 1$, $(n \to \infty)$, it is not hard to show from equation (9) that $R^{(n)}(x) = O\left[\left(\alpha^n L\left(\alpha^n\right)\right)^2\right]$. Hence equation (4) is proven in the case $0 \le x < 1$.

On observing that $\left|1 - B\left(A_n(x)\right)\right| \le 2\left|1 - B\left(A_n(0)\right)\right|$, $(|x| \le 1)$,

and using the inequality $\left|1 - \prod_{m=n}^{\infty} B\left(A_n(x)\right)\right| \le 1 - \prod_{m=n}^{\infty}\left(1 - \left|1-B\left(A_n(x)\right)\right|\right)$,

it is clear from the above that $\left[P_0^{(n)}(x) - \Pi(x)\right]/\alpha^n L\left(\alpha^n\right)$ is uniformly bounded in $|x| \le 1$ and so, by Vitali's theorem, this sequence of functions converges uniformly in compact subsets of the open unit disc to $H(x)$, thus completing the proof of equations (4) and (6).

Letting now $i > 0$, we have

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$$\frac{P_i^{(n)}(x)-\Pi(x)}{\alpha^{n\delta}L(\alpha^n)} = \frac{P_0^{(n)}(x)-\Pi(x)}{\alpha^{n\delta}L(\alpha^n)} - \frac{1-A_n(x)}{\alpha^n} \cdot \frac{P_0^{(n)}(x)\sum_{k=0}^{i-1}\left[A_n(x)\right]^k}{\alpha^{-n(1-\delta)}L(\alpha^n)}$$

and $\alpha^{-n(1-\delta)}L(\alpha^n) \to \infty$ under either of the two conditions given in the last assertion of the theorem. Thus the left hand side of the expression above converges uniformly to $H(x)$ in compact subsets of the open unit disc, thus completing the proof.

In view of equation (7), it is clear from (6) that the states of $\{X_n\}$ are geometrically ergodic, and if $h_j > 0$, the convergence parameter can be taken as any number in the interval $(0, \alpha^\delta)$. Observe that, under the conditions given in the last assertion of the theorem, all moments of the limiting distribution are infinite.

Under the condition $\sum_{j=1}^{\infty} a_j j\log j = \infty$, it is shown in [3] that $c = \infty$. Referring to equation (9), we see that the first factor of the series on the right can be dominated by arbitrary $\varepsilon > 0$ if n is large enough. Thus we have the following.

COROLLARY. *If all the conditions of the theorem hold, except that* $\sum_{j=1}^{\infty} a_j j\log j = \infty$, *then we have*

$$p_{ij}^{(n)} = \pi_j + L(\alpha^n)r_{ij}^{(n)}$$

where $r_{ij}^{(n)} = o(\alpha^{n\delta})$, $(n \to \infty)$.

Once again we see that the states are geometrically ergodic.

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