Canad. Math. Bull. Vol. **36** (3) 1993, pp. 332–343

ON IDEALS ANNIHILATING THE TORAL CLASS OF $\mathrm{BP}_*\big((BZ/P^K)^N\big)$

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ABSTRACT. A sequence of ideals $I_{k,n} \subseteq \mathrm{BP}_*$ is introduced, with the property: $I_{k,n} \subseteq \mathrm{Ann}(\gamma_{k,n})$, where $\gamma_{k,n}$ is the toral class of the Brown-Peterson homology of the n-fold product $BZ/p^k \times \cdots \times BZ/p^k$. These ideals seem to play an interesting and yet unclear role in understanding $\mathrm{Ann}(\gamma_{k,n})$. They are defined by using the formal group law of the Brown-Peterson spectrum BP, and some of their elementary properties are established. By using classical theorems of Landweber and of Ravenel-Wilson, the author computes the radicals of $I_{k,n}$ and $\mathrm{Ann}(\gamma_{k,n})$, and discusses a few examples.

1. **Introduction.** The problem of computing the oriented or complex bordism of finite groups is a classical one. Its solution will help answer many geometric questions on groups actions on manifolds.

The bordism groups of the cyclic groups were easily understood in the early sixties by several researchers. In their seminal work on bordism theories, Conner and Floyd [CF] compute these groups and make an attempt to compute the bordism of the elementary abelian p-groups. The situation here is far more complex. Instead of a "nice" Künneth formula one has a spectral sequence that carries the information on the connection between the bordism of the elementary abelian p-group and that of its factors. Conner and Floyd observed that an obvious action of $(Z/p)^n$ on the torus T^n defines an n-dimensional class in the bordism of $B(Z/p)^n$, known as the toral class. These researchers seemed to have realized, early on, that knowledge of the annihilator of the toral class was very important in understanding the entire structure of the bordism module of the group. They formed a conjecture as to what this annihilator ideal ought to be, known as the Conner-Floyd conjecture.

In general, the study of p-local bordism and cobordism problems can be carried out (equivalently) by the Brown-Peterson homology and cohomology theories. In the case of a finite abelian p-group G, the problem of computing its bordism is equivalent (and reduces) to the problem of computing the Brown-Peterson homology of BG, the classifying space of G, $\mathrm{BP}_*(BG)$. In this setting and by recalling that $\mathrm{BP}_* = Z_{(p)}[v_1, v_2, \ldots]$, the Conner-Floyd conjecture can be stated as follows:

CONJECTURE 1 (CONNER-FLOYD, [CF]). *For any odd prime p, the annihilator of the n-dimensional toral class in $\mathrm{BP}_n(BZ/p)^n$ is $I_n = (p, v_1, \ldots, v_{n-1})$.*

The author has been partially supported by a Naval Academy Research Center grant.
Received by the editors February 19, 1992; revised May 25, 1992.
AMS subject classification: Primary: 55N20.
Key words and phrases: Brown-Peterson spectrum, Annihilator ideals.

Although Conner and Floyd proved that $I_n \subseteq (p, v_1, \ldots, v_{n-1})$, proving the conjecture required another twenty years. This was done by Ravenel and Wilson in [RW], by an amazing application of the Morava K-theories:

THEOREM 2 (RAVENEL-WILSON, [RW]). *The Conner-Floyd conjecture is true.*

A few years later, Steve Mitchell [M], gave a very elegant and conceptual proof of this theorem by using a certain spectrum and some of its elementary properties. His work involved only bordism techniques and he made no use of the Morava K-theories. Furthermore, Mitchell extended the theorem to the case of $p = 2$.

The next natural question is the bordism of finite abelian p-groups. Now, although we still have a toral class there is no corresponding Conner-Floyd conjecture; or at least not an obvious one. The author brought some disturbing news when his study of the cases of the groups $Z/p \times Z/p^n$ and $Z/p^2 \times Z/p^2$ revealed that the annihilators of the corresponding toral classes are rather complicated. Indeed, they are respectively (p, v_1^n) and $(p^2, pv_1, v_1^{p+2}) = (p, v_1)(p, v_1^{p+1})$ [N-1]. Note that the latter ideal is not even regular.

Particular calculations show that the annihilators of the toral class of p-groups have generators that are very complicated (but quite interesting) polynomials in the generators of the coefficient ring. In this note we try to initiate a systematic study of these annihilator ideals by first introducing a sequence of ideals that are, in general, properly contained in the annihilators. These ideals are invariant and their radicals, after proper indexing, are the prime ideals I_n. Indeed, we prove that the radicals of the annihilators are also I_n. Although we can only go as far as computing the radicals, the methods and the approach might be of interest to some workers. What this author is aiming at, is the revitalization of this fundamental problem, a very special case of which, puzzled topologists for at least two decades before it became well understood.

2. **Basic definitions.** Let BP be the Brown-Peterson spectrum associated with the prime p, and let $F(x,y) = x +_F y$ be the corresponding formal group law. The $[n]$-series is inductively defined by $[1]x = x = F(x,0)$ and $[n]x = F(x,[n-1]x)$. Then $[p^k]x = \sum_{i=0}^{\infty} a_{k,i}x^{i+1}$, $a_{k,i} \in \mathrm{BP}_{2i}$ is the $[p^k]$-series, which gives us the defining relation of $\mathrm{BP}^*(BZ/p^k)$. Although interesting, this series is quite complicated; especially when viewed as a k-fold composition of the $[p]$-series. Some information on the coefficients $a_{k,i}$ was obtained in [N-2]. The point of view in this note, will be different. We shall treat the $[p^k]$-series as a (BP-)p-typical series in its own right. By using well known elementary formal group law methods, we shall define elements $v_{k,n} \in \mathrm{BP}_* = Z_{(p)}[v_1, v_2, \ldots]$ such that:

$$(1) \qquad [p^k]x \equiv \sum_{n\geq 1}^{F} v_{k,n}x^{p^n} \bmod (p^n)$$

and ideals $I_{k,n} \subseteq \mathrm{BP}_*$ generated by the $v_{k,i}$'s:

$$(2) \qquad I_{k,n} = (v_{k,0}, v_{k,1}, \ldots, v_{k,n-1}), \text{ where } v_{k,0} = p^k.$$

Basic references for the methods used are [H-2] and [Ar]. Next, we shall recall some well-known basic facts about the BP formal group law F, and its logarithm ([W]):

(3) $$F(x,y) = x +_F y = x + y + \sum_{i>0,j>0} a_{ij}x^i y^j, \text{ (here the } a_{ij}\text{'s generate } \mathrm{BP}_*).$$

(4) $$\log x = \sum_{i\geq 0} m_i x^{p^i}, \text{ where } m_0 = 1, |m_i| = 2(p^i - 1), m_i \in \mathrm{BP}_* \otimes \mathbf{Q}.$$

(5) $$\mathrm{BP}_* \subseteq H_* \mathrm{BP} = Z_{(p)}[m_1, m_2, \ldots], \text{ the Hurewicz map injects.}$$

(6) $$\log(x +_F y) = \log x + \log y, \text{ in particular, } \log([p^k]x) = p^k \log x$$

(7) $$p^i m_i \in I^i_{i+1} \subseteq \mathrm{BP}_{2(p^i-1)}, \quad I_n = (p, v_1, \ldots, v_{n-1}), \quad I_\infty = \bigcup I_n.$$

(8) $$\text{Let } \exp^{\mathrm{BP}} x = \exp x = \sum_{i\geq 0} b_i x^{i+1} = \log^{-1} x, \quad b_0 = 1.$$

Definition 3 and Lemma 4 below are well known ([H-2]). We recall that a *curve* is a formal power series with constant term zero.

DEFINITION 3. A curve $f(z)$ over BP_* is p-typical if $\log f(z) = \sum a_i z^{p^i}$, $a_i \in H_* \mathrm{BP}$.

LEMMA 4. *A curve $f(z)$ is p-typical, if and only if, it has the form $f(z) = \sum^F u_i z^{p^i}$ for some $u_i \in \mathrm{BP}_*$.*

PROOF. If $f(z) = \sum^F u_i z^{p^i}$, then $f(z) \in \mathrm{BP}_*$ and by (6) $\log f(z)$ is

(9) $$\log(u_0 z) + \sum_{i>0} \log(u_i z^{p^i}) = \sum_{j\geq 0} m_j u_0^{p^j} z^{p^j} + \sum_{i>0,j\geq 0} m_j u_i^{p^j} z^{p^{i+j}}$$

Therefore $f(z)$ is p-typical. Conversely, let $\log f(z) = \sum_{i\geq 0} a_i z^{p^i}$. Then (9) holds provided there are elements u_i such that

(10) $$u_0^{p^n} m_n + \sum_{i+j=n, i>0, j>0} m_j u_i^{p^j} + u_n = a_n.$$

By induction, we can solve uniquely for the u_n's in terms of the m_j's and a_j's. The coefficient of z^{p^n} in $f(z)$ gives u_n+ polynomial in u_i for $i < n$ and a few of the a_{st}'s of the formal group law. So by induction, $u_n \in \mathrm{BP}_*$.

LEMMA 5. $\exp(p^k x) \in p^k \mathrm{BP}_*[[x]]$.

PROOF. (7) easily implies that $p^i H_{2i(p-1)} \subseteq \mathrm{BP}_{2i(p-1)}$; hence $p^i b_i \in \mathrm{BP}_*$. Therefore $\exp(p^k x) = \sum_{i\geq 0} b_i (p^k x)^{i+1} \in p^k \mathrm{BP}_*[[x]]$.

LEMMA 6. $[p^k]x +_F [-1]\exp(p^k x)$ *is a p-typical power series over* BP_*.

PROOF. The series has coefficients in BP_* and its logarithm is $p^k \log x - p^k x = \sum_{i>0} m_i x^{p^i}$.

COROLLARY 7. *There exist elements $v_{k,n} \in \mathrm{BP}_{2p^n-2}$ such that*

(11) $$[p^k]x +_F [-1]\exp(p^k x) = \sum_{n>0}^{F} v_{k,n} x^{p^n}$$

If we apply log to (11) we get $p^k \log x = p^k x + \sum_{n>0} \log(v_{k,n} x^{p^n})$. By expanding we have: $p^k x + \sum_{i>0} p^k m_i x^{p^i} = p^k x + \sum_{n>0} v_{k,n} x^{p^n} + \sum_{l>0,n>0} m_l v_{k,n}^{p^n} x^{p^{n+l}}$. The coefficient of x^{p^n} is therefore:

$$p^k m_n = v_{k,n} + \sum_{i=1}^{n-1} m_i v_{k,n-i}^{p^i}. \tag{12}$$

By (12) one can compute $v_{k,n}$ recursively. If $k = 1$ we get precisely the defining relations for the Hazewinkel generators, so $v_{1,n} = v_n$.

$$[p^k]x = \sum_{i=0}^{\infty} a_{k,i} x^{i+1} \equiv \sum_{n\geq 1}^{F} v_{k,n} x^{p^n} \bmod (p^k). \tag{13}$$

$$\text{Let } v_{k,0} = p^k, \quad I_{k,n} = (v_{k,0}, v_{k,1}, \ldots, v_{k,n-1}), \quad I_{k,\infty} = \bigcup I_{k,n}. \tag{14}$$

By the definitions above we have $I_{k,n} \subseteq I_n$. One can actually write the m_i's in terms of the $v_{k,n}$'s, by using the defining relations (12). In fact:

PROPOSITION 8. *(12) is equivalent to*

$$m_n = \sum_{i_1+\cdots+i_r=n} \frac{v_{k,i_1} v_{k,i_2}^{p^{i_1}} \cdots v_{k,i_r}^{p^{i_1+\cdots+i_{r-1}}}}{p^{kr}}, \tag{15}$$

where the sum is over all sequences $(i_1, \ldots, i_r)$, i_j, $r \in \mathbf{N}$ such that $i_1 + \cdots + i_r = n$.

PROOF. (12) easily implies (15) by a straightforward induction on n. Conversely, consider all terms in (15) for m_n for which $i_r = j$ for fixed j, $1 \leq j \leq n$. Then it is clear that these terms of the sum in (15) sum to $p^{-k} m_{n-j} v_{k,j}^{p^{n-j}}$ so (15) implies that $p^k m_n = v_{k,n} + m_1 v_{k,n-1}^p + \cdots + m_{n-1} v_{k,1}^{p^{n-1}}$. We note that when $k = 1$ the above relation is well known ([H-2]).

PROPOSITION 9. *$a_{k,i} \in I_{k,n}$, if $i < p^n - 1$ and $a_{k,p^n-1} \equiv v_{k,n} \bmod I_{k,n}$.*

PROOF. The coefficient of x^{p^n} is $v_{k,n}$+ polynomial in $v_{k,i}$, $i < n$ over BP_* modulo (p^k). For $i + 1 < p^n$ the coefficient of x^{i+1} is in $I_{k,n}$.

Let $\gamma_{k,n}$ be the standard toral class of the homology $\mathrm{BP}_*\big((BZ/p^k)^n\big)$ given by the obvious n-fold product map

$$(S^1)^n = S^1 \times S^1 \times \cdots \times S^1 \longrightarrow (BZ/p^k)^n = BZ/p^k \times BZ/p^k \times \cdots \times BZ/p^k. \tag{16}$$

The Künneth homomorphism:

$$\chi\colon \bigotimes_{i=1}^{n} \mathrm{BP}_*(BZ/p^k) \longrightarrow \mathrm{BP}_*\big((BZ/p^k)^n\big) \tag{17}$$

gives $\chi(\otimes_{i=1}^n \gamma_{k,1}) = \gamma_{k,n}$, hence $\mathrm{Ann}(\otimes_{i=1}^n \gamma_{k,1}) \subseteq \mathrm{Ann}(\gamma_{k,n})$.

$\mathrm{BP}_*(BZ/p^k)$ is well understood as a BP_*-module. It is generated by elements $\gamma_{k,1} = z_1, z_2, \ldots, |z_n| = 2n - 1$ subject to relations:

$$\sum_{i\geq 0} a_{k,i} z_{n-i} = 0, n = 1, 2, \ldots, \text{ where } z_i = 0, \text{ for } i \leq 0. \tag{18}$$

PROPOSITION 10. *$I_{k,n} \subseteq \mathrm{Ann}(\otimes_{i=1}^n \gamma_{k,1})$.*

https://doi.org/10.4153/CMB-1993-046-6 Published online by Cambridge University Press

PROOF. By induction on n. $p^k z_1 = 0$ grounds the induction. Let $I_{k,n} \subseteq \mathrm{Ann}(\otimes_{i=1}^{n} z_1)$ then

$$v_{k,n}\Big(\bigotimes_{i=1}^{n+1} z_1\Big) = a_{k,p^n-1}\Big(\bigotimes_{i=1}^{n+1} z_1\Big) = \Big(-\sum_{i=0}^{p^n-2} a_{k,i}z_{p^n-i}\Big)\bigotimes_{i=1}^{n} z_1 = 0.$$

by induction, (18) and Proposition 9.

PROPOSITION 11. $I_{k,n} \subseteq \mathrm{Ann}(\gamma_{k,n})$.

Let $\iota_{k,n}$ be the canonical element $\iota_{k,n} \in \mathrm{BP}_n\big(K(Z/p^k, n)\big)$. Using the obvious map:

$$\mathop{\times}_{i=1}^{n} BZ/p^k \approx \mathop{\times}_{i=1}^{n} K(Z/p^k, 1) \longrightarrow K(Z/p^k, n).$$

we see that $\gamma_{k,n}$ goes to $\iota_{k,n}$ hence $\mathrm{Ann}(\gamma_{k,n}) \subseteq \mathrm{Ann}(\iota_{k,n})$.

COROLLARY 12. $I_{k,n} \subseteq \mathrm{Ann}(\iota_{k,n})$.

REMARK 13. One can also define "Araki"-type generators for $I_{k,n}$ as follows: $[p^k]x$ is p-typical hence there are elements $u_{k,n}$ in BP_* such that:

$$[p^k]x = \sum_{i=0}^{\infty} a_{k,i}x^{i+1} = \sum_{n\geq 0}^{F} u_{k,n}x^{p^n}. \tag{19}$$

By applying log just as before we easily get:

$$(1 - p^{(p^n-1)k})p^k m_n = u_{k,n} + \sum_{i=1}^{n-1} m_i u_{k,n-i}^{p^i}, \quad n \geq 1, \text{ and } u_{k,0} = p^k. \tag{20}$$

Sometimes the $u_{k,n}$'s can be very handy since in (19) we do not have to go mod p^k. By comparing (12) and (19) reduced mod p^k, we easily see by induction:

PROPOSITION 14. *$I_{k,n} = (u_{k,0}, u_{k,1}, \ldots, u_{k,n-1}) \subseteq \mathrm{BP}_*$, and $v_{k,n} \equiv u_{k,n} \bmod p^k$.*

3. **Elementary properties.** Our motivation for studying the $I_{k,n}$'s is their relation to the annihilator ideals mentioned above, which in turn, as experience shows, seem to encode a lot of information about the bordism groups of the corresponding classifying spaces. But the complexity of the recursive relations (12), especially when one wants the v's in terms of some set of generators of BP_*, seems to be in the way. For example, in terms of the Hazewinkel generators one has:

$$\begin{gathered} v_{k,0} = p^k, \text{ (by definition)}, \\ v_{k,1} = p^{k-1}v_1,\ v_{k,2} = p^{k-1}v_2 + p^{k-2}(1 - p^{(p-1)(k-1)})v_1^{p+1}. \end{gathered} \tag{21}$$

However, at least when $n \leq k$ one can find a "very easy" set of generators for $I_{k,n}$.

PROPOSITION 15. *a. If $n \leq k$ then $I_{k,n} = (p^k, p^k m_1, \ldots, p^k m_{n-1})$, and*
b. $v_{k,n} \equiv p^k m_n$ mod $I_{k,n}$.

https://doi.org/10.4153/CMB-1993-046-6 Published online by Cambridge University Press

PROOF. By finite induction on n, one sees from (12) that p divides $v_{k,n}$ for $n \leq k-1$. Hence $m_i v_{k,n-i}^{p^i} = p^i m_i (p^{-1} v_{k,n-i})^i v_{k,n-i}^{p^i - i}$ is in $I_{k,n}$ since $p^i m_i \in \mathrm{BP}_*$ and $p^i - i > 0$. We are done by (12).

REMARK 16. a. In general, $I_{k,n}$ is properly contained in $\mathrm{Ann}(\otimes_{i=1}^{n} \gamma_{k,1})$.

b. The ideals $I_{k,n}$ are not regular.

c. The ideals $I_{k,2}$ are not primary.

d. The ideals $I_{k,n}$ are invariant.

e. The ideal $I_{2,3}$ is I_2-primary.

Only a) and d) need to be proved. a) follows from Lemmas 17 and 18 below.

Let $n = k = 2$, then:

LEMMA 17. $p^2 | a_{2,2p-2}$, $p | a_{2,i}$ *if* $0 \leq i \leq p^2 - 2$ *and* $a_{2,p^2-1} \equiv pv_2 + v_1^{p+1} \bmod I_{2,2}$.

PROOF. The proof is easy.

LEMMA 18. $v_1^{p+2} z_1 \otimes z_1 = 0$.

PROOF. By Lemma 17 and (18) we may let $v_1^{p+1} z_1 = pt$ for some t. Then by Lemma 17, (18) and Proposition 10 we have (after we drop the tensor product symbol):

$$\begin{aligned} v_1^{p+2} z_1 z_1 &= v_1 (pt) z_1 = t(-p^2 z_p) = -p v_1^{p+1} z_1 z_p = v_1^p z_1 (-p v_1 z_p) \\ &= v_1^p z_1 [p^2 z_{2p-1} + a_{2,2p-2} z_1] = 0. \end{aligned}$$

Actually one can show that the annihilator of the toral class for two factors with $k = 2$ is (p^2, pv_1, v_1^{p+2}), [N-1].

Remark 16 d) is the most important property of the $I_{k,n}$'s, and in fact, one can generalize Ravenel's Theorem [R], into the following:

PROPOSITION 19. $\sum_{i \geq 0, j > 0}^{F} t_i \eta_R (v_{k,j})^{p^i} \equiv \sum_{j \geq 0, i > 0}^{F} v_{k,i} t_j^{p^i} \bmod (p^k)$.

PROOF. The proof follows exactly Ravenel's proof for the v's as presented in [W] p. 44, with the obvious modifications. We include it for completeness. We apply η_R to (12) to get:

$$p^k \eta_R(m_n) = \sum_{0 \leq i < n} \eta_R(m_i) \eta_R(v_{k,n-i})^{p^i}, \text{ or}$$

$$p^k \sum_{i+j=n} m_i t_j^{p^i} = \sum_{0 \leq j+l < n} m_j t_l^{p^j} \eta_R(v_{k,n-j-l})^{p^{j+l}}.$$

Add over all n, substituting $p^k m_i$ in the left hand side;

$$p^k \sum_n t_n + \sum_{h+l+j=n, l>0} m_h v_{k,l}^{p^h} t_j^{p^{h+j}} = \sum_{l \geq 0, s>0} \log t_l \eta_R(v_{k,s})^{p^l} \text{ or,}$$

$$\log\Big(\exp\big(p^k \sum_n t_n\big)\Big) + \sum_{l>0, j\geq 0} \log v_{k,l} t_j^{p^l} = \sum_{l \geq 0, s>0} \log t_l \eta_R(v_{k,s})^{p^l};$$

Apply exp:

$$\exp\Big(p^k \sum_n t_n\Big) +_F \sum_{l>0,j\geq 0}^{F} v_{k,l} t_j^{p^j} = \sum_{l\geq 0,s>0}^{F} t_l \eta_R(v_{k,s})^{p^l}.$$

Now we can reduce mod p^k and we are done by Lemma 5.

PROPOSITION 20. *a. The ideal* $I_{k,n} \subseteq \mathrm{BP}_*$ *is invariant.*
b. $\eta_R(v_{k,n}) \equiv v_{k,n} \bmod I_{k,n}$.

PROOF. The proof is by induction on n. The ideal (p^k) is invariant and $\eta_R(p^{k-1}v_1) = p^{k-1}\eta_R(v_1) = p^{k-1}(v_1 + pt_1) \equiv p^{k-1}v_1 \bmod (p^k)$. Inductively, we can assume that $I_{k,n}$ is invariant so we can work modulo $I_{k,n}$. Lastly, we have to use Proposition 19 in degree $2(p^n - 1)$ modulo $I_{k,n}$. This completes the inductive step of b. In particular, $\eta_R(v_{k,i}) \in I_{k,n+1}\,\mathrm{BP}_*\,\mathrm{BP}$, for $0 \leq i \leq n$. In terms of the u's, the analogue of Proposition 19 takes now the following form:

$$\sum_{i\geq 0,j\geq 0}^{F} t_i \eta_R(u_{k,j})^{p^i} = \sum_{j\geq 0,i\geq 0}^{F} u_{k,i} t_j^{p^i}. \tag{22}$$

Our next goal will be to compute the radicals of the $I_{k,n}$'s. Normally this would be a very difficult task due to the messy generators. But since the $I_{k,n}$'s are invariant, Landweber's work shows otherwise.

LEMMA 21. $[p^k]x \equiv p^k x + p^{k-1}v_1 x^p + p^{k-1}v_2 x^{p^2} + \cdots \bmod I_\infty^{k+1}$.

PROOF. By induction on k. $[p^k]x = [p]([p^{k-1}]x) = \sum_{i\geq 0}^F u_{1,i}([p^{k-1}]x)^{p^i}$ and by induction $u_{1,i}([p^{k-1}]x)^{p^i} \in I_\infty^{k+1}$ for $i > 0$. Hence again by induction:

$$u_{1,0}([p^{k-1}]x) \equiv p(p^{k-1}x + p^{k-1}v_1 x^p + \cdots) \bmod I_\infty^{k+1}.$$

LEMMA 22. *a.* $a_{k,p^{kn}-1} \equiv v_n^{p^{(k-1)n}+\cdots+p^n+1} \bmod I_n$.
b. $a_{k,i} \in I_n$ *for* $i < p^{kn} - 1$.

PROOF. By induction on k. Let $f(k,n) = p^{(k-1)n} + \cdots + p^n + 1$. If $k = 1$, both a) and b) are well known. By induction we have modulo $(I_n, x^{p^{kn}+1})$, $[p^k]x = [p]([p^{k-1}]x) \equiv v_n(v_n^{f(k-1,n)} x^{p^{(k-1)n}})^{p^n}$. This completes the inductive step of both a) and b).

From now on, we denote by P any polynomial, in any number of variables.

LEMMA 23. *a.* $I_{k,kn} \subseteq I_n$.
b. $I_{k,kn+1}$ *is not contained in* I_n.
c. $I_{k,n} \subseteq I_n^k$.
d. $a_{k,p^n-1} \equiv v_{k,n} \equiv u_{k,n} \equiv p^{k-1}v_n \bmod I_n^k$.
e. $a_{k,p^n-1} \equiv v_{k,n} \equiv u_{k,n} \equiv p^{k-1}v_n \bmod I_\infty^{k+1}$.
f. $a_{k,p^{kn}-1} \equiv v_{k,kn} \equiv u_{k,kn} \equiv v_n^{p^{(k-1)n}+\cdots+p^n+1} \bmod I_n$.

PROOF. The coefficient of x^{p^n} is $p^{k-1}v_n \bmod I_\infty^{k+1}$ by Lemma 21 and $u_{k,n} + P(u_{k,0},\ldots,u_{k,n-1})$ by (19). By induction on n, we may assume that $u_{k,i} \equiv p^{k-1}v_i \bmod I_\infty^{k+1}$ for $i = 0,1,\ldots,n-1$. For dimensional reasons and from the formal group law, reduction modulo I_∞^{k+1} gives just $u_{k,n}$. On the other hand, $u_{k,n} = v_{k,n} + p^k t$, where $|t| = 2p^n - 2$. This concludes the proof of e). One can get d) from e) with care: By e) and for dimensional reasons the coefficient of x^{p^n} must be of the form $p^{k-1}v_n + P(p,\ldots,v_{n-1}) + \lambda p^i v_n$ for some constant λ and for $i \geq k$. Reducing mod I_n^k gives us $p^{k-1}v_n$ and trivially $u_{k,n}$ and $v_{k,n}$ (by induction on n). This concludes d), which in turn implies c).

Next, we shall prove f) and a) by induction on n, this time. If $n = 1$, then $I_{k,k} = (p^k, p^k m_1,\ldots,p^k m_{k-1}) \subseteq (p)$ and $a_{k,p^k-1} \equiv v_{k,k} \equiv u_{k,k} \equiv v_1^{f(k,1)} \bmod(p)$ by induction on k. This grounds the induction. Let us assume that $v_{k,k(n-1)} \equiv u_{k,k(n-1)} \equiv v_{n-1}^{f(k,n-1)}$ modulo I_{n-1}. Then by (19), the coefficient of $x^{p^{kn}}$ is $u_{k,kn} + P(u_{k,0},\ldots,u_{k,kn-1})$.By induction, this reduces modulo I_n, to $u_{k,kn} + P(u_{k,(n-1)k+1},\ldots,u_{k,kn-1})$ which reduces to $u_{k,kn}$ for dimensional reasons (since $2\big((n-1)k+1\big) > nk$ for $n > 1$), and from the formal group law. This, along with Lemma 22 a), completes the inductive step for f). To prove a), let $I_{k,kn} \subseteq I_n$. By f), $u_{k,kn} \in I_{n+1}$. The coefficient of $x^{p^{kn+i}}$ for $i = 0,1,\ldots,k-1$ is $u_{k,kn+i} + P(u_{k,0},\ldots,u_{k,kn+i-1})$ which is in I_{n+1} by Lemma 22 b), and P is also in I_{n+1} by finite induction on i, hence $u_{k,kn+i} \in I_{n+1}$. So $I_{k,k(n+1)} \subseteq I_{n+1}$, which concludes the induction of a). Finally, b) follows from f).

Let Rad(I) denote the radical of the ideal I. Then we have:

COROLLARY 24. $\mathrm{Rad}(I_{k,kn}) = I_n$.

PROOF. By induction on n. If $n = 1$, $I_{k,k} = (p^k, p^k m_1,\ldots,p^k m_{k-1})$ has radical (p). Suppose that the radical of $I_{k,k(n-1)}$ is I_{n-1}. Since $I_{k,kn}$ is invariant, the work of Landweber [L-2] shows that its radical should be some I_j ($j = 1,2,\ldots,\infty$). By Lemma 23 a) and by induction we should have $I_{n-1} \subseteq I_j \subseteq I_n$. By Lemma 23 f) $v_{n-1}^{f(k,n-1)} + P(p,\ldots,v_{n-2}) = u_{k,k(n-1)} \in I_{k,kn} \subseteq I_j$. Hence $v_{n-1}^{f(k,n-1)} \in I_j$. Therefore $v_{n-1} \in I_j$, since I_j is prime. This shows that $I_j = I_n$ which completes the induction.

COROLLARY 25. $\mathrm{Rad}(I_{k,k(n-1)+i}) = I_n$, *for* $i = 1,2,\ldots,k$.

PROOF. The proof is as in Lemma 24 by induction, and by using the same lemma.

COROLLARY 26. $I_n \subseteq \mathrm{Rad}\big(\mathrm{Ann}(\otimes_{j=1}^{k(n-1)+i}\gamma_{k,1})\big)$, *for* $i = 1,2,\ldots,k$.

4. **The radical of** $\mathrm{Ann}(\otimes_{i=1}^{n}\gamma_{k,1})$, $\mathrm{Ann}(\gamma_{k,n})$ **and** $\mathrm{Ann}(\iota_{k,n})$. In this section, using Lemma 23 f), and a theorem of Ravenel and Wilson [RW], we show that the radicals of all the annihilators mentioned in the title, are equal to I_n.

LEMMA 27. *If M is a* $BP_*\,BP$*-comodule, and $t \in M$ is v_{n-1}-torsion, then $I_n^r t = 0$ for some integer $r \geq 1$.*

PROOF. v_{n-1}-torsion, implies $p,\ldots,v_{n-2}$-torsion, by [JY]. Let $i_1,\ldots,i_n$ be such that $p^{i_1}t = \cdots = v_{n-1}^{i_n}t = 0$, then any r greater than $n \cdot \max\{i_j, j = 1,\ldots,n\}$ will satisfy the conclusion of the claim.

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LEMMA 28. *For any integer $r \geq 1$, there is an integer $m(r)$ such that $v_n^{m(r)} z_1 \in \mathrm{BP}_*(BZ/p^k)$ is a finite sum in $z_1, z_2, \ldots$ with coefficients in I_n^r.*

PROOF. By induction on r. If $r = 1$, we may take $m(1) = f(k, n)$ and we are done by (18) and Lemma 23 f). If the conclusion holds for $1, 2, \ldots, r-1$, then $v_n^{m(r-1)} z_1$ is a finite sum in $z_1, z_2, \ldots$ with coefficients in I_n^{r-1}. Now by multiplying this equality through by $v_n^{f(k,n)}$, we can reapply (18), via Lemma 23 f), to compute $v_n^{f(k,n)} z_j$ for every z_j that shows up in the sum. This gives us a sum with terms in z_l's with $l < j$, and terms in z_h's with $h \geq j$ whose coefficients are in I_n by Lemma 23 f). For each of the "lower" z's, we may repeat this finite process, if necessary, until we get terms with coefficients in I_n. This concludes the inductive step.

LEMMA 29. *$\otimes_{j=1}^n z_1 \in \otimes_{j=1}^n \mathrm{BP}_*(BZ/p^{i_j})$ is v_{n-1}-torsion.*

PROOF. Induction on n. If $n = 1$ then $p^{i_1} = 0$. Let the statement be true for $1, \ldots, n-1$. So $\otimes_{j=1}^{n-1} z_1$ is v_{n-2}-torsion and by Lemma 27, there is an r such that $I_{n-1}^r \otimes_{j=1}^{n-1} z_1 = 0$. By Lemma 28 there is an m such that $v_{n-1}^m z_1$ is a finite sum with coefficients in I_{n-1}^r, hence $v_{n-1}^m \otimes_{j=1}^n z_1 = 0$. This concludes the induction.

Next, we have immediately, by the Annihilator Ideal Test [JW-2]:

COROLLARY 30. *$\mathrm{hom\,dim}_{\mathrm{BP}_*} \otimes_{j=1}^n \mathrm{BP}_*(BZ/p^{i_j})$ is $\geq n$.*

COROLLARY 31. *$\mathrm{hom\,dim}_{\mathrm{BP}_*} \mathrm{BP}_*(\times_{j=1}^n BZ/p^{i_j})$ is $\geq n$.*

The special case $i_1 = \cdots = i_n = k$ would imply that the homological dimension of $\mathrm{BP}_*\big(K(Z/p^k, n)\big)$ is no less than n. But we already know that it is infinite by [JW-1].

We are finally in a position to compute the radicals of the annihilators mentioned above.

PROPOSITION 32. $\mathrm{Rad}\big(\mathrm{Ann}(\otimes_{i=1}^n \gamma_{k,1})\big) = \mathrm{Rad}\big(\mathrm{Ann}(\gamma_{k,n})\big) = \mathrm{Rad}\big(\mathrm{Ann}(\iota_{k,n})\big) = I_n$.

PROOF. The annihilator of a primitive element must be invariant by [L-1]. The radical of an invariant ideal must be a I_j by [L-2]. Since v_n-torsion implies v_{n-1}-torsion, we have that the radical of all the above annihilators contains I_n, by Lemma 29. We shall show next, that v_n is not in the largest of the radicals. For if it were, then there would be an i, such that $v_n^i \iota_{k,n} = 0$. This contradicts Theorem 13.4 of [RW] which states that $v_n^l p^{k-1} \iota_{k,n}$ is not zero for all integers $l \geq 0$. Therefore the same is true for $\otimes_{i=1}^n \gamma_{k,1}$ and $\gamma_{k,n}$ and the claim follows.

REMARK 33. a. The results of [RW] were proved for p odd, but they were extended to any prime in the Appendix of [JW-3].

b. When $k = 1$ the entire module $\mathrm{BP}_*(\times_{j=1}^n BZ/p^k)$ was computed in [JW-3].

c. The expected answer for the homological dimension of $\mathrm{BP}_*(\times_{j=1}^n BZ/p^k)$, is n.

d. We feel that $\mathrm{Ann}(\otimes_{i=1}^n \gamma_{k,1})$ should be contained in $I_{k,k(n-1)+1}$, but we have too little evidence to upgrade this statement, to the status of a conjecture.

https://doi.org/10.4153/CMB-1993-046-6 Published online by Cambridge University Press

5. **Some examples.** In this section we present a few particular examples that might motivate the interested reader to pursue his or her own computations.

Let $A_{k,n}$ be the annihilator ideal of the toral class of $\mathrm{BP}_*\big((BZ/p^k)^n\big)$. Then we know that $A_{k,1} = (p^k)$, by [CF], $A_{1,n} = I_{1,n} = (v_{1,0}, v_{1,1}, \ldots, v_{1,n-1}) = I_n = (p, v_1, \ldots, v_{n-1})$, by [RW] and [M],and $A_{2,2} = (p^2, pv_1, v_1^{p+2})$, by [N-1].

Let $p = 2$ and $k = 2$. By using the recursion (12) we can compute (by computer) the first few $v_{2,i}$'s:

$$v_{2,0} = 4, \quad v_{2,1} = 2v_1, \quad v_{2,2} = 2v_2 - v_1^3, \quad v_{2,3} = 2v_3 - v_1v_2^2 - 5v_1^4v_2 - 4v_1^7,$$
$$v_{2,4}= 2v_4 - v_1v_3^2 + 2v_1^2v_2^2v_3 + 10v_1^5v_2v_3 - 119v_1^8v_3 - 7v_2^5 + 12v_1^3v_2^4 - 9v_1^6v_2{}^3$$
$$-82v_1^9v_2^2 - 82v_1^{12}v_2 - 40v_1^{15}.$$

Hence the first few $I_{2,i}$'s in simplified generators are:

$$I_{2,1} = (4), \quad I_{2,2} = (4, 2v_1), \quad I_{2,3} = (4, 2v_1, 2v_2 + v_1^3),$$
$$I_{2,4} = (4, 2v_1, 2v_2 + v_1^3, 2v_3 + v_1v_2^2),$$
$$I_{2,5} = (4, 2v_1, 2v_2 + v_1^3, 2v_3 + v_1v_2^2, 2v_4 + v_1v_3^2 + v_2^5).$$

Note that the corresponding radicals are:

$\mathrm{Rad}(I_{2,1}) = \mathrm{Rad}(I_{2,2}) = (2)$, $\mathrm{Rad}(I_{2,3}) = \mathrm{Rad}(I_{2,4}) = (2, v_1)$, $\mathrm{Rad}(I_{2,5}) = (2, v_1, v_2)$, as predicted by Corollary 25. On the other hand since $A_{2,1} = (4)$, $A_{2,2} = (4, 2v_1, v_1^4)$ we have that $\mathrm{Rad}(A_{2,1}) = (2)$, $\mathrm{Rad}(A_{2,2}) = (2, v_1)$, which agrees with Proposition 32. So we see that $A_{1,1} = I_{k,1}$, $I_{2,2} \subseteq A_{2,2}$ and $A_{2,2} \subseteq I_{2,3}$. The last statement agrees with Remark 33 d).

One final observation in this case is that already $v_2^8 \in I_{2,5}$, with some very careful juggling of the relations.

The case $p = 3$, $n = 2$ gives similar results. Calculations reveal that, quite often, studying the case $p = 2$ is enough to predict what happens in general; for it seems that all the pathologies and complications in the $[p^k]$-series show up early on, when $p = 2$.

As a last example we discuss the case $p = 2$ and $k = 3$. This time we can only offer partial information starting with the following lemma that will be stated without proof, and a "natural" conjecture:

LEMMA 34. $(8, 4v_1, 2v_1^4, v_1^{10}) \subseteq A_{3,2}$.

CONJECTURE 35. $(8, 4v_1, 2v_1^4, v_1^{10}) = A_{3,2}$.

Actually one can prove with some care that for any prime p:

LEMMA 36. $(p^3, p^2v_1, pv_1^{p+2}, v_1^{p^2+2p+2}) \subseteq A_{3,2}$, *and form the corresponding conjecture that equality holds.*

If $p = 2$ we have:

$$v_{3,0} = 8, \quad v_{3,1} = 4v_1, \quad v_{3,2} = 4v_2 - 6v_1^3, \quad v_{3,3} = 4v_3 - 6v_1v_2^2 - 102v_1^4v_2 - 81v_1^7,$$

$$v_{3,4} = 4v_4 - 6v_1v_3^2 + 24v_1^2v_2^2v_3 + 408v_1^5v_2v_3 - 32442v_1^8v_3 - 126v_2^5 + 687v_1^3v_2^4 \\ -1956v_1^6v_2^3 - 21207v_1^9v_2^2 - 24429v_1^{12}v_2 - 11796v_1^{15}.$$

Hence the first few $I_{3,i}$'s in simplified generators are:

$$I_{3,1} = (8), \quad I_{3,2} = (8, 4v_1), \quad I_{3,3} = (8, 4v_1, 4v_2 + 2v_1^3),$$
$$I_{3,4} = (8, 4v_1, 4v_2 + 2v_1^3, 4v_3 + 2v_1v_2^2 + v_1^7),$$
$$I_{3,5} = (8, 4v_1, 4v_2 + 2v_1^3, 4v_3 + 2v_1v_2^2 + v_1^7, 4v_4 + 2v_1v_3^2 + 2v_2^5 + v_1^3v_2^4).$$

The corresponding radicals are:

$$\mathrm{Rad}(I_{3,1}) = \mathrm{Rad}(I_{3,2}) = \mathrm{Rad}(I_{3,3}) = (2), \quad \mathrm{Rad}(I_{3,4}) = \mathrm{Rad}(I_{3,5}) = (2, v_1),$$

as predicted by Corollary 25. On the other hand $A_{3,1} = (8), (8, 4v_1, 2v_1^4, v_1^{10}) \subseteq A_{3,2} \subseteq I_{3,4}$ and $\mathrm{Rad}(8, 4v_1, 2v_1^4, v_1^{10}) = \mathrm{Rad}(A_{3,2}) = \mathrm{Rad}(I_{3,4}) = (2, v_1)$.

AKNOWLEDGEMENT. We would like to aknowledge the informative conversations we had with Mike Boardman, J.-P. Meyer, Jack Morava and Steve Wilson on the topics presented above. We have also used freely, ideas and lemmas from our doctoral dissertation written under the advice of, and with help from Mike Boardman, at The Johns Hopkins University.

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