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On Types for Unramified p-Adic Unitary Groups

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Abstract. Let F be a non-archimedean local field of residue characteristic neither 2 nor 3 equipped with a galois involution with fixed field F_0, and let G be a symplectic group over F or an unramified unitary group over F_0. Following the methods of Bushnell–Kutzko for $\mathrm{GL}(N,F)$, we define an analogue of a simple type attached to a certain skew simple stratum, and realize a type in G. In particular, we obtain an irreducible supercuspidal representation of G like $\mathrm{GL}(N,F)$.

Introduction

Let N be an integer ≥ 2, and V an N-dimensional vector space over a non-archimedean local field F. Put $A = \mathrm{End}_F(V)$ and $G = \mathrm{Aut}_F(V) \simeq \mathrm{GL}(N,F)$.

From Bushnell–Kutzko [5], in which a complete classification of the irreducible smooth representations of G is given, we obtain the following results: a stratum in A is a 4-tuple $[\mathfrak{A}, n, 0, \beta]$ which consists of a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in A, an integer $n > 0$, and an element $\beta \in \mathfrak{P}^{-n}$, where $\mathfrak{o}_F$ is the maximal order of F, and $\mathfrak{P}$ is the Jacobson radical of $\mathfrak{A}$. We define a compact open subgroup $J = J(\beta, \mathfrak{A})$ of G and its normal subgroups $H^1(\beta,\mathfrak{A})$, $J^1(\beta,\mathfrak{A})$ [5, (3.1)], associated with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ [5, (1.5)]. Let θ be a simple character which is an abelian character of $H^1 = H^1(\beta,\mathfrak{A})$ [5, (3.2)]. Then there is a unique irreducible representation η of $J^1 = J^1(\beta,\mathfrak{A})$ such that $\eta|H^1$ contains θ [5, (5.1)], and is an irreducible representation κ of J, called a β-extension of η, which is an extension of η and has the G-intertwining $JB^\times J$ [5, (5.2)], where $B^\times$ is the G-centralizer of β.

Suppose that $\mathfrak{A}$ is principal. The group J/J^1 is isomorphic to a Levi subgroup of $\mathrm{GL}(R, k_E)$, where $R = \dim_E(V)$ and k_E denotes the residue class field of E. A certain irreducible cuspidal representation of J/J^1 is chosen and is inflated to the representation σ of J. Then an irreducible representation λ of J is defined by $\lambda = \kappa \otimes \sigma$, which is called a simple type (of positive level) [5, (5.5)]. If $\mathfrak{A} \cap B^\times$ is a maximal compact subgroup of $B^\times$, then the representation (J, λ) is a $[G, \pi]_G$-type in G, for some irreducible supercuspidal representation π of G [5, (6.2)], [6]. Such a simple type (J, λ) is called maximal.

Associated with a simple stratum $[\mathfrak{A}, n, 0, \beta]$, there is a choice of a parabolic subgroup $P = MN$ of G with a Levi component M [5, (7.1)]. From a simple type (J, λ), we can define a certain pair of a compact open subgroups J_P of G and an irreducible representation λ_P of J_P [5, (7.2)]. Then there is an irreducible supercuspidal representation π of M such that $(J_P \cap M, \lambda_P | J_P \cap M)$ is an $[M, \pi]_M$-type in M [5, (7.2)], [6],

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and (J_P, λ_P) is a G-cover of $(J_P \cap M, \lambda_P | J_P \cap M)$ [5, (7.3)], [6]. Hence (J_P, λ_P) is an $[M, \pi]_G$-type in G [6, (8.3)]. Moreover, the Hecke algebra of (J_P, λ_P) is isomorphic to an affine Hecke algebra [5, (5.6)].

Let F be a non-archimedean local field of residual characteristic not 2 equipped with a galois involution with fixed field F_0, and V a finite dimensional F-vector space equipped with a non-degenerate hermitian form h. Let G be the unitary group of (V, h) over F_0. Put $A = \mathrm{End}_F(V)$ and $\widetilde{G} = \mathrm{Aut}_F(V)$ here. From Stevens [28–30], we obtain the following results. A skew semi-simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A is defined, and we obtain the subgroups $H^1(\beta, \mathfrak{A})$, $J^1(\beta, \mathfrak{A})$ and $J(\beta, \mathfrak{A})$ of $\widetilde{G}$ as above. Restricting them to G, we obtain the subgroups $H^1_- = H^1_-(\beta, \mathfrak{A})$, $J^1_- = J^1_-(\beta, \mathfrak{A})$, and $J_- = J_-(\beta, \mathfrak{A})$ of G, respectively. A skew semi-simple character θ_- of H^1_- is defined as well, and we can similarly give a unique irreducible representation η_- of J^1_- such that $\eta_- | H^1_-$ contains θ_-. In particular, if the A-centralizer of β is a maximal commutative semisimple algebra of A, there is an irreducible representation κ_- of J_- such that $\kappa_- | J^1_- = \eta_-$, which is a β-extension of η_- in a sense. The representation (J_-, κ_-) induces an irreducible supercuspidal representation of G, and so it is a type in G [2, 17, 32]. In general, it is very difficult to prove the existence of a β-extension of η_- even for a skew simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A.

Now suppose that h is a non-degenerate alternating form on a $2n$-demensional F-vector space V. Then G is a symplectic group $Sp_{2n}(F)$. Recently, the following results for $G = Sp_{2n}(F)$ were obtained [3]. Let π be a self-contragradient supercuspidal irreducible representation of $\mathrm{GL}(n, F)$ [1, 14], and (J_0, λ_0) a maximal simple type in $\mathrm{GL}(n, F)$ for the inertial class $[\mathrm{GL}(n, F), \pi]_{\mathrm{GL}(n,F)}$. We can take a special simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A = \mathrm{End}_F(V)$ such that the associated parabolic subgroup $P = MN$ of $\mathrm{GL}(2n, F)$ satisfies $M \simeq \mathrm{GL}(n, F) \times \mathrm{GL}(n, F)$ and leads to a Siegel parabolic subgroup $P_0 = M_0 N_0$ of G with $M_0 \simeq \mathrm{GL}(n, F)$. Then there is a simple type (J, λ) in $\mathrm{GL}(2n, F)$ attached to $[\mathfrak{A}, n, 0, \beta]$ such that $J \cap M \simeq J_0 \times J_0$ and $\lambda | (J \cap M) \simeq \lambda_0 \otimes \lambda_0$. Thus we can construct an irreducible representation (J_P, λ_P) in $\mathrm{GL}(2n, F)$ from (J, λ) as above, and restrict (J_P, λ_P) to G so as to obtain an $[M_0, \pi]_G$-type in G as a G-cover of (J_0, λ_0). The methods of [3] construct a type in G without using a simple type for G.

Recently, the problem of constructing (simple) types for $\mathrm{GL}(N, D)$, with D a central division F-algebra, was solved by Sécherre [23–25].

In this paper, let F be a non-archimedean local field of residual characteristic neither 2 nor 3 equipped with a galois involution with fixed field F_0. We assume that F/F_0 is an unramified field extension, and let h be a non-degenerate F/F_0-skew-hermitian form on a vector space V of dimension $2n$ over F such that the anisotropic part is zero. Put $G = U(V, h)$. Following the methods of Bushnell–Kutzko [5], we define a simple type for G attached to a certain skew simple stratum in $A = \mathrm{End}_F(V)$, which is called good (see Definition 2.1), and realize a type in G. A simple type in $Sp_{2n}(F)$, attached to a good skew simple stratum $[\mathfrak{A}, n, 0, \beta]$ with $\mathfrak{A}$ principal and with $e(\mathfrak{B} | \mathfrak{o}_{F[\beta]}) = 2$, gives the one constructed in Blondel [3], where $e(\mathfrak{B} | \mathfrak{o}_{F[\beta]})$ denotes the $\mathfrak{o}_{F[\beta]}$-period of the lattice chain in V defining the $\mathfrak{A}$-centralizer $\mathfrak{B}$ of β.

The contents of this paper are as follows: In Sections 1 and 2, from [5, 29], we recall the definitions of the skew simple stratum $[\mathfrak{A}, n, 0, \beta]$, the compact open sub-

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groups $H^t(\beta,\mathfrak{A})$, $J^t(\beta,\mathfrak{A})$ of G, for $t = 0,1$, and the skew simple character $\theta_- \in \mathcal{C}_-(\mathfrak{A},0,\beta)$. We define a good skew simple stratum $[\mathfrak{A},n,0,\beta]$, which implies that there are hereditary $\mathfrak{o}_F$-orders $\mathfrak{A}_m \subset \mathfrak{A} \subset \mathfrak{A}_M$ in $A = \operatorname{End}_F(V)$ such that $\boldsymbol{U}(\mathfrak{B}_m) = \mathfrak{A}_m \cap B \cap G$ is an Iwahori subgroup of $B \cap G$ and $\boldsymbol{U}(\mathfrak{B}_M) = \mathfrak{A}_M \cap B \cap G$ is a special (good) maximal compact subgroup of $B \cap G$, where B is the A-centralizer of β. This property is used to prove the existence of a β-extension.

In Section 3, let $[\mathfrak{A},n,0,\beta]$ be a good skew simple stratum in A. From [30], there is a unique irreducible representation η_- of $J^1_-(\beta,\mathfrak{A})$ associated with a skew simple character θ_-. Modulo some claim, we can prove that there is a β-extension κ_- of η_-, which is by definition a representation of $J_- = J_-(\beta,\mathfrak{A})$ satisfying (i) $\kappa_-|J_- = \eta_-$, (ii) the G-intertwining of κ_- contains $J_-.B\cap G.J_-$.

In Section 4, we have a parabolic subgroup $P = MN_u$ of G, with Levi component M and unipotent radical N_u, associated with a good skew simple stratum $[\mathfrak{A},n,0,\beta]$ in A. We see that $H^t_-(\beta,\mathfrak{A})$, $J^t_-(\beta,\mathfrak{A})$, $t = 0,1$, have Iwahori decompositions relative to $P = MN_u$, and prove the claim in Section 3.

In Section 5, let $[\mathfrak{A},n,0,\beta]$ be a good skew simple stratum in A with $\mathfrak{A}$ principal. We choose a certain irreducible cuspidal representation σ_- of $J_-(\beta,\mathfrak{A})/J^1_-(\beta,\mathfrak{A})$. From this σ_-, together with a β-extension κ_-, we define an irreducible representation $\lambda_- = \kappa_- \otimes \sigma_-$ of $J_-(\beta,\mathfrak{A})$, that is an analogue of a simple type of positive level for $\mathrm{GL}(N,F)$ of [5, (5.5.10)]. Let $\boldsymbol{W}$ be an affine Weyl group of $B\cap G$ with $B\cap G = \boldsymbol{U}(\mathfrak{B}_m)\boldsymbol{W}\boldsymbol{U}(\mathfrak{B}_m)$, and put $\boldsymbol{W}(\mathfrak{B}) = \{w \in \boldsymbol{W} \mid w \text{ normalizes } \mathfrak{A}\cap M\cap B\}$. We prove that the G-intertwining of the simple type (J_-,λ_-) is contained in $J_-\boldsymbol{W}(\mathfrak{B})J_-$. It follows that if $\mathfrak{A}\cap B$ is a maximal compact subgroup of $G\cap B$, (J_-,λ_-) induces an irreducible supercuspidal representation of G. Moreover, we construct an irreducible representation $(J_{P,-},\lambda_{P,-})$, in the same way as [5], such that $(J_{P,-}\cap M, \lambda_{P,-}|J_{P,-}\cap M)$ is an $[M,\pi]_M$-type in M, for some irreducible supercuspidal representation π of M.

In Section 6, we study the Hecke algebra $\mathcal{H}(G,\lambda_{P,-})$ of $(J_{P,-},\lambda_{P,-})$, and then we prove that $(J_{P,-},\lambda_{P,-})$ is an $[M,\pi]_G$-type in G, and so is (J_-,λ_-).

1 Preliminaries

1.1 Unramified Unitary Groups

Let F be a non-archimedean local field equipped with a galois involution $^-$, with the fixed field F_0. Let $\mathfrak{o}_F$ and $\mathfrak{p}_F$ be its maximal order and the maximal ideal of $\mathfrak{o}_F$, respectively, and $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue class field. Let ϖ_F be a uniformizer of F. We assume that the residual characteristic p is not 2 and that F/F_0 is unramified (possibly $F = F_0$).

Let N be an integer ≥ 4. Let V be an N-dimensional vector space over F, and put $A = \operatorname{End}_F(V) \simeq \mathbb{M}(N,F)$. Let h be a non-degenerate anti-hermitian form on V over F/F_0. We furthermore assume that the anisotropic part of V is zero. Then N must be even. Let $^-$ be the adjoint (anti-)involution on A defined by the form h. Put $\widetilde{G} = \operatorname{Aut}_F(V) \simeq \mathrm{GL}(N,F)$, and define γ to be the involution $x \mapsto \bar{x}^{-1}$ on $\widetilde{G}$. Put $\Gamma = \{1,\gamma\}$.

We put

$$G = \widetilde{G}^{\Gamma} = \{g \in \widetilde{G} \mid h(gv, gw) = h(v, w), \text{ for all } v, w \in V\}.$$

By the assumption, G is a symplectic group over F if $F = F_0$, and is an unramified unitary group over F_0 if $F \neq F_0$. We write $G = U(V, h)$. We also put

$$\mathfrak{G} = \{a \in A \mid a + \bar{a} = 0\}.$$

This is isomorphic to Lie G.

Let $\mathbb{Z}$ and $\mathbb{C}$ denote the ring of rational integers and the field of complex numbers, respectively. For a ring R, let $R^{\times}$ denote the multiplicative group of invertible elements in R. For a finite field extension E/F, we denote by $\mathfrak{o}_E$, $\mathfrak{p}_E$, k_E the objects for E analogous to those above for F.

1.2 Filtrations

We recall the notation in [5, 19].

For an $\mathfrak{o}_F$-lattice in V, we define the dual lattice $L^{\#}$ by

$$L^{\#} = \{v \in V \mid h(v, L) \subset \mathfrak{o}_F\}$$

[19, 1.1]. An $\mathfrak{o}_F$-*lattice chain* in V is a set $\mathcal{L} = \{L_i \mid i \in \mathbb{Z}\}$ of $\mathfrak{o}_F$-lattices in V which satisfies

- $L_i \supsetneq L_{i+1}$, for all $i \in \mathbb{Z}$,
- there is a positive integer e such that $L_{i+e} = \mathfrak{p}_F L_i$, for all $i \in \mathbb{Z}$.

This integer $e = e(\mathcal{L})$ is unique and is called the $\mathfrak{o}_F$-*period* of $\mathcal{L}$.

A $\mathfrak{o}_F$-lattice chain $\mathcal{L}$ in V is called *self-dual* (with respect to the form h) if $L \in \mathcal{L}$ implies $L^{\#} \in \mathcal{L}$. If $\mathcal{L}$ is self-dual, from [19, Proposition 1.4], there is a unique slice of the form:

$$L_{r-1}^{\#} \supsetneq \cdots \supsetneq L_0^{\#} \supset L_0 \supsetneq \cdots \supsetneq L_{r-1} \supset \varpi_F L_{r-1}^{\#},$$

for some integer $r \geq 1$, where possibly $L_0^{\#} = L_0$ and/or $L_{r-1} = \varpi_F L_{r-1}^{\#}$. This slice is called a *self-dual slice* of $\mathcal{L}$.

Associated with an $\mathfrak{o}_F$-lattice chain $\mathcal{L}$ in V, a filtration on A is given by

$$\mathfrak{P}^n = \{x \in A \mid xL_i \subset L_{i+n}, \text{ for all } i \in \mathbb{Z}\},$$

for $n \in \mathbb{Z}$. In particular, $\mathfrak{A} = \mathfrak{A}(\mathcal{L}) = \mathfrak{P}^0$ is a hereditary $\mathfrak{o}_F$-order in A, and $\mathfrak{P}$ is its Jacobson radical. An $\mathfrak{o}_F$-lattice chain $\mathcal{L}$ in V determines a valuation map $\nu_{\mathfrak{A}} \colon A \to \mathbb{Z}$ by

$$\nu_{\mathfrak{A}}(x) = \max\{n \in \mathbb{Z} \mid x \in \mathfrak{P}^n\}, \text{ for } x \in A,$$

with $\nu_{\mathfrak{A}}(0) = \infty$.

We obtain a family of compact open subgroups $\mathfrak{A} \cap \widetilde{G} = \mathfrak{A}^{\times}$ and $1 + \mathfrak{P}^n$ for integers $n \geq 1$, of $\widetilde{G}$. If $\mathcal{L}$ is self-dual, $\mathfrak{A}^{\times}$ and $1 + \mathfrak{P}^n$, $n \geq 1$, are fixed by γ. So we obtain a family of compact open subgroups of G

$$\boldsymbol{U}(\mathfrak{A}) = (\mathfrak{A}^{\times})^{\Gamma} = \mathfrak{A} \cap G, \quad \boldsymbol{U}^n(\mathfrak{A}) = (1 + \mathfrak{P}^n)^{\Gamma} = (1 + \mathfrak{P}^n) \cap G,$$

for integers $n \geq 1$. Then $\{\boldsymbol{U}^n(\mathfrak{A}) \mid n \geq 1\}$ is a filtration on G by normal subgroups of $U(\mathfrak{A})$.

For an $\mathfrak{o}_F$-order $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$ in A, we put

$$\mathfrak{K}(\mathfrak{A}) = \{x \in \widetilde{G} \mid xL \in \mathcal{L}, \text{ for all } L \in \mathcal{L}\}.$$

Then we have $\mathfrak{K}(\mathfrak{A}) = \{x \in \widetilde{G} \mid x\mathfrak{A}x^{-1} = \mathfrak{A}\}$.

1.3 An E-anti-hermitian Form

Suppose that β is an element in the Lie algebra $\mathfrak{G}$ such that the algebra $E = F[\beta]$ is a subfield of A. Then the involution $^-$ on A fixes E. Put $E_0 = \{x \in E \mid \overline{x} = x\}$. We choose an F-linear form $\ell_0 \colon E_0 \to F$ which satisfies

$$\ell_0(\mathfrak{o}_{E_0}) = \mathfrak{o}_{F_0}, \ell_0(\mathfrak{p}_{E_0}^{-1}) = \mathfrak{p}_{F_0}^{-1}$$

as in [3, 2.3]. We define an F-linear form $\ell \colon E \to F$ as follows: if $F = F_0$, put

$$\ell = \ell_0 \circ \operatorname{tr}_{E/E_0}.$$

Otherwise, we extend ℓ_0 to E linearly. In fact, since F/F_0 is unramified and the residual characteristic p of F is not 2, there is an element $\xi \in \mathfrak{o}_F^{\times}$ such that $F = F_0[\xi]$, $E = E_0[\xi]$, and $\xi^2 \in F_0$. We note that E/E_0 is also unramified. Thus we have $\mathfrak{o}_F = \mathfrak{o}_{F_0} + \mathfrak{o}_{F_0}\xi$, $\mathfrak{o}_E = \mathfrak{o}_{E_0} + \mathfrak{o}_{E_0}\xi$. Hence $\ell \colon E \to F$ is given by

$$\ell(x + y\xi) = \ell_0(x) + \ell_0(y)\xi \tag{1.1}$$

for all $x, y \in E_0$. Hereafter we fix this F-linear form $\ell \colon E \to F$.

From the F-linear form ℓ on $E = F[\beta]$ and the form h on V, we can define an E-anti-hermitian form $\widetilde{h}_\beta$ on V by

$$h(av, w) = \ell(a\widetilde{h}_\beta(v, w)) \tag{1.2}$$

for all $v, w \in V$ and all $a \in E$ [26]. Then $\widetilde{h}_\beta$ is non-degenerate. Let $B = B_\beta$ be the A-centralizer of β. Then we may identify B with $\operatorname{End}_E(V)$.

By definition, we have

$$\ell_0^{-1}(\mathfrak{o}_{F_0}) = \mathfrak{o}_{E_0}. \tag{1.3}$$

Proposition 1.1 *The form $\widetilde{h}_\beta$ is a non-degenerate E/E_0-anti-hermitian form on V, and there is a canonical isomorphism*

$$B^{\times} \cap G = \{x \in B^{\times} \mid \gamma(x) = x\} \simeq U(V, \widetilde{h}_\beta).$$

Proof In the case of $F = F_0$, this follows easily [3, 2.3]. Suppose that $F \neq F_0$. By the assumption, E/E_0 is unramified, as was noted above. It follows from the definition of the F-linear form ℓ above that $\ell(\overline{z}) = \overline{\ell(z)}$ for $z \in E$, whence this shows that $\widetilde{h}_\beta$, defined by (1.2), is a non-degenerate E-anti-hermitian form. ■

Lemma 1.2 *We have*

$$\ell^{-1}(\mathfrak{o}_F) = \mathfrak{p}_E^{1-e(E|E_0)},$$

where $e(E \mid E_0)$ denotes the ramification index of E/E_0.

Proof We again note that if G is an unramified unitary group over F_0 (with $F \neq F_0$), E/E_0 must be unramified.

Write $e_0 = e(E \mid E_0)$. Since $p \neq 2$, E/E_0 is tamely ramified. Thus, by [33, VIII, §1, Proposition 4], we have

$$\mathrm{tr}_{E/E_0}^{-1}(\mathfrak{o}_{E_0}) = \mathfrak{p}_E^{1-e_0}. \tag{1.4}$$

Suppose first that $e_0 = 1$, *i.e.*, E/E_0 is unramified. If $F = F_0$, the assertion follows directly from (1.3) and (1.4). Suppose that $F \neq F_0$. Then $\mathfrak{o}_E \subset \ell^{-1}(\mathfrak{o}_F)$ follows immediately. Conversely, let $z = x + y\xi \in \ell^{-1}(\mathfrak{o}_F)$, for $x, y \in E_0$. Then from (1.1), $\ell(z) = \ell_0(x) + \ell_0(y)\xi \in \mathfrak{o}_F$, and so $\ell_0(x), \ell_0(y) \in \mathfrak{o}_{F_0}$. Hence from (1.3) $x, y \in \mathfrak{o}_{E_0}$, that is, $z = x + y\xi \in \mathfrak{o}_E$.

Suppose that $e_0 = 2$, *i.e.*, E/E_0 is ramified. Then we must have $F = F_0$. For, since F/F_0 is assumed to be unramified, it follows from (1.4) that $\mathrm{tr}_{E/E_0}^{-1}(\mathfrak{o}_{E_0}) = \mathfrak{p}_E^{-1}$. Thus from (1.3),

$$\mathrm{tr}_{E/E_0}^{-1}(\ell_0^{-1}(\mathfrak{o}_F)) = \mathrm{tr}_{E/E_0}^{-1}(\mathfrak{o}_{E_0}) = \mathfrak{p}_E^{-1} = \mathfrak{p}_E^{1-e_0}. \qquad \blacksquare$$

1.4 Self-dual Lattice Chains

Suppose that β is an element in the Lie algebra $\mathfrak{G}$ such that the algebra $E = F[\beta]$ is a subfield of A, as in Section 1.3. Let L be an $\mathfrak{o}_E$-lattice in V. Then L is also an $\mathfrak{o}_F$-lattice in V. We define the $\mathfrak{o}_E$-dual $L^\natural$ of L, with respect to $\widetilde{h}_\beta$, by

$$L^\natural = \{v \in V \mid \widetilde{h}_\beta(v, L) \subset \mathfrak{o}_E\}.$$

There is a close relationship between $L^\#$ and $L^\natural$ as follows.

Proposition 1.3 *For an $\mathfrak{o}_E$-lattice L in V, we have*

$$L^\# = \varpi_E^{1-e(E|E_0)} L^\natural,$$

where ϖ_E is a uniformizer of E.

Proof From (1.2), we have an equivalence: $v \in L^\# \Leftrightarrow \mathfrak{o}_F \supset h(v, L) = \ell(\widetilde{h}_\beta(v, L))$. From Lemma 1.2, the latter is equivalent to

$$\mathfrak{p}_E^{1-e_0} \supset \widetilde{h}_\beta(v, L) \Longleftrightarrow \mathfrak{o}_E \supset \widetilde{h}_\beta(\varpi_E^{e_0-1} v, L) \Longleftrightarrow v \in \varpi_E^{1-e_0} L^\natural,$$

where $e_0 = e(E \mid E_0)$. $\blacksquare$

Let $\mathfrak{L}$ be an $\mathfrak{o}_F$-lattice chain in V such that $E^\times \subset \mathfrak{K}(\mathfrak{A})$, with $\mathfrak{A} = \mathfrak{A}(\mathfrak{L})$. Then it follows from [5, (1.2.1)] that $\mathfrak{L}$ is also an $\mathfrak{o}_E$-lattice chain in V, which is denoted by $\mathfrak{L}_{\mathfrak{o}_E}$. Thus, as in Section 1.2, $\mathfrak{L}$ has a unique self-dual slice of the form

$$L_{r-1}^\natural \supsetneq \cdots \supsetneq L_0^\natural \supset L_0 \supsetneq \cdots \supsetneq L_{r-1} \supset \varpi_E L_{r-1}^\natural \tag{1.5}$$

for some integer $r \geq 1$, with respect to the form $\widetilde{h}_\beta$.

Proposition 1.4 *Let $\mathfrak{L}$ be a self-dual $\mathfrak{o}_E$-lattice chain in V with respect to $\widetilde{h}_\beta$. Then it is also a self-dual $\mathfrak{o}_F$-lattice chain in V with respect to h. Moreover, we have the following.*

(i) *Suppose that E/E_0 is unramified. If the self-dual slice of $\mathfrak{L}$ of the form* (1.5) *satisfies $L_0^\natural = L_0$, then $L_0^\# = L_0$ as an $\mathfrak{o}_F$-lattice.*
(ii) *Suppose that E/E_0 is ramified. If the self-dual slice of $\mathfrak{L}$ satisfies $\varpi_E L_{r-1}^\natural = L_{r-1}$, then it contains an $\mathfrak{o}_E$-lattice M in V such that $M^\# = M$ as an $\mathfrak{o}_F$-lattice.*

Proof The first assertion and (i) follow immediately from Proposition 1.3. We show (ii). Write $e = e(\mathfrak{L}_{\mathfrak{o}_E})$ for the $\mathfrak{o}_E$-period of $\mathfrak{L}$. From Lemma 1.2, it follows that $M = \varpi_E^{-1} L_{r-1}$ is the desired lattice. For we have

$$\begin{aligned}(\varpi_E^{-1} L_{r-1})^\# &= (L_{-e+r-1})^\# = \varpi_E^{-1} L_{-e+r-1}^\natural \\ &= (\varpi_E L_{-e+r-1})^\natural = L_{r-1}^\natural = \varpi_E^{-1} L_{r-1}. \qquad \blacksquare\end{aligned}$$

2 Skew Simple Strata

2.1 Skew Simple Strata

We recall the definition of a skew simple stratum in [5, 29], and define a good skew simple stratum in A.

A stratum in A is a 4-tuple $[\mathfrak{A}, n, r, b]$, which consists of a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in A, integers $n > r$, and an element $b \in A$ such that $\nu_{\mathfrak{A}}(b) \geq -n$.

Definition 2.1 ([29, (1.7)]) A stratum $[\mathfrak{A}, n, r, b]$ in A is called *skew* if the lattice chain $\mathfrak{L}$, with $\mathfrak{A} = \mathfrak{A}(\mathfrak{L})$, is self-dual and $b \in \mathfrak{G} \simeq \mathrm{Lie}(G)$.

Definition 2.2 ([5, (1.5.5)]) A stratum $[\mathfrak{A}, n, r, \beta]$ in A is *pure* if

(i) the algebra $E = F[\beta]$ is a field,
(ii) $E^\times \subset \mathfrak{K}(\mathfrak{A})$,
(iii) $\nu_{\mathfrak{A}}(\beta) = -n$.

For a pure stratum $[\mathfrak{A}, n, r, \beta]$ in A, the integer $k_0(\beta, \mathfrak{A})$ of [5, (1.4.5)] is defined.

Definition 2.3 ([5, (1.5.5)]) A pure stratum $[\mathfrak{A}, n, r, \beta]$ in A is *simple* if it satisfies $r < -k_0(\beta, \mathfrak{A})$.

Let $[\mathfrak{A}, n, r, \beta]$ be a pure stratum in A. Then the rings $\mathfrak{H}(\beta, \mathfrak{A})$, $\mathfrak{J}(\beta, \mathfrak{A})$ of [5, (3.1)] are defined. We define

$$H(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A})^\times, \quad J(\beta, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A})^\times$$

subgroups of G, and for an integer $m \geq 1$,

$$H^m(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A}) \cap (1 + \mathfrak{P}^m), \quad J^m(\beta, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A}) \cap (1 + \mathfrak{P}^m)$$

normal subgroups of $H(\beta, \mathfrak{A})$ and $J(\beta, \mathfrak{A})$, respectively. A simple character set $\mathcal{C}(\mathfrak{A}, m, \beta)$, for an integer $m \geq 0$, of [5, (3.2)] is defined. An element of $\mathcal{C}(\mathfrak{A}, m, \beta)$ is a certain abelian character of the group $H^{m+1}(\beta, \mathfrak{A})$.

Let $[\mathfrak{A}, n, 0, \beta]$ be a skew simple stratum in A, with $r = -k_0(\beta, \mathfrak{A})$. Then $\mathfrak{H}(\beta, \mathfrak{A})$ and $\mathfrak{J}(\beta, \mathfrak{A})$ are fixed by Γ. For $0 \leq m \leq r - 1$, the subset $\mathcal{C}^\Gamma(\mathfrak{A}, m, \beta)$ of $\mathcal{C}(\mathfrak{A}, m, \beta)$ is defined in [28, 3.2] by $\mathcal{C}^\Gamma(\mathfrak{A}, m, \beta) = \{\theta \in \mathcal{C}(\mathfrak{A}, m, \beta) \mid \theta^\gamma = \theta\}$, where $\theta^\gamma(x) = \theta(\gamma(x))$, for $x \in H^{m+1}(\beta, \mathfrak{A})$.

We define two families of compact open subgroups of G as follows:

$$H^m_-(\beta, \mathfrak{A}) = H^m(\beta, \mathfrak{A})^\Gamma = H^m(\beta, \mathfrak{A}) \cap G,$$

$$J^m_-(\beta, \mathfrak{A}) = J^m(\beta, \mathfrak{A})^\Gamma = J^m(\beta, \mathfrak{A}) \cap G,$$

for integers $m \geq 0$. From [28, (2.1)], there is a correspondence $\mathbf{g}$, which is called *Glauberman's correspondence*, between the set of equivalence classes of irreducible representations of $H^{m+1}(\beta, \mathfrak{A})$ fixed by Γ and the set of equivalence classes of irreducible representations of $H^{m+1}_-(\beta, \mathfrak{A})$. In particular, for $\theta \in \mathcal{C}^\Gamma(\mathfrak{A}, m, \beta)$, we have $\mathbf{g}(\theta) = \theta | H^{m+1}_-(\beta, \mathfrak{A})$. We put $\mathcal{C}_-(\mathfrak{A}, m, \beta) = \{\mathbf{g}(\theta) \mid \theta \in \mathcal{C}^\Gamma(\mathfrak{A}, m, \beta)\}$.

An element of $\mathcal{C}_-(\mathfrak{A}, m, \beta)$ is called a *skew simple character*.

2.2 Good Skew Simple Strata

Suppose that $[\mathfrak{A}, n, 0, \beta]$ is a skew simple stratum in A, with $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$. Let $E = F[\beta]$ and $B = B_\beta$ the A-centralizer of β. Let E_0 be the fixed field of E under the involution $^-$ on A. From Proposition 1.3, $\mathcal{L}$ is a self-dual $\mathfrak{o}_E$-lattice chain in V with respect to the form $\widetilde{h}_\beta$. Thus $\mathcal{L}_{\mathfrak{o}_E}$ has a self-dual slice of the form (1.5).

Definition 2.4 A skew simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A, with $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$, is called *good* if it satisfies

(i) E/E_0 is unramified;
(ii) $R = \dim_E(V)$ is even;
(iii) the self-dual slice of $\mathcal{L}_{\mathfrak{o}_E}$ of the form (1.5) contains the L_0 satisfying $L_0^\natural = L_0$.

Proposition 2.5 *If the conditions* (i), (ii) *and* (iii) *in Definition* 2.4 *are satisfied, the anisotropic part of* $(V, \widetilde{h}_\beta)$ *is zero.*

Proof A proof is found in [3, 2.3]. ∎

If $[\mathfrak{A}, n, 0, \beta]$ is a good skew simple stratum in A, from [5, (5.5.2), (7.1.2)(ii)], we have an E-decomposition of V subordinated to $\mathcal{L}_{\mathfrak{o}_E}$, with $e = e(\mathcal{L}_{\mathfrak{o}_E})$:

$$V = \bigoplus_{i=1}^{e} V^i \tag{2.1}$$

such that

(i) $L_k = \coprod_{i=1}^{e} L_k^i$, where $L_k^i = L_k \cap V^i$, for $1 \le i \le e,\ k \in \mathbb{Z}$;
(ii) $L_{i+me}^i = L_{i+me+1}^i = \cdots = L_{i+(m+1)e-1}^i \ne L_{i+(m+1)e}^i$, for $1 \le i \le e,\ m \in \mathbb{Z}$.

Lemma 2.6 *Let $[\mathfrak{A}, n, 0, \beta]$ be a good skew simple stratum in A, with $\mathfrak{A} = \mathfrak{A}(\mathfrak{L})$, $E = F[\beta]$ and $e = e(\mathfrak{L}_{\mathfrak{o}_E})$. For the self-dual slice of $\mathfrak{L}_{\mathfrak{o}_E}$ of the form* (1.5)*, there is a Witt basis for L_0,*

$$\mathcal{V} = \{v_1, v_2, \dots, v_R\}, \tag{2.2}$$

such that $L_0 = \mathfrak{o}_E v_1 \oplus \mathfrak{o}_E v_2 \oplus \cdots \oplus \mathfrak{o}_E v_R$, and each pair $\{v_j, v_{R-j+1}\}$ generates a hyperbolic E-subspace of V relative to $\widetilde{h}_\beta$. Write $L_0 = \mathfrak{o}_E\langle\mathcal{V}\rangle$. For the E-decomposition (2.1) *of V. Each V^i is spanned by $\mathcal{V}^i = \mathcal{V} \cap V^i = \{v_{j_{i-1}+1}, v_{j_{i-1}+2}, \dots, v_{j_i}\}$ over E, and $L_k = \coprod_i L_k^i$, $0 \le k \le [e/2]$, satisfies*

$$L_k^i = \begin{cases} \mathfrak{o}_E\langle\mathcal{V}^i\rangle & \text{for } i \le e-k, \\ \mathfrak{p}_E\langle\mathcal{V}^i\rangle & \text{for } i \ge e-k+1, \end{cases}$$

where $j_0, j_1, \dots, j_e$ are integers with $0 = j_0 < j_1 < \cdots < j_e = R$ and for a real number r, $[r]$ denotes the largest integer $\le r$.

Proof This follows directly from Proposition 1.1 and [19, Proposition 1.7]. ■

Proposition 2.7 *Suppose that $[\mathfrak{A}, n, 0, \beta]$ is a good skew simple stratum in A with $\mathfrak{A} = \mathfrak{A}(\mathfrak{L})$. Let $E = F[\beta]$ and $B = B_\beta$ the A-centralizer of β, and $e = e(\mathfrak{L}_{\mathfrak{o}_E})$. Put $t = [(e+1)/2]$. Then the E-vector space V is decomposed into an orthogonal decomposition $V = \perp_{i=1}^t V_i$, $\widetilde{h}_\beta = \perp_{i=1}^t \widetilde{h}_i$ such that for $1 \le i \le [e/2]$, $(V_i, \widetilde{h}_i)$ is a hyperbolic space, where V^i and V^{e-i+1} are totally isotropic subspaces of V_i.*

Proof From (2.1), for $1 \le i \le [e/2]$, put $V_i = V^i \oplus V^{e-i+1}$, $\widetilde{h}_i = \widetilde{h}_\beta | V_i$, and if $t = (e+1)/2$ is an integer, put $V_t = V^t$, $\widetilde{h}_t = \widetilde{h}_\beta | V_t$. Then the assertion follows directly from [19, Propositions 1.7, 1.12]. ■

Let $\mathfrak{A}$, $E = F[\beta]$ be as above, and $B = B_\beta$ be the A-centralizer of β. Put $\mathfrak{B} = B \cap \mathfrak{A}$. We define a compact open subgroup of G by $\boldsymbol{U}(\mathfrak{B}) = \mathfrak{A} \cap B^\times \cap G$, and a family of normal subgroups of $\boldsymbol{U}(\mathfrak{B})$ by $\boldsymbol{U}^m(\mathfrak{B}) = (1 + \mathfrak{P}^m) \cap B^\times \cap G = (1 + \mathfrak{Q}^m) \cap G$, for integers $m \ge 1$, where $\mathfrak{Q} = \mathfrak{P} \cap B$.

Proposition 2.8 *Suppose that $[\mathfrak{A}, n, 0, \beta]$ is a good skew simple stratum in A with $\mathfrak{A} = \mathfrak{A}(\mathfrak{L})$. Let $E = F[\beta]$, $B = B_\beta$ the A-centralizer of β, and $e = e(\mathfrak{L}_{\mathfrak{o}_E})$. Put $t = [(e+1)/2]$. Suppose moreover that the lattice chain $\mathfrak{L}_{\mathfrak{o}_E}$ has self-dual slice of the form* (1.5)*. Then there is a canonical isomorphism*

$$\boldsymbol{U}(\mathfrak{B})/\boldsymbol{U}^1(\mathfrak{B}) \simeq \begin{cases} \prod_{i=1}^{e/2} \mathrm{Aut}_{k_E}(\overline{V}^i) & \text{if } e \text{ is even}, \\ \prod_{i=1}^{(e-1)/2} \mathrm{Aut}_{k_E}(\overline{V}^i) \times \boldsymbol{U}(\overline{V}_t, \overline{h}_t) & \text{if } e \text{ is odd}, \end{cases}$$

where $\overline{V}^i = L_{i-1}/L_i$, for $1 \le i \le [e/2]$, and if $t = (e+1)/2$ is an integer, $\overline{V}_t = L_{t-1}/\varpi_E L_{t-1}^\natural$ and $\overline{h}_t$ is a non-degenerate form, induced naturally from $\widetilde{h}_\beta$. Moreover, $(\overline{V}_t, \overline{h}_t)$ is a k_E/k_{E_0}-anti-hermitian space whose anisotropic part is zero.

Proof This follows at once from Proposition 2.7 and [19, 1.10 and Proposition 1.12]. In particular, the last assertion follows from Proposition 2.5 and [19, 1.10]. ■

3 Beta Extensions

3.1 Heisenberg Representations

Following the methods of [5, 30], we prove the existence of a beta extension for our classical group G. Hereafter, we assume that the residual characteristic p of F is neither 2 nor 3.

If ρ is a representation of a compact open subgroup K of G, and $g \in G$, we write $I_g(\rho) = \mathrm{Hom}_{K^g \cap K}(\rho, \rho^g)$, where $K^g = g^{-1}Kg$ and $\rho^g(x) = \rho(gxg^{-1})$, for $x \in K^g \cap K$.

Proposition 3.1 ([5, (5.1.1)]) *Let $[\mathfrak{A}, n, 0, \beta]$ be a skew simple stratum in A, and $\theta_- \in \mathcal{C}_-(\mathfrak{A}, 0, \beta)$. Then there is a unique irreducible representation $\eta_- = \eta(\theta_-)$ of $J^1_-(\beta, \mathfrak{A})$ such that $\eta_- | H^1_-(\beta, \mathfrak{A})$ contains θ_-. We have*

$$\dim(\eta_-) = (J^1_-(\beta, \mathfrak{A}) : H^1_-(\beta, \mathfrak{A}))^{\frac{1}{2}},$$

and for $g \in G$,

$$\dim(I_g(\eta_-)) = \begin{cases} 1 & \text{if } g \in J^1_-(B^\times \cap G)J^1_-, \\ 0 & \text{otherwise.} \end{cases}$$

Proof This is a special case of [30, (3.29) and (3.31)]. ∎

Proposition 3.2 ([5, (5.1.2)]) *For $i = 1, 2$, suppose that $[\mathfrak{A}_i, n_i, 0, \beta]$ is a skew simple stratum in A, and let $\theta^i_- \in \mathcal{C}_-(\mathfrak{A}_i, 0, \beta)$. Let η^i_- be the unique irreducible representation of $J^1_-(\beta, \mathfrak{A}_i)$ which contains θ^i_-. Then we have*

$$\dim(\eta^1_-)(\boldsymbol{U}^1(\mathfrak{B}_1) : \boldsymbol{U}^1(\mathfrak{B}_2)) = \dim(\eta^2_-)(J^1_-(\beta, \mathfrak{A}_1) : J^1_-(\beta, \mathfrak{A}_2)),$$

where $\mathfrak{B}_i$ denotes the $\mathfrak{A}$-centralizer of β, for $i = 1, 2$.

Proof Using the exact sequence of [30, (3.17)] and the Cayley map $C(x) = (1 + \frac{1}{2}x)(1 - \frac{1}{2}x)^{-1}$, we can prove the assertion in the same way as the proof of [5, (5.1.2)] (see [3, 4.2]). ∎

Suppose that $[\mathfrak{A}, n, 0, \beta]$ is a good skew simple stratum in A, with $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$. Let $E = F[\beta]$, and $B = B_\beta$ be the A-centralizer of β. Then $\mathcal{L} = \mathcal{L}_{\mathfrak{o}_E}$ is a self-dual $\mathfrak{o}_E$-lattice chain in V, with $e = e(\mathcal{L}_{\mathfrak{o}_E})$. From Definition 2.4, its self-dual slice of the form (1.5) contains the $\mathfrak{o}_E$-lattice L_0 in V such that $L_0^\natural = L_0$. Thus we can put

$$\mathcal{L}_M = \{\varpi_E^i L_0 \mid i \in \mathbb{Z}\}.$$

This is a self-dual $\mathfrak{o}_E$-lattice chain in V satisfying $\mathcal{L}_M \subset \mathcal{L}$, and the $\mathfrak{o}_E$-period of $\mathcal{L}_M$ is equal to one. We can choose a (maximal) self-dual $\mathfrak{o}_E$-lattice chain $\mathcal{L}_m$ in V satisfying $\mathcal{L} \subset \mathcal{L}_m$ with $\mathfrak{o}_E$-period equal to $R = \dim_E(V)$. From $\mathcal{L}_M$ and $\mathcal{L}_m$ we obtain $\mathfrak{o}_E$-orders $\mathfrak{B}_M$ and $\mathfrak{B}_m$ in $B = B_\beta$ as follows:

$$\mathfrak{B}_M = \mathrm{End}^0_{\mathfrak{o}_E}(\mathcal{L}_M) = \{x \in B \mid xL \subset L, \text{ for all } L \in \mathcal{L}_M\}$$

and similarly $\mathfrak{B}_m = \mathrm{End}^0_{\mathfrak{o}_E}(\mathcal{L}_m)$. Then $\mathfrak{B}_M$ (resp. $\mathfrak{B}_m$) is a maximal (resp. minimal) hereditary $\mathfrak{o}_E$-order of B. Moreover, $\mathfrak{B} = B \cap \mathfrak{A}$ satisfies $\mathfrak{B}_m \subset \mathfrak{B} \subset \mathfrak{B}_M$. From Proposition 1.3, $\mathcal{L}_M$ and $\mathcal{L}_m$ are also self-dual $\mathfrak{o}_F$-lattice chains in V. Write

$$\mathfrak{A}_M = \mathrm{End}^0_{\mathfrak{o}_F}(\mathcal{L}_M), \quad \mathfrak{A}_m = \mathrm{End}^0_{\mathfrak{o}_F}(\mathcal{L}_m).$$

Then we have $\mathfrak{B}_M = \mathfrak{A}_M \cap B$, $\mathfrak{B}_m = \mathfrak{A}_m \cap B$.

We denote by $\nu_E(\beta)$ the normalized valuation of β in E. Then, since we have $\nu_{\mathfrak{A}_M}(\beta) = -\nu_E(\beta)$ and $\nu_{\mathfrak{A}_m}(\beta) = -\nu_E(\beta)R$, strata

$$[\mathfrak{A}_M, -\nu_E(\beta), 0, \beta] \quad \text{and} \quad [\mathfrak{A}_m, -\nu_E(\beta)R, 0, \beta]$$

in A are both (good) skew simple. From [30, (3.26)], there is a transfer

$$\tau_{\mathfrak{A}_m,\mathfrak{A}_M,\beta,0} : \mathcal{C}_-(\mathfrak{A}_m, 0, \beta) \to \mathcal{C}_-(\mathfrak{A}_M, 0, \beta),$$

(see [5, (3.6.2)]). Similarly, there is a transfer $\tau_{\mathfrak{A}_m,\mathfrak{A},\beta,0}$.

Let $\theta_{M,-} \in \mathcal{C}_-(\mathfrak{A}_M, 0, \beta)$, $\theta_{m,-} \in \mathcal{C}_-(\mathfrak{A}_m, 0, \beta)$, and $\theta_- \in \mathcal{C}_-(\mathfrak{A}, 0, \beta)$. Assume that these characters are related as follows:

$$\theta_{M,-} = \tau_{\mathfrak{A}_m,\mathfrak{A}_M,\beta,0}(\theta_{m,-}), \quad \theta_- = \tau_{\mathfrak{A}_m,\mathfrak{A},\beta,0}(\theta_{m,-}),$$

as in [5, (5.1.13)].

For an integer $t \geq 1$, write simply $J^t_- = J^t_-(\beta, \mathfrak{A})$, $J^t_{m,-} = J^t_-(\beta, \mathfrak{A}_m)$, $J^t_{M,-} = J^t_-(\beta, \mathfrak{A}_M)$, $J_- = J_-(\beta, \mathfrak{A})$, and so on, with similar conventions for the group H_-. Let η_- (resp. $\eta_{m,-}$, resp. η_M) be the unique irreducible representation in Proposition 3.1 which contains θ_- (resp. $\theta_{m,-}$, resp. $\theta_{M,-}$). Analogous results for $\mathrm{GL}(N, F)$ in [5, Propositions (5.1.14)–(5.1.19)] can be proved for G in a quite similar way.

Proposition 3.3 ([5, (5.1.14)–(5.1.18)]) *Let notation and assumptions be as above.*

(i) *There is a unique irreducible representation $\widetilde{\eta}_{M,-}$ of $\boldsymbol{U}^1(\mathfrak{B}_m)J^1_{M.-}$ such that*
  - (a) $\widetilde{\eta}_{M,-}|J^1_{M,-} = \eta_{M,-}$*;*
  - (b) *the representations $\widetilde{\eta}_{M,-}$ and $\eta_{m,-}$ induce equivalent irreducible representations of $\boldsymbol{U}^1(\mathfrak{A}_m)$.*

(ii) *There is a unique irreducible representation $\widetilde{\eta}_-$ of $\boldsymbol{U}^1(\mathfrak{B}_m)J^1_-$ such that*
  - (a) $\widetilde{\eta}_-|J^1_- = \eta_-$*;*
  - (b) *the representations $\widetilde{\eta}_-$ and $\eta_{m,-}$ induce equivalent irreducible representations of $\boldsymbol{U}^1(\mathfrak{A}_m)$.*

(iii) *There is a unique irreducible representation $\hat{\eta}_{M,-}$ of $\boldsymbol{U}^1(\mathfrak{B})J^1_{M,-}$ such that*
  - (a) $\hat{\eta}_{M,-}|J^1_{M,-} = \eta_{M,-}$*;*
  - (b) *the representations $\hat{\eta}_{M,-}$ and η_- induce equivalent irreducible representations of $\boldsymbol{U}^1(\mathfrak{A})$.*

If ρ is a representation of a compact open subgroup K of G, put

$$I_G(\rho) = \{g \in G \mid I_g(\rho) \neq (0)\}.$$

We say that an element g of G *intertwines* ρ, if $g \in I_G(\rho)$.

Proposition 3.4 ([5, (5.1.19)]) *Let notation and assumptions be as in Proposition 3.3. Then we have*

$$I_G(\widetilde{\eta}_{M,-}) = J^1_{M,-}(B^\times \cap G) J^1_{M,-}, \quad I_G(\eta_-) = J^1_-(B^\times \cap G) J^1_-.$$

Proof By using [29, Theorem 2.2], we can prove the assertion in the same way as the proof of [5, (5.1.19)]. ■

3.2 Beta Extensions

Let $[\mathfrak{A}, n, 0, \beta]$ be a skew simple stratum in A, and $\theta_- \in \mathcal{C}_-(\mathfrak{A}, 0, \beta)$. Let $E = F[\beta]$ and $B = B_\beta$ be the A-centralizer of β. Let η_- be the unique irreducible representation of $J^1_-(\beta, \mathfrak{A})$ which contains θ_-.

Definition 3.5 ([5, (5.2.1)]) A representation κ_- of $J_-(\beta, \mathfrak{A})$ is called a *β-extension* of η_-, if it satisfies $\kappa_-|J^1_-(\beta, \mathfrak{A}) = \eta_-$ and $B^\times \cap G \subset I_G(\kappa_-)$.

We show that if a skew simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A is good, there is a β-extension of η_-.

Lemma 3.6 *Let U, V be subgroups of $\widetilde{G}$ fixed by Γ. Suppose that U normalizes V, and that $U \cap V$ is a pro p-group. Then we have $(UV)^\Gamma = U^\Gamma V^\Gamma$.*

Proof The groups $UV, U \cap V$ are both Γ-sets. Then we obtain a short sequence

$$1 \longrightarrow U \cap V \xrightarrow{\delta} U \times V \xrightarrow{\pi} UV \longrightarrow 1,$$

where $\delta(x) = (x, x)$, for $x \in U \cap V$, and $\pi(x, y) = xy^{-1}$, for $x \in U, y \in V$. This is an exact sequence of Γ-sets. For we have

$$\delta(\gamma(x)) = (\gamma(x), \gamma(x)) = \gamma(x, x),$$
$$\pi(\gamma(x), \gamma(y)) = \gamma(x)\gamma(y)^{-1} = \gamma(xy^{-1}) = \gamma(\pi(x, y)),$$

for $x \in U, y \in V$. From [22, Proposition 3.6], we thus obtain an exact sequence

$$1 \longrightarrow (U \cap V)^\Gamma \longrightarrow (U \times V)^\Gamma \longrightarrow (UV)^\Gamma \longrightarrow H^1(\Gamma, U \cap V) \longrightarrow H^1(\Gamma, U \times V).$$

Since $U \cap V$ is pro p-group and p is not 2, we hence have $H^1(\Gamma, U \cap V) = 1$, whence $(UV)^\Gamma = U^\Gamma V^\Gamma$. ■

Proposition 3.7 ([5, (5.2.4)]) *Let $[\mathfrak{A}, n, 0, \beta]$ be a good skew simple stratum in A, and $\widetilde{\eta}_{M,-}$ the representation of $\boldsymbol{U}^1(\mathfrak{B}_M) J^1_{M,-}$, as in Proposition 3.3. Then there is a representation $\kappa_{M,-}$ of $J_{M,-}$ such that $\kappa_{M,-}|\boldsymbol{U}^1(\mathfrak{B}_m) J^1_{M,-} = \widetilde{\eta}_{M,-}$.*

Proof Following the methods of the proof of [5, (5.2.4)], we prove the assertion. We sketch the proof.

Put $r = -k_0(\beta, \mathfrak{A})$. From Lemma 3.6 and [30, (3.12)], we get

$$J^1_{M,-} = \boldsymbol{U}^1(\mathfrak{B}_M) J_-^{[(r+1)/2]}(\beta, \mathfrak{A}_M), \quad J_{M,-} = \boldsymbol{U}(\mathfrak{B}_M) J^1_{M,-}.$$

From the case where $e = e(\mathcal{L}_{\mathfrak{o}_E}) = 1$ in Proposition 2.8, we have

$$J_{M,-}/J^1_{M,-} \simeq \boldsymbol{U}(\mathfrak{B}_M)/\boldsymbol{U}^1(\mathfrak{B}_M) \simeq U(\overline{V}, \overline{h}),$$

where $\overline{V} = L_0/\varpi L_0^\natural$ for $L_0 \in \mathcal{L}_{\mathfrak{o}_E}$ in (1.5) and $\overline{h}$ is a non-degenerate k_E/k_{E_0}-anti-hermitian form, which is naturally induced from the form $\widetilde{h}_\beta$. It follows from Proposition 2.8 that $\mathcal{G} = U(\overline{V}, \overline{h})$ is a unitary group over k_{E_0} of type A^2_{R-1}. The canonical image of $\boldsymbol{U}^1(\mathfrak{B}_m)/\boldsymbol{U}^1(\mathfrak{B}_M)$ into $\mathcal{G}$ is the unipotent radical $\mathcal{N}$ of a Borel subgroup of $\mathcal{G}$. Thus $\boldsymbol{U}^1(\mathfrak{B}_m)J^1_{M,-}$ is a Sylow pro p-subgroup of $J_{M,-}$. Since, from [30, (3.31)], $J_{M,-}$ normalizes $\eta_{M,-}$, we obtain a projective representation of $J_{M,-}$ which is an extension of $\eta_{M,-}$. We can adjust this projective representation to be a linear representation λ of $J_{M,-}$. Then we have

$$\lambda | \boldsymbol{U}^1(\mathfrak{B}_m) J^1_{M,-} = \widetilde{\eta}_{M,-} \otimes \phi,$$

where ϕ is a character of $\boldsymbol{U}^1(\mathfrak{B}_m)$ which is trivial on $\boldsymbol{U}^1(\mathfrak{B}_M)$. This ϕ is a character of $\mathcal{N}$ which is intertwined by all the elements of $\mathcal{G}$. Let Φ be a root system of $\mathcal{G}$ and Δ the set of simple roots in Φ, associated with $\mathcal{N}$. We denote by U_a the root subgroup of $\mathcal{G}$ associated with $a \in \Phi$, and by $[\mathcal{N}, \mathcal{N}]$ the commutator group of $\mathcal{N}$. Let ht be the height function on Φ with respect to the basis Δ. Then, under the assumption $p \neq 2, 3$, by using the commutator relations in the twisted group $\mathcal{G}$ of $\mathrm{GL}(R, k_E)$, we can easily see that $[\mathcal{N}, \mathcal{N}] = \prod_a U_a$, where a runs through roots in Φ with $ht(a) \geq 2$, (see [27, §11], [11, §13]) and see that there is a canonical isomorphism

$$\mathcal{N}/[\mathcal{N}, \mathcal{N}] \simeq \prod_{a \in \Delta} U_a.$$

As in [11, 8.1], this fact holds for any finite group of Lie type. Thus ϕ is trivial on $\mathcal{N}$ and can be extended to a character ϕ' of $\mathcal{G}$, as in the proof of [5, (5.2.4)] for $\mathrm{GL}(N, F)$. We regard ϕ' as a character of $J_{M,-}$, and put $\kappa_{M,-} = \lambda \otimes \phi'^{-1}$. It easily seen that the representation $\kappa_{M,-}$ is the desired one. ∎

Proposition 3.8 ([5, (5.2.5)]) *Let $\kappa_{M,-}$ be the representation as in Proposition 3.7. Then there is a representation κ_- of J_- which is uniquely determined by the following properties:*

(i) $\kappa_- | J^1_- = \eta_-$ *;*
(ii) κ_- *and* $\kappa_{M,-} | \boldsymbol{U}(\mathfrak{B}) J^1_{M,-}$ *induce equivalent irreducible representations of* $\boldsymbol{U}(\mathfrak{A})$*;*
(iii) $\mathrm{Ind}(\kappa_- : J_-, \boldsymbol{U}(\mathfrak{B})\boldsymbol{U}^1(\mathfrak{A}))$ *is equivalent to*

$$\mathrm{Ind}(\kappa_{M,-} | \boldsymbol{U}(\mathfrak{B}) J^1_{M,-} : \boldsymbol{U}(\mathfrak{B}) J^1_{M,-}, \boldsymbol{U}(\mathfrak{B})\boldsymbol{U}^1(\mathfrak{A})).$$

Proof Using Proposition 3.2, we can prove the assertion in the same way as the proof of [5, (5.2.5)]. ∎

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We show that the representation κ_- in Proposition 3.8 is a β-extension.

Proposition 3.9 ([5, (5.2.7)]) *Let κ_- be the representation of J_- constructed as in Proposition* 3.8. *Then we have*

$$I_G(\kappa_-) = J_-(B^\times \cap G)J_- = J_-^1(B^\times \cap G)J_-^1.$$

Proof The proof of [5, (5.2.7)] for $\mathrm{GL}(N,F)$ remains valid for our classical G, as well. We also sketch the proof.

Using the Witt basis $\mathcal{V}$ of (2.2), we express elements of $B^\times \cap G$ in matrix form, that is, $B^\times \cap G$ is embedded in $\mathrm{GL}(R,E)$ where $R = \dim_E(V)$. Moreover, $\boldsymbol{U}(\mathfrak{B}_M)$ is embedded in $\mathrm{GL}(R,\mathfrak{o}_E)$, and it is a special maximal compact subgroup of $B^\times \cap G$. Thus $B^\times \cap G$ has a Cartan decomposition relative to $\boldsymbol{U}(\mathfrak{B})$.

From [30, (3.13)], $I_G(\kappa_-) \subset I_G(\eta_-) = J_-(B^\times \cap G)J_-$. So it is enough to prove that any element y of $B^\times \cap G$ intertwines κ_-. Moreover, by Proposition 3.8(ii), it is enough to treat the case where $\mathcal{L} = \mathcal{L}_M$ and $\kappa_- = \kappa_{M,-}$. Since $\boldsymbol{U}(\mathfrak{B}_M) \subset J_- \cap B^\times \cap G$, we can choose y in a $(\boldsymbol{U}(\mathfrak{B}_M), \boldsymbol{U}(\mathfrak{B}_M))$-double coset, and reduce it to a diagonal element $\mathrm{Diag}(\varpi_E^{n_1}, \dots, \varpi_E^{n_r}, \varpi_E^{-n_r}, \dots, \varpi_E^{-n_1})$, where $r = R/2$ and $n_1, n_2, \dots, n_r$ are integers with $n_1 \geq n_2 \geq \cdots \geq n_r$. Here we recall that E/E_0 is unramified. As in the proof of [5, (5.2.7)], we can choose a self-dual $\mathfrak{o}_E$-lattice chain $\mathcal{L}'$ in V, with $e(\mathcal{L}'_{\mathfrak{o}_E}) = e'$, for some integer $e' \geq 1$, which satisfies the following properties:

(P1) the self-dual slice of $\mathcal{L}'$ of the form (1.5) satisfies $L_0^\natural = L_0$,
(P2) this lattice L_0 is the same as that of $\mathcal{L}$,
(P3) for the E-decomposition $V = \bigoplus_{i=1}^{e'} V^i$ subordinated to $\mathcal{L}'$, the element y has a diagonal block form (y_i), and each y_i in $\mathrm{End}_E(V^i)$ is central, for $1 \leq i \leq e'$.

From Proposition 1.4, $\mathcal{L}'$ is also a self-dual $\mathfrak{o}_F$-lattice chain in V. Put $\mathfrak{B}' = \mathrm{End}^0_{\mathfrak{o}_F}(\mathcal{L}') \cap B$. From (P2), elements of $\mathfrak{B}'$ are written in the following block form: (x_{jk}), $1 \leq j,k \leq e'$, such that coefficients of the $n_j \times n_k$-matrix x_{jk} are all in $\mathfrak{o}_E$ if $j \leq k$, and all in $\mathfrak{p}_E$ otherwise, where $R = n_1 + n_2 + \cdots + n_{e'}$ is the partition of R associated with $\mathcal{L}'$. Put $\widetilde{\mathfrak{M}}(\mathfrak{B}') = \{(x_{jk}) \in \mathfrak{B}' \mid x_{jk} = 0, \text{for all } j \neq k\}$. Then it follows from Proposition 2.7 that the involution $^-$ fixes $\widetilde{\mathfrak{M}}(\mathfrak{B}')$. Thus we have

$$\mathfrak{M}(\mathfrak{B}')^\times = (\widetilde{\mathfrak{M}}(\mathfrak{B}')^\times)^\Gamma = \widetilde{\mathfrak{M}}(\mathfrak{B}') \cap G.$$

From the proof of [5, (5.2.7)], we have

$$y \text{ centralizes } \widetilde{\mathfrak{M}}(\mathfrak{B}') \quad \text{and} \quad \mathfrak{B}_M \cap \mathfrak{B}_M^y \subset \mathfrak{p}_F\mathfrak{B}_M + (\mathfrak{B}' \cap (\mathfrak{B}')^y),$$

where $L^y = y^{-1}Ly$. We denote by ${}^t\mathfrak{B}'$ the transpose of $\mathfrak{B}'$. Then we also have

$$y^{-1} \text{ centralizes } {}^t\widetilde{\mathfrak{M}}(\mathfrak{B}') \quad \text{and} \quad \mathfrak{B}_M \cap {}^y\mathfrak{B}_M \subset \mathfrak{p}_F\mathfrak{B}_M + {}^y({}^t\mathfrak{B}' \cap ({}^t\mathfrak{B}')),$$

where ${}^yL = yLy^{-1}$.

If $\mathfrak{B}' = \mathfrak{B}_M$, clearly $y = 1$. We note that this fact never occurs for the case of $\mathrm{GL}(N,F)$. Thus $y = 1$ trivially intertwines $\kappa_{M,-}$.

From [5, p. 173] together with Lemma 3.6, we obtain

$$(\mathfrak{M}(\mathfrak{B}')^{\times}\boldsymbol{U}^1(\mathfrak{B}')J_M^1)\cap(\mathfrak{M}(\mathfrak{B}')^{\times}\boldsymbol{U}^1(\mathfrak{B}')J_M^1)^y = \mathfrak{M}(\mathfrak{B}')^{\times}(\boldsymbol{U}^1(\mathfrak{B}')J_M^1\cap(\boldsymbol{U}^1(\mathfrak{B}')J_M^1)^y)$$

in $\widetilde{G}$. It follows from Lemma 3.6 and [5, (5.2.11)] that the element y intertwines $\kappa_{M,-}|\boldsymbol{U}(\mathfrak{B}')J_{M,-}^1$ with $\kappa_{M,-}|\boldsymbol{U}(\mathfrak{B}')J_{M,-}^1\otimes\phi$, where ϕ is an abelian character of $\mathfrak{M}(\mathfrak{B}')^{\times}/(\mathfrak{M}(\mathfrak{B}')^{\times}\cap\boldsymbol{U}^1(\mathfrak{B}')J_{M,-}^1)$. For the lattice chain $\mathfrak{L}'$ in V, we can choose the minimal self-dual $\mathfrak{o}_E$-lattice chain $\mathfrak{L}'_M=\mathfrak{L}_M$, given in Section 3.1, and a maximal self-dual $\mathfrak{o}_E$-lattice chain $\mathfrak{L}'_m$ in V, such that $\mathfrak{L}'_m\subset\mathfrak{L}'\subset\mathfrak{L}'_M$. Then we can see that ϕ is factored through the determinant in a suitable sense [5, p. 173]. Let κ_- be the representation of $J_-(\beta,\mathfrak{A}')$ given by Proposition 3.8, where $\mathfrak{A}'=\mathrm{End}^0_{\mathfrak{o}_F}(\mathfrak{L}')$. We can form the representation $\kappa_-\otimes\phi$, and by using Propositions 3.8 and 3.1, we can prove that y intertwines κ_- with $\kappa_-\otimes\phi$.

Claim 3.10 There is an extension μ_- of η_- intertwined by y.

We shall prove the claim in Section 4.2 below. We now assume that the claim is true. We also apply $H=J_-^1$, $N=\mathfrak{M}(\mathfrak{B}')^{\times}$, $g=y$, $\rho=\eta_-$ to [5, (5.2.11)]. Then these satisfy those hypotheses. In particular, we apply κ_- to $\widetilde{\rho}$ there. We now apply μ_- to ρ' [5, (5.2.11)(a)] so that y intertwines μ_- with $\mu_-\otimes\phi$. Thus the uniqueness of ϕ shows that ϕ is trivial. Hence we have seen that y intertwines $\kappa_{M,-}|\boldsymbol{U}(\mathfrak{B}')J_{M,-}^1$.

From the proof of [5, (5.2.7)] and Lemma 3.6, we obtain

$$J_{M,-}\cap J_{M,-}^y=(\boldsymbol{U}^1(\mathfrak{B}_M)\cap\boldsymbol{U}(\mathfrak{B}_M)^y)(\boldsymbol{U}(\mathfrak{B}')J_{M,-}^1\cap(\boldsymbol{U}(\mathfrak{B}')J_{M,-}^1)^y).$$

Similarly,

$$(\boldsymbol{U}^1(\mathfrak{B}_M)\cap\boldsymbol{U}(\mathfrak{B}_M)^y)\subset(\boldsymbol{U}(\mathfrak{B}_M)\cap\boldsymbol{U}^1(\mathfrak{B}_M)^y)(\boldsymbol{U}(\mathfrak{B}')\cap\boldsymbol{U}(\mathfrak{B}')^y).$$

Hence we can prove that y intertwines $\kappa_{M,-}$ in the same way as the proof of [5, (5.2.7)]. This completes the proof modulo the claim. ■

Theorem 3.11 *Let $[\mathfrak{A},n,0,\beta]$ be a good skew simple stratum in A, and*

$$\theta_-\in\mathcal{C}_-(\mathfrak{A},0,\beta).$$

Let η_- be the unique irreducible representation of $J_-^1(\beta,\mathfrak{A})$ which contains θ_-. Then there is a β-extension of η_-.

Proof The assertion follows directly from Propositions 3.8 and 3.9 (modulo the claim). ■

To prove the claim, the following lemma will be used in the next section.

Lemma 3.12 *Let $\mathfrak{L}'$ be the self-dual $\mathfrak{o}_E$-lattice chain in V associated with $y\in B^{\times}\cap G$ in the proof of Proposition 3.9. Let $\mathfrak{A}'=\mathrm{End}^0_{\mathfrak{o}_F}(\mathfrak{L}')$ and $n'=-\nu_{\mathfrak{A}'}(\beta)$. Then $[\mathfrak{A}',n',0,\beta]$ is a good skew simple stratum in A.*

Proof Straightforward. ■

4 Iwahori Decompositions

4.1 Iwahori Decompositions

We prove the claim in the proof of Proposition 3.9.

Suppose that $[\mathfrak{A}, n, 0, \beta]$ is a skew simple stratum in A, with $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$. Let $E = F[\beta]$, and $B = B_\beta$ be the A-centralizer of β. Put $e = e(\mathcal{L}_{\mathfrak{o}_E})$. For the E-decomposition $V = \bigoplus_{i=1}^{e} V^i$ of (2.1) subordinated to $\mathcal{L}_{\mathfrak{o}_E}$, put

$$A^{ij} = \mathrm{Hom}_F(V^j, V^i),\ A^i = A^{ii},\ \text{for } 1 \leq i, j \leq e.$$

We define subgroups of $\widetilde{G}$ as follows:

$$\widetilde{P} = \widetilde{G} \cap \Bigl(\prod_{1\leq i<j\leq e} A^{ij}\Bigr) \qquad \widetilde{M} = \widetilde{G} \cap \Bigl(\prod_{1\leq i\leq e} A^{i}\Bigr)$$

$$\mathbb{N}_u = \prod_{1\leq i<j\leq e} A^{ij},\ \widetilde{N}_u = 1 + \mathbb{N}_u \qquad \mathbb{N}_\ell = \prod_{1\leq j<i\leq e} A^{ij},\ \widetilde{N}_\ell = 1 + \mathbb{N}_\ell.$$

Each $\mathfrak{o}_E$-lattice L_k in $\mathcal{L}_{\mathfrak{o}_E}$ has a decomposition $L_k = \coprod_{1\leq i\leq e} L_k^i$, with $L_k^i = L_k \cap V^i$, for $k \in \mathbb{Z}$. From [5, (7.1.12)], there is a canonical isomorphism

$$H^1(\beta, \mathfrak{A}) \cap \widetilde{M} \simeq \prod_{i=1}^{e} H^i(\beta, \mathfrak{A}^{(i)}),$$

where $\mathfrak{A}^{(i)} = \mathrm{End}^0_{\mathfrak{o}_F}(\{L_k^i \mid k \in \mathbb{Z}\})$, for $1 \leq i \leq e$.

Proposition 4.1 ([5, (7.1.19)]) *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A, with $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$ and $e = e(\mathcal{L}_{F[\beta]})$, and $\theta \in (\mathfrak{A}, 0, \beta)$. Then θ is trivial on*

$$H^1(\beta, \mathfrak{A}) \cap \mathrm{Hom}_F(V^j, V^i),$$

for $i \neq j$. Under the identification $H^1(\beta, \mathfrak{A}) \cap \widetilde{M} = \prod_i H^1(\beta, \mathfrak{A}^{(i)})$, we have

$$\theta | (H^1(\beta, \mathfrak{A}) \cap \widetilde{M}) = \theta^{(1)} \otimes \cdots \otimes \theta^{(e)},$$

where $\theta^{(i)} \in \mathcal{C}(\mathfrak{A}^{(i)}, 0, \beta)$ and $\theta^{(i)} = \tau_{\mathfrak{A}, \mathfrak{A}^{(i)}, \beta, 0}(\theta)$, for $1 \leq i \leq e$.

Suppose that a skew simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A is good. Let $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$, $E = F[\beta]$, $e = e(\mathcal{L}_{\mathfrak{o}_E})$, and $B = B_\beta$ be the A-centralizer of β. Put $t = [(e+1)/2]$. For the orthogonal decomposition $(V, \widetilde{h}_\beta) = \perp_i (V_i, \widetilde{h}_i)$ in Proposition 2.7, we define

$$h_i = \ell \circ \widetilde{h}_i,$$

for $1 \leq i \leq t$, where $\ell \colon E \to F$ is the F-linear form defined in Section 1.3. Then for $1 \leq i \leq [e/2]$, (V_i, h_i) is a hyperbolic F-space such that V^i, V^{e-i+1} are totally

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isotropic F-subspaces of V_i, and if $t = (e+1)/2$ is an integer, then $V_t = V^t$ and $h_t = h|V_t$. Moreover, we have an orthogonal F-decomposition of V:

$$V = \perp_{i=1}^{t} V_i, \quad h = \perp_{i=1}^{t} h_i,$$

Thus the involution $^-$ on A defined by h, induces involutions $A^i \to A^{e-i+1}$, $A^{ij} \to A^{e-i+1,e-j+1}$, for $1 \le i, j, \le e$, where if $i \equiv j \pmod{e}$, we set $i = j$. We denote by $x \mapsto \bar{x}$ the induced involution $A^i \to A^{e-i+1}$. Hence the involution $^-$ on A fixes $\prod_i A^i, \mathbb{N}_u$ and $\mathbb{N}_\ell$, respectively, whence the involution γ on $\widetilde{G}$ fixes the subgroups $\widetilde{P}$, $\widetilde{M}$, $\widetilde{N}_u$ and $\widetilde{N}_\ell$. Let $\widetilde{\mathcal{G}}$ be one of these subgroups. Put $\mathcal{G} = \widetilde{\mathcal{G}}^\Gamma = \widetilde{\mathcal{G}} \cap G$. Then $P = MN_u$ is a parabolic subgroup of G, with Levi component M and unipotent radical N_u. We also have the opposite parabolic subgroup $P_\ell = MN_\ell$ with respect to M. We say that the parabolic subgroup $P = MN_u$ is *associated with* a good skew simple stratum $[\mathfrak{A}, n, 0, \beta]$.

Lemma 4.2 *Let $[\mathfrak{A}, n, 0, \beta]$ be a good skew simple stratum in A, and $P = MN_u$ a parabolic subgroup of G associated with $[\mathfrak{A}, n, 0, \beta]$. Let $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$, $E = F[\beta]$, and $e = e(\mathcal{L}_{\mathfrak{o}_E})$. Let $V = \bigoplus_{i=1}^{e} V^i$ be the E-decomposition of* (2.1) *subordinated to $\mathcal{L}_{\mathfrak{o}_E}$. Then there is a canonical isomorphism*

$$M \simeq \begin{cases} \prod_{i=1}^{e/2} \mathrm{Aut}_F(V^i) & \text{if } e \text{ is even,} \\ \left(\prod_{i=1}^{(e-1)/2} \mathrm{Aut}_F(V^i)\right) \times U(V_t, h_t) & \text{if } e \text{ is odd,} \end{cases}$$

where $t = (e+1)/2$.

Proof The assertion follows easily from the above argument (see Proposition 2.8). ■

We write simply $H_-^m = H_-^m(\beta, \mathfrak{A})$ and $J_-^m = J_-^m(\beta, \mathfrak{A})$, for $m = 0, 1$. From [5, (7.1.14), (7.1.16)–(7.1.18)], we obtain Iwahori decompositions of H_-^m, J_-^m, for $m = 0, 1$, as follows.

Proposition 4.3 ([5, (7.1.14)]) *Let $\mathcal{G}_-$ denote any of the groups H_-^m, J_-^m, for $m = 0, 1$. Then we have the Iwahori decomposition:*

$$\mathcal{G}_- = (\mathcal{G}_- \cap N_\ell) \cdot (\mathcal{G}_- \cap M) \cdot (\mathcal{G}_- \cap N_u),$$

$$\mathcal{G}_- \cap P = (\mathcal{G}_- \cap M) \cdot (\mathcal{G}_- \cap N_u),$$

Put $t = [(e+1)/2]$. According to the decomposition of M in Lemma 4.2, for $m = 0, 1$, we have

$$J_-^m(\beta, \mathfrak{A}) \cap M \simeq \prod_{i=1}^{t} J^m(\beta, \mathfrak{A}^{(i)}),$$

where if $t = (e+1)/2$ is an integer, we understand $J^m(\beta, \mathfrak{A}^{(t)}) = J_-^m(\beta, \mathfrak{A}^{(t)})$. Likewise for $H_-^m(\beta, \mathfrak{A})$, for $m = 0, 1$. Moreover, we have

$$(J_- \cap M)H_-^1 = (H_-^1 \cap N_\ell)(J_- \cap M)(H_-^1 \cap N_u),$$

$$(J_- \cap P)H_-^1 = (H_-^1 \cap N_\ell)(J_- \cap M)(J_-^1 \cap N_u).$$

4.2 The Proof of Claim 3.10

We are ready to prove Claim 3.10.

Proposition 4.4 *Let $[\mathfrak{A}, n, 0, \beta]$ be a good skew simple stratum in A, with $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$ and $e = e(\mathcal{L}_{\mathfrak{o}_E})$, and $\theta_- \in \mathcal{C}_-(\mathfrak{A}, 0, \beta)$. Let $P = MN_u$ be a parabolic subgroup of G associated with $[\mathfrak{A}, n, 0, \beta]$. Put $t = [(e+1)/2]$. Then θ_- is trivial on both $H^1_-(\beta, \mathfrak{A}) \cap N_\ell$ and $H^1_-(\beta, \mathfrak{A}) \cap N_u$. After the identification $H^1_-(\beta, \mathfrak{A}) \cap M = \prod_{i=1}^t H^1(\beta, \mathfrak{A}^{(i)})$, we have*

$$\theta_-|(H^1_-(\beta, \mathfrak{A}) \cap M) = \theta^{(1)} \otimes \cdots \otimes \theta^{(t)},$$

where $\theta^{(i)} \in \mathcal{C}(\mathfrak{A}^{(i)}, 0, 2\beta)$, for $1 \le i \le [e/2]$, and if $t = (e+1)/2$ is an integer, we understand $\theta^{(t)} = \theta_-^{(t)}$ and $\mathcal{C}(\mathfrak{A}^{(t)}, 0, \beta) = \mathcal{C}_-(\mathfrak{A}^{(t)}, 0, \beta)$. Further, $\theta^{(i)}$ is a simple character of $H^1(2\beta, \mathfrak{A}^{(i)}) = H^1(\beta, \mathfrak{A}^{(i)})$ for $1 \le i \le [e/2]$.

Proof The first assertion follows directly from Proposition 4.1. As in Section 2.1, we have $\theta_- = \boldsymbol{g}(\theta)) = \theta|H^1_-(\beta, \mathfrak{A})$, for some $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ with $\theta^\gamma = \theta$. From Proposition 4.1, $\theta|(H^1(\beta, \mathfrak{A} \cap \widetilde{M}) = \theta^{(1)\prime} \otimes \cdots \otimes \theta^{(e)\prime}$. We restrict this character to $\widetilde{G} \cap (A^i \times A^{e-i+1})$ for $1 \le i \le [e/2]$ and so have

$$(\widetilde{G} \cap (A^i \times A^{e-i+1}))^\Gamma = \{(x, \overline{x}^{\,-1}) \mid x \in (A^i)^\times = \mathrm{Aut}_F(V^i)\},$$

where $x \mapsto \overline{x}$ is the involution $A^i \to A^{e-i+1}$ defined in Section 4.1. Since $\theta((x,1)) = \theta^\gamma((x,1))$, for $x \in H^1(\beta, \mathfrak{A}^{(i)})$, we have $\theta^{(i)\prime}(x) = \theta^{(e-i+1)\prime}(\overline{x}^{\,-1})$. Thus θ_- restricted to the factor $H^1(\beta, \mathfrak{A}^{(i)})$ is equal to $(\theta^{(i)\prime})^2$. Denote this character by $\theta^{(i)}$. Then $\theta^{(i)}$ belongs to $\mathcal{C}(\mathfrak{A}^{(i)}, 0, 2\beta)$. Since it follows from [3, §4.3, Lemma 1] that $H^1(2\beta, \mathfrak{A}^{(i)}) = H^1(\beta, \mathfrak{A}^{(i)})$, $\theta^{(i)}$ is a simple character of $H^1(\beta, \mathfrak{A}^{(i)})$ as in the assertion. Moreover, if $t = (e+1)/2$ is an integer, clearly $\theta^{(t)} = \theta_-^{(t)} \in \mathcal{C}^1_-(\mathfrak{A}^{(t)}, 0, \beta)$. ∎

Suppose that $[\mathfrak{A}, n, 0, \beta], \theta_- \in \mathcal{C}_-(\mathfrak{A}, 0, \beta)$, and $P = MN_u$ is as in Proposition 4.4. From [5, (5.1.1)] and Proposition 3.1, we obtain the unique irreducible representation η_- (resp. $\eta^{(i)}$, resp. $\eta_-^{(t)}$) of $J^1_-(\beta, \mathfrak{A})$ (resp. $J^1(\beta, \mathfrak{A}^{(i)})$, resp. $J^1_-(\beta, \mathfrak{A}^{(t)})$) which contains θ_- (resp. $\theta^{(i)}$, resp. $\theta_-^{(t)}$). We define a subgroup of J_- by

$$J^1_{P,-} = (J^1_-(\beta, \mathfrak{A}) \cap P)H^1_-(\beta, \mathfrak{A}).$$

Proposition 4.5 *Let notation and assumptions be as above. Then there is an irreducible representation $\eta_{P,-}$ of $J^1_{P,-}$ which satisfies the following conditions:*

(i) $\eta_{P,-}|(J^1_-(\beta, \mathfrak{A}) \cap M) \simeq \eta^{(1)} \otimes \cdots \otimes \eta^{(t)}$,
(ii) $\eta_{P,-}|H^1_-(\beta, \mathfrak{A})$ *is a multiple of* θ_-,
(iii) $\eta_{P,-}|(J^1_-(\beta, \mathfrak{A}) \cap N_u)$ *is the trivial character,*
(iv) $\eta_- = \mathrm{Ind}(\eta_{P,-} : J_{P,-}, J_-)$,

where in (i), *if $t = (e+1)/2 \in \mathbb{Z}$, we understand $\eta^{(t)} = \eta_-^{(t)}$.*

Proof By using Proposition 4.4, we can prove the proposition in the same way as the proofs of [5, (7.2.3), (7.2.4)]. ∎

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Let y be the element in the proof Proposition 3.9. From Lemma 3.12, we may replace $[\mathfrak{A}', n', 0, \beta]$ in that proposition by $[\mathfrak{A}, n, 0, \beta]$ in this subsection. From Lemma 4.2, we can write y in the form $y = (y_1, \dots, y_t)$, where if $t = (e+1)/2 \in \mathbb{Z}$, $y_t = 1$.

Lemma 4.6 *Let notation and assumptions be as above. For $1 \le i \le [e/2]$, there is an irreducible representation $\mu^{(i)}$ of $J(\beta, \mathfrak{A}^{(i)})$ which is intertwined by y_i and is an extension of $\eta^{(i)}$. Moreover, if $t = (e+1)/2$ is an integer, there is an irreducible representation $\mu^{(t)} = \mu_-^{(t)}$ of $J_-(\beta, \mathfrak{A}^{(t)})$ which is an extension of $\eta^{(t)}$.*

Proof In case $1 \le i \le [e/2]$, the assertion is just [5, (7.2.10)]. In case $t = (e+1)/2 \in \mathbb{Z}$, since $y_t = 1$, the assertion follows from Proposition 3.8. ■

The following proposition is nothing but Claim 3.10.

Proposition 4.7 *There is an irreducible representation μ of $J_-(\beta, \mathfrak{A})$ which is intertwined by y and such that $\mu | J_-^1 = \eta_-$.*

Proof For $\eta^{(i)}$ in Lemma 4.6, put $\eta_{N_u,-} = \eta^{(1)} \otimes \cdots \otimes \eta^{(t)}$, where if $t = (e+1)/2 \in \mathbb{Z}$, we understand $J^1(\beta, \mathfrak{A}^{(t)}) = J_-^1(\beta, \mathfrak{A}^{(t)})$, $\eta^{(t)} = \eta_-^{(t)}$. From Lemma 4.6, we obtain an irreducible representation of $J_-(\beta, \mathfrak{A}) \cap M = \prod_i J(\beta, \mathfrak{A}^{(i)})$ by

$$\mu_{N_u,-} = \mu^{(1)} \otimes \cdots \otimes \mu^{(t)}.$$

Then $y = (y_i)$ clearly intertwines $\mu_{N_u,-}$. From the Iwahori decomposition in Section 4.1, we can inflate $\mu_{N_u,-}$ to a representation $\mu_{P,-}$ of $(J_-(\beta, \mathfrak{A}) \cap P) H_-^1(\beta, \mathfrak{A})$ by putting

$$\mu_{P,-}(hmj) = \mu_{N_u,-}(m), \quad \text{for } h \in H_-^1 \cap N_\ell, m \in J_- \cap M, j \in J_-^1 \cap N_u.$$

So put $\mu_- = \operatorname{Ind}(\mu_{P,-} : (J_- \cap P) H_-^1, J_-)$. From Proposition 4.5, $\eta_{P,-}$ induces η_-. Hence, from the Mackey restriction formula, we get $\mu_- | J_-^1 = \eta_-$, and from [5, (4.1.5)], we can at once see that y intertwines μ_-. ■

The proposition completes the proof of Proposition 3.9, and hence that of Theorem 3.11.

5 Simple Types

5.1 Affine Weyl Groups

In this section, we define an analogue of a simple type for $\mathrm{GL}(N, F)$ defined by [5, (5.5.10)].

Suppose that $[\mathfrak{A}, n, 0, \beta]$ is a good skew simple stratum in $A = \operatorname{End}_F(V)$. Let $E = F[\beta]$, and $B = B_\beta$ the A-centralizer of β. Put $R = \dim_E(V)$. Let $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$, $\mathfrak{B} = \mathfrak{A} \cap B$, and put $e = e(\mathcal{L}_{\mathfrak{o}_E})$.

From Proposition 1.1, $B^\times \cap G$ is the unramified unitary group of the non-degenerated E-anti-hermitian space $(V, \widetilde{h}_\beta)$, and from Proposition 2.5, it is of type C in the sense of [8, (10.1.2)]. In this paragraph, we recall the structure of the affine Weyl group of $B^\times \cap G$ by [8, 10.1] and [31]. Denote by $\boldsymbol{G}_1$ the algebraic group defined

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over E_0 such that the group of E_0-rational points in $\boldsymbol{G}_1$, denoted by $G_1 = \boldsymbol{G}_1(E_0)$, is equal to $B^\times \cap G$.

In order to quote [8, 10.1] and [31], we rewrite the Witt basis $\mathcal{V}$ of (2.2) for $(V, \widetilde{h}_\beta)$ as follows: let $r = R/2$ and $I = \{\pm 1, \dots, \pm r\}$. Put $\mathcal{V} = \{e_i | \ i \in I\}$ with $e_{-r} = v_1, e_{-r+1} = v_2, \dots, e_{-1} = v_r, e_1 = v_{r+1}, \dots, e_r = v_{2r} = v_R$.

We express elements of G_1 in the matrix form by this basis $\mathcal{V}$. Let $\boldsymbol{S}$ be the maximal E_0-split torus of $\boldsymbol{G}_1$ defined by

$$\boldsymbol{S}(E_0) = \{\mathrm{Diag}(d_{-r}, \dots, d_{-1}, d_1, \dots, d_r) | \ d_i \in E_0 \text{ and } d_{-i} d_i = 1 \ (i \in I)\}.$$

Let $\boldsymbol{Z}$ be the centralizer of $\boldsymbol{S}$, and $\boldsymbol{N}$ the normalizer of $\boldsymbol{S}$. Then we have

$$\boldsymbol{Z}(E_0) = \{\mathrm{Diag}(d_{-r}, \dots, d_{-1}, d_1, \dots, d_r) | \ d_i \in E \text{ and } \overline{d_{-i}} d_i = 1 \ (i \in I)\}.$$

Write $H = \boldsymbol{Z}(E_0)$ for simplicity. Then H has the maximal compact open subgroup

$$H_0 = \{\mathrm{Diag}(d_{-r}, \dots, d_{-1}, d_1, \dots, d_r) | \ d_i \in \mathfrak{o}_E^\times \text{ and } \overline{d_{-i}} d_i = 1 \ (i \in I)\},$$

which coincides with Z_c in the notation of [31, 1.2]. Let $\boldsymbol{W}_0 = \boldsymbol{N}(E_0)/H$ and $\boldsymbol{W} = \boldsymbol{N}(E_0)/H_0$.

For $i, j \in I$, denote by $\delta_{i,j}$ the Kronecker delta. Then the group $\boldsymbol{N}(E_0)$ consists of all matrices of the form $n = n(\sigma; d_{-r}, \dots, d_r) = (g_{ij})$ with $g_{ij} = \delta_{i,\sigma(j)} d_j$, where (i) σ is a permutation of I which preserves the partition of I in pairs $(-i, i)$, (ii) $d_i \in E$ such that $\overline{d_{-i}} d_i = 1$, and (iii) $\det(n) = \pm \prod_{i \in I} d_i = 1$.

For an integer i, $1 \le i \le r$, we define a character $a_i \colon \boldsymbol{S} \to \boldsymbol{GL}_1$ by

$$a_i(\mathrm{Diag}(d_{-r}, \dots, d_r)) = d_{-i},$$

where $\boldsymbol{GL}_1$ denotes the multiplicative group defined over E_0. Then $(a_i)_{1 \le i \le r}$ is a $\mathbb{Z}$-basis of the character group $X^* = \mathrm{Hom}_{E_0}(\boldsymbol{S}, \boldsymbol{GL}_1)$. Put $a_{-i} = -a_i, \ a_{ij} = a_i + a_j$ in X^*. Then $\Phi = \{a_{ij} | \ i, j \in I, i \ne \pm j\} \cup \{2a_i | \ i \in I\}$ is the root system of $(\boldsymbol{G}_1, \boldsymbol{S})$. Let $\boldsymbol{U}_a$ be the root subgroup of $\boldsymbol{G}_1$ associated with a root $a \in \Phi$. Associated with a_{ij} and $2a_i$, we define elements $u_{ij}(c)$ $(c \in E)$ and $u_i(0, d)$ $(d \in E_0)$ of $G_1 = \boldsymbol{G}_1(E_0)$ respectively as follows: $u_{ij}(c) = 1 + (g_{k\ell})$ with $g_{-j,i} = \overline{c}, \ g_{-i,j} = -c$ and all other $g_{k\ell} = 0$, and $u_i(0, d) = 1 + (g_{k\ell})$ with $g_{-i,i} = d$ and all other $g_{k\ell} = 0$ [8, (10.2.1)], where we recall that $2 \in E_0$ is invertible. Then $\boldsymbol{U}_{a_{ij}}(E_0) = \{u_{ij}(c) | \ c \in E\}$ and $\boldsymbol{U}_{2a_i}(E_0) = \{u_i(0, d) \mid \ d \in E_0\}$. Further, we define elements $m(u_{ij}(c))$ $(c \in E^\times)$ and $m(u_i(0, d))$ $(d \in E_0^\times)$ of $\boldsymbol{N}(E_0)$ by

$$m(u_{ij}(c)) = u_{-j,-i}(-c^{-1}) u_{ij}(c) u_{-j,-i}(-c^{-1}) = n(\sigma; d_{-r}, \dots, d_r),$$

where $\sigma = (i, -j)(j, -i)$, $d_{-i} = c^{-1}$, $d_{-j} = -(\overline{c})^{-1}$, $d_j = -c$, $d_i = \overline{c}$ and all other $d_k = 1$, and

$$m(u_i(0, d)) = u_{-i}(0, -d^{-1}) u_i(0, d) u_{-i}(0, -d^{-1}) = n(\sigma; d_{-r}, \dots, d_r),$$

where $\sigma = (i, -i)$, $d_{-i} = -d^{-1}$, $d_i = d$ and all other $d_k = 1$. For each integer i, $1 \le i \le r$, we define an element h_i of H_0 by $h_i = \mathrm{Diag}(d_{-r}, \dots, d_r)$ with $d_{-r+i-1} = d_{r-i+1} = -1$ and all other $d_k = 1$. Put

$$n_{s_i} = \begin{cases} m(u_{-(r-i),r-i+1}(1))h_i & (1 \le i \le r-1), \\ m(u_{-1}(0,1))h_r & (i = r). \end{cases}$$

Then it follows from [8, (10.1.2), (10.1.6)] that $n_{s_r}, n_{s_{r-1}}, \dots, n_{s_1} \in \boldsymbol{N}(E_0)$ correspond to the roots $2a_{-1}, a_{1,-2}, \cdots, a_{r-1,-r}$, respectively, which form a basis Δ of Φ. The root $2a_{-r}$ is the highest root with respect to Δ. Associated with this $2a_{-r}$, put $n_{s_0} = n(\sigma; d_{-r}, \dots, d_r)$ where $\sigma = (-r, r)$, $d_{-r} = -\varpi_E^{-1}$, $d_r = \varpi_E$ and all other $d_i = 0$.

We now denote by N_0 the subgroup of $\boldsymbol{N}(E_0)$ generated by $\{n_{s_1}, \dots, n_{s_r}\}$, and by $N_{\mathfrak{o}}$ the subgroup of $\boldsymbol{N}(E_0)$ generated by N_0 and H_0. Then $N_{\mathfrak{o}}$ consists of all $n(\sigma; d_{-r}, \dots, d_r) \in \boldsymbol{N}(E_0)$ with $d_i \in \mathfrak{o}_E^\times$, and $\boldsymbol{N}(E_0)$ is generated by $N_{\mathfrak{o}}$ and $H = \boldsymbol{Z}(E_0)$. We define a subgroup $\boldsymbol{D}$ of H by

$$\boldsymbol{D} = \{\mathrm{Diag}(\varpi_E^{m_r}, \dots, \varpi_E^{m_1}, \varpi_E^{-m_1}, \dots, \varpi_E^{-m_r}) \mid m_1, \dots, m_r \in \mathbb{Z}\}.$$

Then, since $E^\times = \varpi_E^{\mathbb{Z}} \times \mathfrak{o}_E^\times$, we have semi-direct products $H = \boldsymbol{D} \cdot H_0$ and

$$\boldsymbol{N}(E_0) = \boldsymbol{D} \rtimes N_{\mathfrak{o}}.$$

Since the derived subgroup of $\boldsymbol{G}_1$ is semi-simple and simply-connected, $\boldsymbol{W} = \boldsymbol{N}(E_0)/H_0$ is an affine Weyl group [31, 1.13]. Since E/E_0 is unramified, it follows from [31, 1.6, 1.8] that

$$\Phi_{af} = \{a_{ij} + \gamma \mid i, j \in I, i \ne \pm j, \gamma \in \mathbb{Z}\} \cup \{2a_i + \gamma \mid i \in I, \gamma \in \mathbb{Z}\}$$

(see [31, 1.15]). The set $\{2a_{-1}, a_{1,-2}, \cdots, a_{r-1,-r}, 2a_r + 1\}$ is a basis of Φ_{af}. For each i, $0 \le i \le r$, denote by $s_i \in \boldsymbol{W}$ the image of $n_{s_i} \in \boldsymbol{N}(E_0)$ under the canonical map $\boldsymbol{N}(E_0) \to \boldsymbol{W} = \boldsymbol{N}(E_0)/H_0$. Then it follows that $s_r, s_{r-1}, \dots, s_1, s_0$ are the affine reflections associated with $2a_{-1}, a_{1,-2}, \cdots, a_{r-1,-r}, 2a_r + 1\}$, respectively.

Proposition 5.1 *Let notation and assumptions be as above. Then $\boldsymbol{W}$ is a Coxeter group with a set of generators $\{s_0, s_1, \dots, s_r\}$, and there is an isomorphism*

$$\boldsymbol{W} \simeq \boldsymbol{D} \rtimes \boldsymbol{W}_0.$$

Identifying $\boldsymbol{W}$ with $\boldsymbol{D} \rtimes \boldsymbol{W}_0$ via this isomorphism, we can regard $\boldsymbol{W}_0$ as a finite Coxeter group with a set of generators $\{s_1, \dots, s_r\}$.

Proof The first assertion has been proved above. For the second, from the above arguments, we have

$$\boldsymbol{W} = (\boldsymbol{D} \rtimes N_{\mathfrak{o}})/H_0 = \boldsymbol{D} \rtimes (N_{\mathfrak{o}}/H_0),$$

(see [16, 2.1]). By definition, $\{s_1, \dots, s_r\}$ is contained in N_0 and so in $N_{\mathfrak{o}}$. Thus from [8, (10.1.6), (10.1.7)] there is an isomorphism $N_{\mathfrak{o}}/H_0 \simeq \boldsymbol{W}_0$, which shows the second assertion. The last is clear. ∎

5.2 Intertwining

Suppose that $[\mathfrak{A}, n, 0, \beta]$ is a good skew simple stratum in $A = \operatorname{End}_F(V)$ as in Section 5.1. Let $E = F[\beta]$, and $B = B_\beta$ the A-centralizer of β. Let $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$, $\mathfrak{B} = \mathfrak{A} \cap B$, and put $e = e(\mathcal{L}_{\mathfrak{o}_E})$. Hereafter we assume that $\mathfrak{A}$ is principal. Then for $R = \dim_E(V)$, there is a positive integer f such that $R = fe$.

We choose self-dual $\mathfrak{o}_E$-lattice chains $\mathcal{L}_M$, $\mathcal{L}_m$ in V such that $e(\mathcal{L}_M|\mathfrak{o}_E) = 1$, $e(\mathcal{L}_m|\mathfrak{o}_E) = R$, and $\mathcal{L}_M \subset \mathcal{L} \subset \mathcal{L}_m$, as in Section 3.1. In $B = B_\beta$, put $\mathfrak{B}_M = \operatorname{End}^0_{\mathfrak{o}_E}(\mathcal{L}_M)$ and $\mathfrak{B}_m = \operatorname{End}^0_{\mathfrak{o}_E}(\mathcal{L}_m)$, as in Section 3.1. Then $B^\times \cap G$ contains an Iwahori subgroup $\boldsymbol{U}(\mathfrak{B}_m) = \mathfrak{B}_m \cap G$. From Proposition 5.1, we have the semi-direct product $\boldsymbol{W} = \boldsymbol{D} \rtimes \boldsymbol{W}_0$ and an Iwahori–Bruhat decomposition of $B^\times \cap G$:

$$B^\times \cap G = \boldsymbol{U}(\mathfrak{B}_m)\boldsymbol{W}\boldsymbol{U}(\mathfrak{B}_m). \tag{5.1}$$

Let $V = \bigoplus_{i=1}^e V^i$ be the E-decomposition of V subordinated to $\mathcal{L}_{\mathfrak{o}_E}$, and write $\mathcal{V} = \{v_i\}$ again. For each integer i, $1 \le i \le e$, we may set

$$\mathcal{V}^i = \mathcal{V} \cap V^i = \{v_{(i-1)f+1}, v_{(i-1)f+2}, \dots, v_{if}\}.$$

For each i, $1 \le i \le e$, define an integer $\bar{i}$, with $1 \le \bar{i} \le e$ by

$$\bar{i} = e - i + 1. \tag{5.2}$$

For each i, $1 \le i \le [(e+1)/2]$, we rewrite the basis $\mathcal{V}^i$ and $\mathcal{V}^{\bar{i}}$ as follows: $\mathcal{V}^i = \{v^i_1, v^i_2, \dots, v^i_f\}$, $\mathcal{V}^{\bar{i}} = \{v^{\bar{i}}_1, v^{\bar{i}}_2, \dots, v^{\bar{i}}_f\}$, and

$$\begin{aligned} &v^i_1 = v_{(i-1)f+1}, v^i_2 = v_{(i-1)f+2}, \dots, v^i_f = v_{if}, \\ &v^{\bar{i}}_1 = v_{\bar{i}f}, v^{\bar{i}}_2 = v_{\bar{i}f-1}, \dots, v^{\bar{i}}_f = v_{(\bar{i}-1)f+1}. \end{aligned} \tag{5.3}$$

If $i \ne \bar{i}$, each $Ev^i_j + Ev^{\bar{i}}_j$ is a hyperbolic subspace of V by Lemma 2.6. If $i = \bar{i}$, e is odd and $i = (e+1)/2$. Since $R = ef$ is even, so f is also even. In this case, each $Ev^i_j + Ev^i_{f-j+1}$ is a hyperbolic subspace of V as well.

Put $\widetilde{\mathfrak{M}}(\mathfrak{B}) = \bigoplus_{i=1}^e \mathfrak{B}^i$ as in the proof of Proposition 3.9, where $\mathfrak{B}^i = \mathfrak{A}^{(i)} \cap \operatorname{End}_E(V^i)$ for $\mathfrak{A}^{(i)}$, defined in Section 4.1. Denote by $\boldsymbol{D}(\mathfrak{B})$ the $\boldsymbol{D}$-centralizer of $\widetilde{\mathfrak{M}}(\mathfrak{B})^\times$. We define elements $n_{\boldsymbol{s}_1}, n_{\boldsymbol{s}_2}, \dots, n_{\boldsymbol{s}_{[e/2]}}$ of N_0 as follows: for $1 \le i \le [e/2] - 1$,

$$n_{\boldsymbol{s}_i} \colon v^i_j \leftrightarrow v^{i+1}_j,\ v^{\bar{i}}_j \leftrightarrow v^{\overline{i+1}}_j, \quad \text{for } 1 \le j \le f,$$
$$n_{\boldsymbol{s}_i}|V^k \equiv I, \quad \text{for } k \ne i,\ \bar{i},$$

and

$$n_{\boldsymbol{s}_{[e/2]}} \colon v^{[e/2]}_j \mapsto v^{\overline{[e/2]}}_j,\ v^{\overline{[e/2]}}_j \mapsto -v^{[e/2]}_j, \quad \text{for } 1 \le j \le f,$$
$$n_{\boldsymbol{s}_{[e/2]}}|V^k \equiv I, \quad \text{for } k \ne [e/2]$$

Let $\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_{[e/2]}$ be the canonical image of $n_{\boldsymbol{s}_1}, n_{\boldsymbol{s}_2}, \dots, n_{\boldsymbol{s}_{[e/2]}}$, respectively, under the canonical map $N_0 \to \boldsymbol{W}_0$. Denote by $\boldsymbol{W}_0(\mathfrak{B})$ the subgroup of $\boldsymbol{W}_0$ generated by $\boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_{[e/2]}$. From Proposition 5.1, we can define a subgroup, $\boldsymbol{W}(\mathfrak{B})$, of $\boldsymbol{W}$ by $\boldsymbol{W}(\mathfrak{B}) = \boldsymbol{D}(\mathfrak{B}) \rtimes \boldsymbol{W}_0(\mathfrak{B})$. This group is the $\boldsymbol{W}$-normalizer of $\widetilde{\mathfrak{M}}(\mathfrak{B})^\times$.

5.3 Simple Types

Suppose that $[\mathfrak{A}, n, 0, \beta]$ is a good skew simple stratum in A, with $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$ principal. Let $E = F[\beta]$, $e = e(\mathcal{L}_{\mathfrak{o}_E})$, and $B = B_\beta$ be the A-centralizer of β. We have $R = \dim_E(V) = ef$, for some positive integer f, as in Section 5.2. We note that f must be even if e is odd, since R is even. Since $J_-(\beta, \mathfrak{A})/J_-^1(\beta, \mathfrak{A}) \simeq \boldsymbol{U}(\mathfrak{B})/\boldsymbol{U}^1(\mathfrak{B})$, from Proposition 2.8, there is a canonical isomorphism

$$J_-(\beta, \mathfrak{A})/J_-^1(\beta, \mathfrak{A}) \simeq \begin{cases} \mathrm{GL}(f, k_E)^{e/2} & \text{if } e \text{ is even,} \\ \mathrm{GL}(f, k_E)^{(e-1)/2} \times U(f, k_{E_0}) & \text{if } e \text{ is odd,} \end{cases}$$

where $U(f, k_{E_0})$ is the unitary group of a non-degenerate k_E/k_{E_0}-anti-hermitian form.

Suppose that σ_0 (resp. σ_1) is an irreducible cuspidal representation of $\mathrm{GL}(f, k_E)$ (resp. $U(f, k_{E_0})$). If e is even, we define an irreducible representation σ_- of $\mathrm{GL}(f, k_E)^{e/2}$ by

$$\sigma_- = \sigma_0 \otimes \cdots \otimes \sigma_0 = \bigotimes^{e/2} \sigma_0,$$

and if e is odd, we define an irreducible representation σ_- of $\mathrm{GL}(f, k_E)^{(e-1)/2} \times U(f, k_{E_0})$ by

$$\sigma_- = \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_1 = \Big(\bigotimes^{(e-1)/2} \sigma_0\Big) \otimes \sigma_1.$$

Via the above isomorphism, we lift σ_- to an irreducible representation, say again σ_-, of $J_-(\beta, \mathfrak{A})$. We can also regard σ_- as an irreducible representation of $\boldsymbol{U}(\mathfrak{B})$.

Let $[\mathfrak{A}, n, 0, \beta]$ be a good skew simple stratum in A, with $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$ principal, and $\theta_- \in \mathcal{C}_-(\mathfrak{A}, 0, \beta)$. Then there is a unique irreducible representation η_- of $J_-^1(\beta, \mathfrak{A})$ which contains θ_-, and from Theorem 3.11, there is an irreducible representation κ_- of $J_-(\beta, \mathfrak{A})$ which is a β-extension of η_-.

Definition 5.2 Let notation and assumptions be as above. We say that a representation λ_- is a *simple type* (of positive level) in G, if it has the form $\lambda_- = \kappa_- \otimes \sigma_-$ for a β-extension κ_- and an irreducible representation σ_- of $J_-(\beta, \mathfrak{A})$ as above.

The representation λ_- is an analogue of a simple type for $\mathrm{GL}_N(F)$ defined by [5, (5.5.10)(a)].

Proposition 5.3 ([5, (5.3.2)]) *Let $\lambda_- = \kappa_- \otimes \sigma_-$ be a simple type in G. Let $E = F[\beta]$, $B = B_\beta$, and $\mathfrak{B} = \mathfrak{A} \cap B$. Then λ_- is irreducible and*

$$I_G(\lambda_-) = J_-(\beta, \mathfrak{A}) I_{B^\times \cap G}(\sigma_- | \boldsymbol{U}(\mathfrak{B})) J_-(\beta, \mathfrak{A}),$$

Proof By using Propositions 3.1 and 3.9, we can prove the assertion in the same way as the proof of [5, (5.3.2)]. ∎

Let $\boldsymbol{W}(\mathfrak{B})$ be as in Section 5.2, and σ_- be an irreducible representation of $\boldsymbol{U}(\mathfrak{B})$ defined as above. Put $\boldsymbol{W}(\sigma_-) = \{w \in \boldsymbol{W}(\mathfrak{B}) \mid (\sigma_-)^w \simeq \sigma_-\}$, where $(\sigma_-)^w(x) = \sigma_-(wxw^{-1})$ for $x \in \boldsymbol{U}(\mathfrak{B})/\boldsymbol{U}^1(\mathfrak{B})$.

The involution $x \mapsto \overline{x}$: $A^i \to A^{e-i+1}$, defined in Section 4.1, induces an involution $B^i \to B^{e-i+1}$. This is also induced by the involution on B which is defined by $\widetilde{h}_\beta$. Under the identification $B^1 = \cdots = B^e = \mathbb{M}(f, E)$ via the Witt basis $\mathcal{V}$, the involution $B^i \to B^{e-i+1}$ induces naturally the involution on the $\mathrm{GL}(f, \mathfrak{o}_E)$, and induces ones on $\mathrm{GL}(f, k_E)$ and $U(f, k_{E_0})$. We write again by $^-$ these involutions. In particular, we have $U(f, k_{E_0}) = \{x \in \mathrm{GL}(f, k_E) \mid x\overline{x} = 1\}$.

Definition 5.4 Let σ_0 be an irreducible cuspidal representation of $\mathrm{GL}(f, k_E)$. We define a representation σ_0^* by $\sigma_0^*(x) = \sigma_0(\overline{x}^{\,-1})$ for $x \in \mathrm{GL}(f, k_E)$. We say that the representation σ_0 is *self-dual*, if $\sigma_0 \simeq \sigma_0^*$.

In this definition, the definition of σ_0^* depends on the choice of the Witt basis $\mathcal{V}$. But the definition of self-dual does not depend on it. For another Witt basis induces an involution on each $\mathrm{GL}(f, \mathfrak{o}_E)$ which differs by a conjugation from the above involution $x \mapsto \overline{x}$.

If the component σ_0 of σ_- is self-dual, it is easy to see that $\boldsymbol{W}(\sigma_-)$ is equal to $\boldsymbol{W}(\mathfrak{B})$.

In the next paragraph, we shall show the existence of a self-dual irreducible cuspidal representation σ_0 of $\mathrm{GL}(f, k_E)$.

Remark 5.5. Any irreducible cuspidal representation σ_1 of $U(f, k_{E_0})$ is automatically self-dual.

5.4 Self-dual Irreducible Cuspidal Representations

Suppose that f is an integer ≥ 2. For simplicity, write $k_0 = k_{E_0}$ and $k = k_E$. Let $k_0 = \mathbb{F}_q$ be the finite field of order q. Then $k = \mathbb{F}_{q^2}$ is the quadratic extension of k_0. Let $x \mapsto \overline{x} = x^q$ be the non-trivial Galois involution of k/k_0. Let $\boldsymbol{G} = \boldsymbol{GL}_f$ be the general linear group of rank f defined over k, and $G = \boldsymbol{G}(k)$ the group of k-rational points in $\boldsymbol{G}$. We define a Frobenius map F_0 on $\boldsymbol{G}$ as follows: for $g = (g_{ij}) \in \boldsymbol{G}$,

$$F_0(g) = (\overline{g}_{ij}) = (g_{ij}^q).$$

Let $(\sigma_0, \mathcal{V})$ be an irreducible cuspidal representation of $G = \boldsymbol{G}(k)$. From Remark 5.5, we may set the representation $(\sigma_0^*, \mathcal{V})$ of G to be one defined by

$$\sigma_0^*(g) = \sigma_0({}^t(F_0(g))^{-1}), \quad g \in G,$$

where ${}^t g$ denotes the transpose of g.

Put $\boldsymbol{G}_1 = \mathrm{Res}_{k/k_0}(\boldsymbol{G})$, where Res denotes the functor of restrictions of scalars. We may identify $\boldsymbol{G}_1$ with $\boldsymbol{G} \times \boldsymbol{G} = \boldsymbol{G} \times F_0(\boldsymbol{G})$. We define a Frobenius map F_1 on $\boldsymbol{G}_1$ as follows: for $(x, y) \in \boldsymbol{G}_1 = \boldsymbol{G} \times \boldsymbol{G}$, $F_1(x, y) = (F_0(y), F_0(x))$. Then we have $\boldsymbol{G}_1(k_0) = \boldsymbol{G}(k)$ and $\boldsymbol{G}_1(k_0) = \boldsymbol{G}_1^{F_1} = \{g \in \boldsymbol{G}_1 | \, F_1(g) = g\}$.

We define automorphisms δ and τ of $\boldsymbol{G}_1$ by $\delta(x, y) = (y, x)$ for $x, y \in \boldsymbol{G}$ and $\tau(g) = {}^t\delta(g)^{-1}$ for $g \in \boldsymbol{G}_1$, where ${}^t(x, y) = ({}^t x, {}^t y)$ for $(x, y) \in \boldsymbol{G}_1 = \boldsymbol{G} \times \boldsymbol{G}$. Then for $g = (g, F_0(g)) \in \boldsymbol{G}_1(k_0) = \boldsymbol{G}(k) = G$, we have $\delta(g) = F_0(g)$ and

$$\tau(g) = {}^t(F_0(g))^{-1}.$$

Let χ_{σ_0} be the character of σ_0, *i.e.*, $\chi_{\sigma_0}(g) = \operatorname{Tr}(\sigma_0(g)),\ g \in G$. Then by Deligne–Lusztig theory [13, Proposition 8.3] (*cf.* [10, Ch. 7]), it is well known that there are a minisotropic maximal k-torus $\boldsymbol{T}$ of $\boldsymbol{G}$ and a regular (in general position) character θ of $T = \boldsymbol{T}(k)$ such that

$$\chi_{\sigma_0} = \pm R_{T,\theta} \quad \text{(Deligne–Lusztig character).}$$

Then there are an extension $k_f = \mathbb{F}_{q^{2f}}$ of k of degree f and the multiplicative group $\boldsymbol{GL}_1$ defined over k_f such that $\boldsymbol{T}$ is isomorphic to $\operatorname{Res}_{k_f/k}(\boldsymbol{GL}_1)$. We identify $\boldsymbol{T} = \operatorname{Res}_{k_f/k}(\boldsymbol{GL}_1)$. Put $\boldsymbol{T}_1 = \operatorname{Res}_{k/k_0}(\boldsymbol{T})$. Then we have $T = \boldsymbol{T}(k) = T_1(k_0)$.

We study $\chi_{\sigma_0^*}$. The automorphism τ of $\boldsymbol{G}_1$ satisfies the following properties:

- τ is defined over k_0,
- $\tau \circ F_1 = F_1 \circ \tau$,
- $\tau^2 = \mathrm{Id}$.

Since $\sigma_0^*(g) = \sigma_0(\tau(g)),\ g \in G$, by definition, we have

$$\chi_{\sigma_0^*}(g) = \chi_{\sigma_0}(\tau(g)) = \pm R_{T,\theta}(\tau(g)), \quad g \in G.$$

We prove the following.

Proposition 5.6 *We have $R_{T,\theta}(\tau(g)) = R_{\tau(T),\theta\circ\tau}(g)$, $g \in G$.*

Proof We first note that $T = \boldsymbol{T}_1(k_0) = \boldsymbol{T}(k)$ and $G = \boldsymbol{G}_1(k_0) = \boldsymbol{G}(k)$. We adapt Deligne–Lusztig theory [13] (cf. [10, Ch. 7]) to the groups $\boldsymbol{G}_1 \supset \boldsymbol{T}_1$ defined over k_0. Let $g \in G = \boldsymbol{G}_1(k_0)$ and $g = us = su$ be the Jordan decomposition of g, where u is the unipotent part of g and s is the semisimple part of g. Then we have the character formula [13, Theorem 4.2] (*cf.* [10, Theorem 7.2.8]) as follows:

$$R_{T,\theta}(g) = \frac{1}{|C^0(s)^{F_1}|} \sum_{x \in G, x^{-1}sx \in T_1} \theta(x^{-1}sx) Q^{C^0(s)}_{xT_1x^{-1}}(u),$$

where $C^0(s)$ denotes the connected centralizer of s in $\boldsymbol{G}_1$, and $Q^{G_1}_{T_1}(u) = R_{T_1,1}(u)$.

For the decomposition $g = us$, $\tau(g) = \tau(u)\tau(s)$ is also the Jordan decomposition with $\tau(u)$ unipotent and $\tau(s)$ semisimple. Thus we obtain

$$R_{T,\theta}(\tau(g)) = \frac{1}{|C^0(\tau(s))^{F_1}|} \sum_{x \in G, x^{-1}\tau(s)x \in T_1} \theta(x^{-1}\tau(s)x) Q^{C^0(\tau(s))}_{xT_1x^{-1}}(\tau(u)) \tag{5.4}$$

as well.

(i) From the properties of τ, we have $\tau(C^0(\tau(s))^{F_1}) = C^0(s)^{F_1}$ and

$$|C^0(\tau(s))^{F_1}| = |C^0(s)^{F_1}|.$$

(ii) Similarly, from $\tau(x^{-1}\tau(s)x) = \tau(x)^{-1}s\tau(x)$, we obtain

$$\theta(x^{-1}\tau(s)x) = \theta \circ \tau(\tau(x)^{-1}s\tau(x)),$$

and if $x \in G = \boldsymbol{G}_1(k_0),\ x^{-1}\tau(s)x \in T = \boldsymbol{T}_1(k_0)$, we have

$$\tau(x) \in G,\ \tau(x)^{-1}s\tau(x) \in \tau(T).$$

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(iii) We again have $\tau(C^0(\tau(s))) = C^0(s)$, $\tau(\tau(x)\tau(\boldsymbol{T}_1)\tau(x)^{-1}) = x\boldsymbol{T}_1x^{-1}$. The Lang variety $\widetilde{X} = L^{-1}(\boldsymbol{U})$ is associated with a Borel subgroup $B = \boldsymbol{T}_1\boldsymbol{U}$ of $\boldsymbol{G}_1$, where $\boldsymbol{U}$ is the unipotent radical of B. Thus, $\tau(\widetilde{X}) = \tau(L^{-1}(\boldsymbol{U})) = L^{-1}(\tau(\boldsymbol{U}))$ is associated with $\tau(B) = \tau(\boldsymbol{T}_1)\tau(\boldsymbol{U})$. Hence we get

$$Q^{C^0(\tau(s))}_{xT_1x^{-1}}(\tau(u)) = Q^{C^0(s)}_{\tau(x)\tau(T_1)\tau(x)^{-1}}(u).$$

From (i)–(iii), it follows that the right-hand side of $R_{T,\theta}(\tau(g))$ in (5.4) is equal to $R_{\tau(T),\theta\circ\tau}(g)$. ∎

We further study the right-hand side of the equality in Proposition 5.6 and obtain the following.

Proposition 5.7 *We have $\chi_{\sigma_0^*} = \pm R_{T,\overline{\theta}_1}$ for the unique character θ_1 of $T = \boldsymbol{T}(k)$ with $\theta_1^q = \theta$.*

Proof From Proposition 5.6, we have $\chi_{\sigma_0^*} = \pm R_{\tau(T),\theta\circ\tau}$.

We can represent $T = \boldsymbol{T}(k)$ in $G = \boldsymbol{G}(k)$ as follows. We choose an element $\alpha \in k_f^\times = k_f - \{0\}$ satisfying

- $\{1, \alpha, \dots, \alpha^{f-1}\}$ is a basis of k_f as a k-vector space,
- for the regular representation $\rho\colon k_f^\times \to G = \boldsymbol{GL}_f(k)$ with respect to the basis, we may set $T = \{\rho(x) | x \in k_f^\times\}$.

Write $\overline{\rho(x)} = F_0(\rho(x))$ for simplicity. We have $\overline{\alpha} = F_0(\alpha) \in k_f^\times$ and $\{1, \overline{\alpha}, \dots, \overline{\alpha}^{f-1}\}$ is also a k-basis of k_f. Let $\rho'\colon k_f^\times \to G$ be the regular representation of $k_f^\times$ with respect to this new basis. Then for $x \in k_f^\times$, we can check that $\overline{\rho(x)} = \rho'(x^q) = \rho'(x)^q$ and that there is an element $g_0 \in G$ such that $\rho'(x) = g_0\rho(x)g_0^{-1}$, $x \in k_f^\times$. Hence we have $\overline{\rho(x)} = g_0\rho(x)^q g_0^{-1}$, $x \in k_f^\times$, and $\overline{T} = \{\overline{\rho(x)} | x \in k_f^\times\} = g_0Tg_0^{-1}$. However, for $g \in \tau(T) = {}^t\overline{T}$, we have $\theta\circ\tau(g) = \theta((\overline{{}^tg})^{-1}) = \overline{\theta}(\overline{{}^tg})$. Since the Pontrjagin dual $\hat{T}$ of T is (non-canonically) isomorphic to $k_f^\times = (\mathbb{F}_{q^{2f}})^\times$, it is a cyclic group of order $q^{2f}-1$. It follows that there is a chracter θ_1 of T with $\theta_1^q = \theta$ as in the assertion. Thus we have $\theta\circ\tau(g) = \overline{\theta}_1^q(\overline{{}^tg})$. We can write $\overline{{}^tg} = \rho(x)$ for some $x \in k_f^\times$, so that

$${}^tg = \overline{\rho(x)} = g_0\rho(x)^q g_0^{-1}.$$

From $\overline{T} = g_0Tg_0^{-1}$ above, it follows that ${}^{g_0}\overline{\theta}_1$ is a unique character of $\overline{T}$. Thus

$$({}^{g_0}\overline{\theta}_1)({}^tg) = \overline{\theta}_1(g_0^{-1}({}^tg)g_0) = \overline{\theta}_1(\rho(x)^q) = \overline{\theta}_1^q(\overline{{}^tg}) = \overline{\theta}(\overline{{}^tg}).$$

Hence, for $g \in \tau(T) = {}^t\overline{T}$, we have $\theta\circ\tau(g) = {}^{g_0}\overline{\theta}_1({}^tg)$.

Let h be a generator of the group $\tau(T) = {}^t\overline{T}$. Then the elements $h \in \tau(T)$ and ${}^th \in \overline{T}$ are both regular semisimple, and have the same characteristic polynomial. Thus there is an element $g_1 \in G$ such that $h = g_1({}^th)g_1^{-1}$, and it does not depend on the choice of h. So we have $\tau(T) = {}^t\overline{T} = g_1(\overline{T})g_1^{-1}$. Hence, since ${}^tg = g_1^{-1}gg_1$ for $g \in \tau(T)$, we have ${}^{g_0}\overline{\theta}_1({}^tg) = {}^{g_0}\overline{\theta}_1(g_1^{-1}gg_1) = {}^{g_1g_0}(\overline{\theta}_1)(g)$. Consequently, it follows that $\theta \circ \tau(g) = {}^{g_1 g_0}(\overline{\theta}_1)(g)$, $g \in \tau(T)$ and that $(g_1 g_0)^{-1}\tau(T)(g_1 g_0) = T$. By the orthogonality relation [13, Theorem 6.8]for $R_{T,\theta}$, we obtain

$$R_{\tau(T),\theta\circ\tau} = R_{\tau(T),{}^{g_1 g_0}\overline{\theta}_1} = R_{T,\overline{\theta}_1},$$

which completes the proof. ■

Corollary 5.8 *If the integer f is odd, there is an irreducible cuspidal representation σ_0 of $G = \mathrm{GL}(f, k_E)$ such that σ_0 is equivalent to σ_0^*.*

Proof Let $\boldsymbol{T}$ be a minisotropic maximal k-torus of $\boldsymbol{G}$, and θ be a regular character of $T = \boldsymbol{T}(k)$ such that $\chi_\sigma = \pm R_{T,\theta}$. We have $\sigma_0 \simeq \sigma_0^*$ if and only if $\chi_{\sigma_0} = \chi_{\sigma_0^*}$. Thus it follows from Proposition 5.7 that $\sigma_0 \simeq \sigma_0^*$ is equivalent to $R_{T,\theta} = R_{T,\overline{\theta}_1}$, where $\theta_1^q = \theta$. By the orthogonality relations for $R_{T,\theta}$, the last condition is equivalent to the condition that there is a non-negative integer ℓ such that $\theta^{q^{2\ell}} = \overline{\theta}_1$, that is, $\theta^{q^{2\ell+1}} = \theta^{-1}$.

Let ξ be a generator of $\hat{T} \simeq k_f^\times$. Take $\theta = \xi^{q^f-1}$ in $\hat{T}$. Then we have $\theta^{q^f+1} = (\xi^{q^f-1})^{q^f+1} = \xi^{q^{2f}-1} = 1$. Further we can show directly that $\theta^{q^{2i}} \neq \theta$ for any integer i, $1 \leq i \leq f-1$, that is, θ is regular. ■

5.5 The G-intertwining of a Simple Type

We moreover study the G-intertwining of a simple type $(J_-(\beta, \mathfrak{A}), \lambda_-)$ in G.

Proposition 5.9 ([[5, (5.5.11)]]) *Let $[\mathfrak{A}, n, 0, \beta]$ be a good skew simple stratum in A, with $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$ principal, and $\lambda_- = \kappa_- \otimes \sigma_-$ a simple type in G attached to $[\mathfrak{A}, n, 0, \beta]$. Then we have $I_G(\lambda_-) \subset J_-(\beta, \mathfrak{A})\boldsymbol{W}(\mathfrak{B})J_-(\beta, \mathfrak{A})$.*

Proof If $g \in G$ intertwines λ_-, from Proposition 5.3, $g \in J_- y J_-$ for some $y \in B^\times \cap G$ and y intertwines $\sigma_-|\boldsymbol{U}(\mathfrak{B})$. Since J_- contains the Iwahori subgroup $\boldsymbol{U}(\mathfrak{B}_m)$ of $B^\times \cap G$, by the Iwahori–Bruhat decomposition of (5.1), we may take $y \in \boldsymbol{W}$. Thus the result follows from the following lemma, which is an analogue of [5, (5.5.5)]. ■

Lemma 5.10 *If $w \in \boldsymbol{W}$ intertwines $\sigma_-|\boldsymbol{U}(\mathfrak{B})$, then $w \in \boldsymbol{W}(\mathfrak{B})$.*

Proof It is hard to prove this lemma, (see [5, (5.5.5)]).

It follows from the argument in Section 5.2 that the $\boldsymbol{W}$-normalizer of $\widetilde{\mathfrak{M}}(\mathfrak{B})^\times$ is equal to $\boldsymbol{W}(\mathfrak{B}) = \boldsymbol{D}(\mathfrak{B}) \rtimes \boldsymbol{W}_0(\mathfrak{B})$. Thus, if $w \in \boldsymbol{W}$ intertwines $\sigma_-|\boldsymbol{U}(\mathfrak{B})$, it is enough to prove that w normalizes $\widetilde{\mathfrak{M}}(\mathfrak{B})^\times$.

We now assume that $w \in \boldsymbol{W}$ does not normalize $\widetilde{\mathfrak{M}}(\mathfrak{B})^\times$. Put $\mathcal{L}_{\mathfrak{o}_E} = \{L_k \mid k \in \mathbb{Z}\}$ with $L_0^\natural = L_0$. Let $V = \bigoplus_{i=1}^e V^i$ be the E-decomposition of V subordinated to $\mathcal{L}_{\mathfrak{o}_E}$, $L_k = \coprod_{i=1}^e L_k^i$, $L_k^i = L_k \cap V^i$, for $k \in \mathbb{Z}$, $\mathcal{V} = \{v_1, v_2, \ldots, v_R\}$ and let $\mathcal{V} = \coprod_{i=1}^e \mathcal{V}^i$ be as in Lemma 2.6. Let $L_k \in \mathcal{L}$. Then for each integer i, $1 \leq i \leq e$, there is an integer $m(i,k)$ such that

$$L_k \cap V^i = L_k^i = \mathfrak{p}_E^{m(i,k)}\langle \mathcal{V}^i \rangle.$$

We denote this lattice by $\langle \mathfrak{p}_E^{m(i,k)} \rangle^i$. Thus we have

$$L_k = \bigoplus_{i=1}^e L_k^i = \bigoplus_{i=1}^e \langle \mathfrak{p}_E^{m(i,k)} \rangle^i$$

We prepare the following three lemmas.

Lemma 5.11 *The function $m(i,k)$ on $\{1,\dots,e\}\times\mathbb{Z}$ satisfies the following conditions:*

(i) $m(1,0)=m(2,0)=\cdots=m(e,0)=0$,

(ii) $m(1,k)\le m(2,k)\le\cdots\le m(e,k)\le m(1,k)+1$, *for $k\in\mathbb{Z}$, and precisely one of these inequalities is strict,*

(iii) *for each i, $m(i,k)$ jumps at k, with $k\equiv -i \pmod e$, that is, $m(i,k+1)=m(i,k)+1$.*

Proof Straightforward. ■

Lemma 5.12 *Let $w\in\boldsymbol{W}$. Then for each integer j, $1\le j\le R/2$, there are integers d_j and $k=k(j)$, determined uniquely by j, such that*

$$w(\mathfrak{o}_E v_j)=\mathfrak{p}_E^{d_j}v_k,\ w(\mathfrak{o}_E v_{R-j+1})=\mathfrak{p}_E^{-d_j}v_{R-k+1}.$$

Proof This follows straightforwardly by the definition of $\boldsymbol{W}$ in Section 5.1. ■

We recall $\bar{i}=e-i+1$, for $i\in\{1,2,\dots,e\}$, defined by (5.2).

Lemma 5.13 *Let $w\in\boldsymbol{W}$. The element w permutes $\{L_k^i\mid i\in\{1,2,\dots,e\},k\in\mathbb{Z}\}$ if and only if for each $L_k^i=\langle\mathfrak{p}_E^{m(i,k)}\rangle^i$, $L_k^{\bar{i}}=\langle\mathfrak{p}_E^{m(\bar{i},k)}\rangle^{\bar{i}}$, there are integers δ_i,j,k',k'' such that*

$$w(L_k^i)=L_{k'}^j=\langle\mathfrak{p}_E^{m(i,k)+\delta_i}\rangle^j,\quad w(L_k^{\bar{i}})=L_{k''}^{\bar{j}}=\langle\mathfrak{p}_E^{m(\bar{i},k)-\delta_i}\rangle^{\bar{j}}.$$

Proof This follows directly from Lemma 5.12. ■

By Lemma 5.13, we may assume that the element w does not permute $\{L_k^i\}$ as in the proof of [5, (5.5.5)].

For $i\in\{1,\dots,e\}$ and $j\in\{1,\dots,f\}$, let the basis $\mathcal{V}^i=\{v_j^i\}$ to be as in (5.3), and define an integer $\nu(i,j)$ in $\{1,\dots,e\}$ by $w^{-1}(v_j^i)\in V^{\nu(i,j)}$. Let k be any integer, and L_k be the lattice in $\mathcal{L}$ as above. Then $wL_k\cap Ev_j^i\subset w(L_k\cap V^{\nu(i,j)})$, and from Lemma 5.12, there is an integer d_j^i such that

$$wL_k\cap Ev_j^i=\mathfrak{p}_E^{m(\nu(i,j),k)+d_j^i}v_j^i.$$

We remark that the integers $\nu(i,j)$ and d_j^i depend on the element w of $\boldsymbol{W}$, but they do not depend on k of L_k.

Let i be an integer with $1\le i\le[(e+1)/2]$. Then, for each integer k, we have

$$wL_k\cap(V^i+V^{\bar{i}})=(wL_k\cap V^i)+(wL_k\cap V^{\bar{i}}).$$

If $i\ne\bar{i}$, then, again by Lemma 5.12, we have $w^{-1}(v_j^{\bar{i}})\in V^{\overline{\nu(i,j)}}$, so that $\nu(\bar{i},j)=\overline{\nu(i,j)}$, and similarly $d_j^{\bar{i}}=-d_j^i$. If $i=\bar{i}$, then we have $\nu(i,f-j+1)=\overline{\nu(i,j)}$ and $d_{f-j+1}^i=-d_j^i$ as well. We put

$$f'=\begin{cases} f & \text{if } i\ne\bar{i},\\ f/2 & \text{if } i=\bar{i},\end{cases}$$

and for each $j \in \{1, \dots, f'\}$, rewrite

$$v^i_{-j} = \begin{cases} v^{\bar{i}}_j & \text{if } i \neq \bar{i}, \\ v^i_{2f'-j+1} & \text{if } i = \bar{i}. \end{cases}$$

Then $\{v^i_j,\ v^i_{-j} \mid j \in \{1, \dots, f'\}\}$ form a basis of $V^i + V^{\bar{i}}$, and for each integer k, we have

$$wL_k \cap (V^i + V^{\bar{i}}) = \sum_{j=1}^{f'} \mathfrak{p}_E^{m(\nu(i,j),k)+d^i_j} v^i_j + \sum_{j=1}^{f'} \mathfrak{p}_E^{m(\overline{\nu(i,j)},k)-d^i_j} v^i_{-j}. \tag{5.5}$$

Lemma 5.14 *There is an integer i,* $1 \leq i \leq [(e+1)/2]$*, which satisfies the condition, "not* $\nu(i,1) = \cdots = \nu(i,f)$ *or not* $d^i_1 = \cdots = d^i_f$*".*

Proof Suppose that there is no integer i as in the assertion. Then for $i = \bar{i} = (e+1)/2$, we have $\nu(i,1) = \cdots = \nu(i,f') = (e+1)/2$ and $d^i_1 = \cdots = d^i_{f'} = 0$, so that $w(L^i_k) = L^i_k$, for $k \in \mathbb{Z}$. For i, with $i \neq \bar{i}$, put $\nu = \nu(i,1) = \cdots = \nu(i,f')$ and $d = d^i_1 = \cdots = d^i_{f'}$. For each integer k, it follows from the above argument that

$$wL^\nu_k = wL_k \cap V^i = \langle \mathfrak{p}_E^{m(\nu,k)+d} \rangle^i,$$

whence, by Lemma 5.11, we have $wL^\nu_k = \langle \mathfrak{p}_E^{m(i,\ell)} \rangle^i = L^i_\ell$ for some integer ℓ. Hence the element w permutes $\{L^i_k\}$, which contradicts the assumption on w. ■

We fix such an integer i as in Lemma 5.14, and for each $j \in \{1, \dots, f'\}$, write $\mu(j), d_j$, and ν_j for $\nu(i,j), d^i_j$, and v^i_j, respectively. Put $W = V^i + V^{\bar{i}}$, and

$$W_+ = \sum_{j=1}^{f'} E v_j, \quad W_- = \sum_{j=1}^{f'} E v_{-j}.$$

Then we have $W = W_+ \oplus W_-$, and W_+ and W_- are both maximal totally isotropic subspaces of W with respect to $\widetilde{h}_\beta | W$.

Remarks 5.15. (i) In case $i = \bar{i}$, the condition in Lemma 5.14 is divided into the following two cases:

(a) not $\nu(1) = \cdots = \nu(f')$ or not $d_1 = \cdots = d_{f'}$,
(b) $\nu(1) = \cdots = \nu(f'), d_1 = \cdots = d_{f'}$, and either $\nu(f') \neq \overline{\nu(1)}$ or $d_1 \neq 0$.

(ii) In case $i \neq \bar{i}$, it is nothing but (a) above, since $f' = f$.

For $wL_k \cap W$ of (5.5), put

$$M = \{(\nu(j), d_j), (\overline{\nu(j)}, -d_j) \mid j \in \{1, \dots, f'\}\},$$

where the $(\nu(j), d_j)$ do not depend on k of L_k as remarked above. We define a linear order $\prec$ on the set M by $(\nu', d') \prec (\nu, d)$ if and only if either $d' < d$ or both $d' = d$ and $\nu' < \nu$.

Lemma 5.16 *If elements (ν, d) and (ν', d') in M_i satisfy $(\nu', d') \prec (\nu, d)$, then $m(\nu', k) + d' \le m(\nu, k) + d$ and $m(\overline{\nu}, k) - d \le m(\overline{\nu'}, k) - d'$, for any integer k.*

Proof This follows directly from Lemma 5.11(ii). ∎

Denote by $\tau_{j\ell}$ the product of the transposition of ν_j and ν_ℓ in $\mathcal{V}^i$ with that of ν_{-j} and $\nu_{-\ell}$ in $\mathcal{V}^{\bar{i}}$. By Lemma 5.16, multiplying an element u which is a product of appropriate $\tau_{j\ell}$'s, we can permute $\{\nu_1, \dots, \nu_{f'}\}$ (so $\{\nu_{-1}, \dots, \nu_{-f'}\}$) so as to have

$$uwL_k \cap W = \sum_{j=1}^{f'} \mathfrak{p}_E^{\mu(j,k)} \nu_j + \sum_{j=1}^{f'} \mathfrak{p}_E^{\mu'(j,k)} \nu_{-j},$$

with $\mu(1, k) \le \cdots \le \mu(f', k)$, $\mu'(f', k) \le \cdots \le \mu'(1, k)$. for each k.

Let (ν_0, d_0) be the maximal element in the set M with respect to the order $\prec$. Then we have $d_0 \ge 0$, and $\mu(f', k) = m(\nu_0, k) + d_0$ or $\mu'(1, k) = m(\nu_0, k) + d_0$. We may assume $\mu(f', k) = m(\nu_0, k) + d_0$, up to the transposition of W_+ and W_-. Put $\kappa = e - \nu_0$, and for $uwL_\kappa \cap W$ and $uwL_{\kappa+1} \cap W$, write

$$a_j = \mu(j, \kappa), \quad a'_j = \mu'(j, \kappa) \quad \text{and} \quad b_j = \mu(j, \kappa + 1), \quad b'_j = \mu'(j, \kappa + 1)$$

for $j \in \{1, \dots, f'\}$. Then from the choice of κ, we have

$$m(1, \kappa) = \cdots = m(\nu_0, \kappa) = 0, \quad m(\nu_0 + 1, \kappa) = \cdots = m(e, \kappa) = 1,$$
$$m(\nu_0, \kappa + 1) = 1.$$

Thus, by definition, we have

$$a_{f'} = \mu(f', \kappa) = m(\nu_0, \kappa) + d_0 = d_0,$$
$$b_{f'} = \mu(f', \kappa + 1) = m(\nu_0, \kappa + 1) + d_0 = 1 + d_0 = a_{f'} + 1.$$

This implies $uwL_\kappa \cap W \supsetneq uwL_{\kappa+1} \cap W$.

Lemma 5.17 (i) *In case $i \ne \bar{i}$, there is an integer s, $1 \le s \le f'$, such that $b_1 \le \cdots \le b_s < b_{s+1} = \cdots = b_{f'}$.*

(ii) *In case $i = \bar{i}$, we can replace the element u of $\boldsymbol{W}$ so that there is an integer s, $0 \le s \le f'$, such that $b_1 \le \cdots \le b_s < b_{s+1} = \cdots = b_{f'}$ and $b'_1 < b_{s+1}$. In particular, if $s = 0$, then $b_1 = \cdots = b_{f'} > b'_{f'} = \cdots = b'_1$.*

Proof We first assume (1) not $\nu(1) = \cdots = \nu(f')$ or not $d_1 = \cdots = d_{f'}$ in Remark 5.15. Then there is an integer s, $1 \le s \le f'$, which satisfies $b_1 \le \cdots \le b_s < b_{s+1} = \cdots = b_{f'}$. For if not all the $\nu(j)$ are equal, then there is some s such that $a_s = b_s$. Thus the maximal one of these is the desired. If all the $\nu(j)$ are equal, not all the d_j are equal. Thus, if $a_s < a_{f'}$, then $b_s \le a_s + 1 < a_{f'} + 1 = b_{f'}$. Hence, similarly, we get s as claimed. If $i \ne \bar{i}$, then, since the assumption (1) is satisfied, the assertion (i) is proved.

So, let $i = \bar{i}$. Denote by τ_j the transposition of v_j and v_{-j}. If we have $b_{s+1} = b_{f'} = b'_1$, we can replace u by the product of appropriate $\tau_{j\ell}$'s and τ_m's so that $b'_{f'} \leq \cdots \leq b'_1 < b_{s+1}$. Then we have $0 \leq s \leq f'$ and $b_1 \leq \cdots \leq b_s < b_{s+1} = \cdots = b_{f'}$ as the assertion says.

We next assume (2) $\nu(1) = \cdots = \nu(f')$, $d_1 = \cdots = d_{f'}$, and "$\nu(f') \neq \overline{\nu(1)}$ or $d_1 \neq 0$" in Remark 5.15. Then similarly we can replace u so that $\mu(1,k) = \cdots = \mu(f',k) > \mu'(f',k) = \cdots = \mu'(1,k)$, for any integer k. In particular, for $k = \kappa + 1$, $b_1 = \cdots = b_{f'} > b'_{f'} = \cdots = b'_1$. ■

Via the integer s in Lemma 5.17, we decompose the spaces W_+ and W_- into

$$W_+ = W_1 \oplus W_2,\ W_- = W_2^\natural \oplus W_1^\natural$$

by setting

$$W_1 = \sum_{j=1}^{s} Ev_j, \quad W_2 = \sum_{j=s+1}^{f'} Ev_j, \quad W_2^\natural = \sum_{j=s+1}^{f'} Ev_{-j}, \quad W_1^\natural = \sum_{j=1}^{s} Ev_{-j}.$$

Here, if $s = 0$, we understand $W_1 = W_1^\natural = (0)$. Then we have $W = W_2 \oplus (W_1^\natural \oplus W_1) \oplus W_2$. We produce a self-dual $\mathfrak{o}_E$-lattice chain in W of $\mathfrak{o}_E$-period equal to 2 or 3. We first define $\mathfrak{o}_E$-lattices in W_+ by

$$\overline{L}_0 = \sum_{j=1}^{f'} \mathfrak{o}_E v_j \supsetneq \overline{L}_1 = \sum_{j=1}^{s} \mathfrak{o}_E v_j + \sum_{j=s+1}^{f'} \mathfrak{p}_E v_j \supsetneq \varpi_E \overline{L}_0,$$

and in W_- by

$$\overline{L}_0^\natural = \sum_{j=1}^{f'} \mathfrak{o}_E v_{-j} \supsetneq \varpi_E \overline{L}_1^\natural \sum_{j=s+1}^{f'} \mathfrak{o}_E v_{-j} + \sum_{j=1}^{s} \mathfrak{p}_E v_{-j} \supsetneq \varpi_E \overline{L}_0^\natural.$$

Multiplying these $\mathfrak{o}_E$-lattices by ϖ_E^m, $m \in \mathbb{Z}$, we obtain an $\mathfrak{o}_E$-lattice chain $\overline{\mathcal{L}}$ in V^i. Further, in W we define

$$M_0 = \overline{L}_0^\natural \oplus \overline{L}_0, \quad M_1 = \overline{L}_0^\natural \oplus \varpi_E \overline{L}_1, \quad M_2 = \varpi_E \overline{L}_1^\natural \oplus \varpi_E \overline{L}_0.$$

Then we have $M_0 \supsetneq M_1 \supset M_2 \supsetneq \varpi_E M_0$, and these $\mathfrak{o}_E$-lattices generate a self-dual $\mathfrak{o}_E$-lattice chain $\overline{\mathcal{M}}$ in W. The $\mathfrak{o}_E$-period of $\overline{\mathcal{M}}$ is equal to 3 if $s \neq 0$, and to 2 if $s = 0$.

Let $\overline{\mathfrak{B}} = \mathrm{End}^0_{\mathfrak{o}_E}(\overline{\mathcal{M}})$ be the hereditary $\mathfrak{o}_E$-order in $\mathrm{End}_E(W)$ defined by $\overline{\mathcal{M}}$, and $\overline{\mathfrak{Q}}$ its Jacobson radical. In $\mathrm{End}_E(W) \cap \mathfrak{G}$, put

$$\mathfrak{n} = \left\{ \mathrm{Hom}_E(W_1^\natural \oplus W_1 \oplus W_2, W_2^\natural) \coprod \mathrm{Hom}_E(W_2, W_1^\natural \oplus W_1) \right\} \cap \mathfrak{G},$$

if $i = \bar{i}$, and put

$$\mathfrak{n} = \left\{ \mathrm{Hom}_E(W_1^\natural, W_2^\natural) \coprod \mathrm{Hom}_E(W_2, W_1) \right\} \cap \mathfrak{G},$$

if $i \neq \bar{i}$. Take any element $x \in \mathfrak{n} \cap \overline{\mathfrak{B}} = \mathfrak{n} \cap \overline{\mathfrak{Q}}$.

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Lemma 5.18 *There is an integer ℓ, with $0 \leq \ell < e$, such that*

$$x(uwL_{\kappa+1} \cap W) \subset uwL_{\kappa+\ell+1} \cap W, \tag{5.6}$$

$$x(uwL_{\kappa+\ell+1} \cap W) \subset \varpi_E(uwL_{\kappa+1} \cap W). \tag{5.7}$$

Since we have chosen the element $u \in \boldsymbol{W}$ so as to have $b'_1 \leq b_{s+1}$, we have $b_1 \geq b'_{s+1}$ by Lemma 5.16. Thus $b_1 \geq b'_{s+1} \leq b'_s$. To prove Lemma 5.18, we will consider the following two cases:

Case 1. $b'_{s+1} < b'_s$, if $i \neq \bar{i}$, and $b_1 > b'_{s+1} < b'_s$, if $i = \bar{i}$,
Case 2. $b'_{s+1} = b'_s$, if $i \neq \bar{i}$, and $b_1 = b'_{s+1}$ or $b'_{s+1} = b'_s$, if $i = \bar{i}$.

In Case 1, by definition, we see that $x(uwL_{\kappa+1} \cap W)$ is contained in

$$\begin{cases} \displaystyle\sum_{j=s+1}^{f'} \mathfrak{p}_E^{b'_s} v_{-j} + \sum_{j=1}^{s} \mathfrak{p}_E^{b_{s+1}} v_j, & \text{if } i \neq \bar{i}, \\ \displaystyle\sum_{j=s+1}^{f'} \mathfrak{p}_E^{\min\{b'_s, b_1\}} v_{-j} + \sum_{j=1}^{s} (\mathfrak{p}_E^{b_{s+1}} v_{-j} + \mathfrak{p}_E^{b_{s+1}} v_j), & \text{if } i = \bar{i}. \end{cases} \tag{5.8}$$

By Lemma 5.17, we have

$$\begin{aligned} &b'_{f'} + 1 \leq \cdots \leq b'_{s+1} + 1 \leq \min\{b'_s, b_1\} \leq b'_s, \\ &b'_s + 1 \leq \cdots \leq b'_1 + 1 \leq b_{s+1}, \quad \text{if } i = \bar{i}, \\ &b_1 + 1 \leq \cdots \leq b_s + 1 \leq b_{s+1}. \end{aligned}$$

Hence we obtain $x(uwL_{\kappa+1} \cap W) \subset \varpi_E(uwL_{\kappa+1} \cap W)$, which is (5.7) with $\ell = 0$ in Lemma 5.18.

We consider Case 2. For an integer ℓ, $0 \leq \ell < e$, put

$$c_j = \mu(j, \kappa + \ell + 1), c'_j = \mu'(j, \kappa + \ell + 1)$$

for $j \in \{1, \ldots, f'\}$. Then we see that $x(uwL_{\kappa+\ell+1} \cap W)$ is contained in (5.8) in which b'_s, b_1, and b_{s+1} are replaced by c'_s, c_1, and c_{s+1}, respectively. To prove (5.6), we must prove the following inequalities:

(I-1) $c'_{s+1} \leq b'_s$ if $i \neq \bar{i}$, and $c'_{s+1} \leq \min\{b_1, b'_s\}$ if $i = \bar{i}$,
(I-2) $c'_1 \leq b_{s+1}$ if $i = \bar{i}$,
(I-3) $c_s \leq b_{s+1}$,

and for (5.7),

(II-1) $b'_{s+1} < c'_s$ if $i \neq \bar{i}$, and $b'_{s+1} < \min\{c_1, c'_s\}$ if $i = \bar{i}$,
(II-2) $b'_1 < c_{s+1}$ if $i = \bar{i}$,
(II-3) $b_s < c_{s+1}$.

By Lemma 5.17, we easily obtain (I-2), (I-3), (II-2), and (II-3), for any integer ℓ, $0 \leq \ell < e$, in Case 2. Thus it remains for us to prove that there is an integer ℓ, $0 \leq \ell < e$, such that (I-1) and (II-1) hold.

Lemma 5.19 *If $b'_{s+1} = b'_s$, then there is an integer ℓ, $0 \le \ell < e$, such that $c'_{s+1} = b'_{s+1}$ and $c'_s = b'_s + 1$.*

Proof Put $b'_s = m(a, \kappa+1)+d$, for some integers a and d. Then $b_s = m(\overline{a}, \kappa+1)-d$. On the other hand, $b_{s+1} = b_{f'} = m(\nu_0, \kappa+1)+d_0 = 1+d_0$ and $b'_{s+1} = m(\overline{\nu_0}, \kappa+1)-d_0$. From $b_s < b_{s+1}$ and $b'_{s+1} = b'_s$, we easily get $\overline{\nu_0} < a$. For if $\overline{\nu_0} = a$, then $\overline{a} = \nu_0$. It follows that $b_s < b_{s+1}$ implies $-d < d_0$ and that $b'_s = b'_{s+1}$ implies $d = -d_0$. This is a contradiction. Thus, if $\nu_0 \le \overline{\nu_0}$, then $\nu_0 \le \overline{\nu_0} < a$. On the other hand, if $\overline{\nu_0} < \nu_0$, then we have $a < \nu_0$. For suppose $\nu_0 \le a$. Then $\overline{a} \le \overline{\nu_0}$, so that $m(\overline{\nu_0}, \kappa + 1) = m(\overline{a}, \kappa + 1) = 0$ and $m(a, \kappa + 1) = 1$. Thus, again from the above condition, we obtain $-d < 1 + d_0$ and $-d_0 = 1 + d$. This is a contradiction. Hence we have obtained

$$\begin{cases} \nu_0 \le \overline{\nu_0} < a, & \text{if } \nu_0 \le \overline{\nu_0}, \\ \overline{\nu_0} < a < \nu_0, & \text{if } \overline{\nu_0} < \nu_0. \end{cases}$$

It follows from Lemma 5.11 that $m(a, k)$ jumps at $k = \kappa + \ell + 1$ for some integer ℓ, $0 \le \ell < e$, and that $m(\overline{\nu_0}, k)$ is constant for $\kappa+1 \le k \le \kappa+\ell+1$. Hence the assertion follows. ∎

If $i \ne \overline{i}$, for the integer ℓ of Lemma 5.19, we have

$$c'_{s+1} = b'_{s+1} = b'_s < b'_s + 1 = c'_s.$$

Thus (I-1) and (II-1) hold. Hence, the proof of Lemma 5.18 is complete in Case 2 with $i \ne \overline{i}$.

To prove Lemma 5.18 in Case 2 with $i = \overline{i}$, let $i = \overline{i}$, and $b_1 = b'_{s+1}$ or $b'_{s+1} = b'_s$.

Lemma 5.20 *If $b_1 = b'_{s+1}$, then there is an integer ℓ, $0 \le \ell < e$, such that $c'_{s+1} = b'_{s+1}$ and $c_1 = b_1 + 1$.*

Proof The proof is quite similar to that of Lemma 5.19. We sketch the outline. Put $b_1 = m(a, \kappa + 1) + d$. Then $b'_1 = m(\overline{a}, \kappa + 1) - d$. We have $b_{s+1} = 1 + d_0$ and $b'_{s+1} = m(\overline{\nu_0}, \kappa + 1) - d_0$. By Lemma 5.17(ii), we have $b_1 < b_{s+1}$ and $b_1 = b'_{s+1}$. Similarly, it follows that

$$\begin{cases} a \le \nu_0 \text{ or } \overline{\nu_0} < a, & \text{if } \nu_0 \le \overline{\nu_0}, \\ \overline{\nu_0} < a \le \nu_0, & \text{if } \overline{\nu_0} < \nu_0. \end{cases}$$

This shows the assertion. ∎

Denote by ℓ_1 (resp. ℓ_2) the integer ℓ in Lemma 5.19 (resp. Lemma 5.20). Put $\ell = \max\{\ell_1, \ell_2\}$. Then for this ℓ, we have $c'_{s+1} = b'_{s+1}$, $c'_s = b'_s + 1$, and $c_1 = b_1 + 1$. Since $b_1 \ge b'_{s+1} \le b'_s$, we obtain $c'_{s+1} = b'_{s+1} \le \min\{b_1, b'_s\}$ (I-1). Further, $c_1 > b_1 \ge b'_{s+1} \le b'_s < c'_s$, so that $b'_{s+1} = c'_{s+1} < \min\{c_1, c'_s\}$ (II-1). Hence the proof of Lemma 5.18 is complete.

By Lemma 5.18, we have

$$(5.9) \qquad (uw)^{-1}x(uw) \in \mathfrak{Q} = \mathrm{rad}(\mathfrak{B}),$$

and by definition

$$u^{-1}xu \in u^{-1}\overline{\mathfrak{Q}}u = \mathrm{End}^1_{\mathfrak{o}_E}(u^{-1}\overline{\mathcal{M}}) \tag{5.10}$$

in $\mathrm{End}_E(W)$ as well.

Let $i = \bar{i}$. Then $u^{-1}\overline{\mathcal{M}}$ is a self-dual $\mathfrak{o}_E$-lattice chain in $W = V^i = V^{\bar{i}}$ of $\mathfrak{o}_E$-period equal to 2 or 3. Let $h = C(x) = (1 - \frac{1}{2}x)(1 + \frac{1}{2}x)^{-1}$ in G. Then from (5.9), we have $w^{-1}u^{-1}huw \in \boldsymbol{U}^1(\mathfrak{B})$. Take an operator T in $I_w(\sigma_-|\boldsymbol{U}(\mathfrak{B}))$. Then it follows that

$$\sigma_-(u^{-1}hu) \circ T = \sigma_-^w(w^{-1}u^{-1}huw) \circ T = T \circ \sigma_-(w^{-1}u^{-1}huw) = T.$$

In $B^i = \mathrm{End}_E(V^i)$, let $\mathfrak{B}^i = \mathrm{End}_{\mathfrak{o}_E}(\{L^i_k \mid k \in \mathbb{Z}\})$ with $\mathfrak{Q}^i$ its Jacobson radical. By the choice of the element u of $\boldsymbol{W}$, it follows from (5.10) that the set of $\{u^{-1}hu \mid h = C(x), x \in \mathfrak{n} \cap \overline{\mathfrak{Q}}\}$ projects onto the unipotent radical of a proper parabolic subgroup of $\boldsymbol{U}(\mathfrak{B}^i)/\boldsymbol{U}^1(\mathfrak{B}^i)$. Thus $\sigma_-(u^{-1}hu) \circ T = T$ above contradicts the cuspidality of σ_1. Hence the element w never intertwines $\sigma_-|\boldsymbol{U}(\mathfrak{B})$.

Let $i \neq \bar{i}$. Then $u^{-1}\overline{\mathcal{M}}$ is a self-dual $\mathfrak{o}_E$-chain in $W = V^i \oplus V^{\bar{i}}$ of $\mathfrak{o}_E$-period equal to 3. For the $\mathfrak{o}_E$-lattice chain $\overline{\mathcal{L}}$ in V^i defined above, let $\overline{\mathfrak{B}}^i = \mathrm{End}^0_{\mathfrak{o}_E}(\overline{\mathcal{L}})$ and $\overline{\mathfrak{Q}}^i$ its Jacobson radical, in $B^i = \mathrm{End}_E(V^i)$. As an element $x \in \mathfrak{n} \cap \overline{\mathfrak{B}} = \mathfrak{n} \cap \overline{\mathfrak{Q}}$ above, we take $x = (x_1, x_1^\natural) \in (B^i)^\times \times (B^{\bar{i}})^\times$ and let $h = C(x)$. Then this is written in the form (y, y'), with $y = C(x_1) = 1 - x_1 \in \boldsymbol{U}^1(\overline{\mathfrak{B}}^i)$. If x_1 varies, the set of the $y = C(x_1)$ projects onto $\boldsymbol{U}^1(\overline{\mathfrak{B}}^i)/\boldsymbol{U}^1(\mathfrak{B}^i)$. The quotient $\boldsymbol{U}(\overline{\mathfrak{B}}^i)/\boldsymbol{U}^1(\mathfrak{B}^i)$ is a proper parabolic subgroup of $\boldsymbol{U}(\mathfrak{B}^i)/\boldsymbol{U}^1(\mathfrak{B}^i)$, and $\boldsymbol{U}^1(\overline{\mathfrak{B}}^i)/\boldsymbol{U}^1(\mathfrak{B}^i)$ is its unipotent radical, as in the proof of [5, 5.5.7]. Hence, similarly, we have $\sigma_-(u^{-1}hu) \circ T = T$ for $T \in I_w(\sigma_-|\boldsymbol{U}(\mathfrak{B}))$, and this contradicts the cuspidality of σ_0. This completes the proof of Lemma 5.10. ■

5.6 Types

From Proposition 5.9, we obtain an analogue of a maximal simple type for $\mathrm{GL}(N, F)$ of [5, (6.1)] as follows.

Theorem 5.21 *Let $[\mathfrak{A}, n, 0, \beta]$ be a good skew simple stratum in A, with $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$ principal, and let (J_-, λ_-) be a simple type in G attached to $[\mathfrak{A}, n, 0, \beta]$. Let $\mathfrak{B}$ be the $\mathfrak{A}$-centralizer of β. Suppose that $\mathfrak{B}$ is maximal,* i.e., *$e(\mathcal{L}_{\mathfrak{o}_E}) = 1$. Then (J_-, λ_-) is a $[G, \pi]_G$-type in G for some irreducible supercuspidal representation π of G, and π is given by* $\mathrm{Ind}(\lambda_- : J_-, G)$.

Proof From Proposition 5.1, we have $\boldsymbol{W}(\mathfrak{B}) = \{1\}$, and from Proposition 5.9, $I_G(\lambda_-) \subset J_-$. Thus $\mathrm{Ind}(\lambda_- : J_-, G)$ is an irreducible supercuspidal representation of G (see [9, (1.5)]). If an irreducible representation π of G contains λ_-, from Frobenius reciprocity (see [9, (1.6)]), π is equivalent to $\mathrm{Ind}(\lambda_- : J_-, G)$. Hence the assertion follows from [6, Section 2] (see also [21, Definition 7.3]). ■

Such a simple type (J_-, λ_-) in G as in Theorem 5.21 is called a *supercuspidal type* in G.

Suppose that $[\mathfrak{A}, n, 0, \beta]$ is a good simple stratum in A, with $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$ principal, and $\theta_- \in \mathcal{C}_-(\mathfrak{A}, 0, \beta)$. Let $E = F[\beta]$ and $e = e(\mathcal{L}_{\mathfrak{o}_E})$.

Definition 5.22 Let $P = MN_u$ be a parabolic subgroup of G associated with $[\mathfrak{A}, n, 0, \beta]$. Let (J_-, λ_-) be a simple type in G attached to $[\mathfrak{A}, n, 0, \beta]$. We write

$$J_{P,-} = (J_- \cap P)H^1_-$$

as in Subsection 4.2, and define $\lambda_{P,-}$ to be the natural representation on the subspace of $(J_- \cap N_u)$-fixed vectors in the representation space of λ_-. Moreover, we define a representation $(J_{P,-} \cap M, \lambda_{M,-})$ by $\lambda_{M,-} = \lambda_{P,-}|(J_{P,-} \cap M)$.

We note $J_{P,-} \cap M = J_- \cap M$. Put $t = [(e+1)/2]$. We have seen in Subsection 4.2 that

$$J_- \cap M = \prod_{i=1}^{t} J(\beta, \mathfrak{A}^{(i)}), \tag{5.11}$$

where if $t = (e+1)/2 \in \mathbb{Z}$, we understand $J(\beta, \mathfrak{A}^{(t)}) = J_-(\beta, \mathfrak{A}^{(t)})$ in $U(V^t, h_t)$ by Lemma 4.2. According to this decomposition, the representation $\lambda_{M,-}$ will be decomposed.

From Proposition 4.3, under the identification $H^1_-(\beta, \mathfrak{A}) = \prod_i H^1(\beta, \mathfrak{A}^{(i)})$, we have $\theta_- = \theta^{(1)} \otimes \cdots \otimes \theta^{(t)}$, where $\theta^{(i)} \in \mathcal{C}(\mathfrak{A}^{(i)}, 0, 2\beta)$, $1 \le i \le t$, (see Proposition 4.4). From Proposition 3.2, there is a unique irreducible representation η_- which contains θ_-, and from Theorem 3.11, we have an irreducible representation κ_- of J_-, which is a β-extension of η_-. From Proposition 4.5, we obtain $\eta_{P,-}$ of $J^1_{P,-} = (J^1_- \cap P)H^1_-$ such that $\eta_{P,-}|(J^1_- \cap M) \simeq \eta^{(1)} \otimes \cdots \otimes \eta^{(t)}$, where $\eta^{(i)}$ is the unique irreducible representation of $J^1(\beta, \mathfrak{A}^{(i)})$ which contains $\theta^{(i)}$, and if $t = (e+1)/2 \in \mathbb{Z}$, we understand $J^1(\beta, \mathfrak{A}^{(t)}) = J^1_-(\beta, \mathfrak{A}^{(t)})$, $\eta^{(t)} = \eta^{(t)}_-$.

Let $\kappa_{P,-}$ be the natural representation on the subspace of $(J^1_- \cap N_u)$-fixed vectors in the representation space of κ_-. Then, as in [5, (7.2)], we obtain the results for $\kappa_{P,-}$ as follows: $\kappa_{P,-}$ is irreducible and $\kappa_{P,-}|J^1_{P,-} = \eta_{P,-}$. We have

$$\kappa_{P,-}|(J_- \cap M) \simeq \kappa^{(1)} \otimes \cdots \otimes \kappa^{(t)},$$

where $\kappa^{(i)}$ is an irreducible representation of $J(\beta, \mathfrak{A}^{(i)})$ and a β-extension of $\eta^{(i)}$, and if $t = (e+1)/2 \in \mathbb{Z}$, we understand $J(\beta, \mathfrak{A}^{(t)}) = J_-(\beta, \mathfrak{A}^{(t)})$, $\kappa^{(t)} = \kappa^{(t)}_-$. Moreover, we have $\kappa_- = \mathrm{Ind}(\kappa_{P,-} : (J_- \cap P)H^1_-, J_-)$. By definition, elements of $\boldsymbol{W}(\mathfrak{B})$ normalize the Levi subgroup M of G (cf. Subsections 4.1 and 5.1). We can easily show that the analogues of [5, (7.2.10), (7.1.15)] hold for G. Thus it follows from [5, (7.2.16)] that some element of $\boldsymbol{W}(\mathfrak{B})$ may induce an equivalence $\kappa^{(i)} \simeq \kappa^{(j)}$. Hence we have $\kappa^{(i)} \simeq \kappa^{(j)}$, for $1 \le i, j \le [e/2]$. We note that the involution $^-$ on A induces an involution on $J(\beta, \mathfrak{A}^{(i)})$, for $1 \le i \le t$, by (5.11). Furthermore, if the component σ_0 of σ_- is self-dual, we have $\kappa^{(i)} \simeq (\kappa^{(i)})^*$, for $1 \le i \le t$, where $(\kappa^{(i)})^*(x) = \kappa^{(i)}(\bar{x}^{\,-1})$, for $x \in J(\beta, \mathfrak{A}^{(i)})$. This leads to $\theta^{(i)} \simeq (\theta^{(i)})^*$, for $1 \le i \le t$. In particular, if $t = (e+1)/2 \in \mathbb{Z}$, $\kappa^{(t)} = \kappa^{(t)}_-$, and automatically, $\kappa^{(t)}_- = (\kappa^{(t)}_-)^*$, and $\theta^{(t)}_- = (\theta^{(t)}_-)^*$.

Theorem 5.23 ([5, (7.2.17)]) *Let $[\mathfrak{A}, n, 0, \beta]$ be a good skew simple stratum in A, with $\mathfrak{A} = \mathfrak{A}(\mathfrak{L})$ principal, and let (J_-, λ_-) be a simple type in G attached to $[\mathfrak{A}, n, 0, \beta]$. Let $P = MN_u$ be a parabolic subgroup of G associated with $[\mathfrak{A}, n, 0, \beta]$, and $(J_{P,-}, \lambda_{P,-})$, $(J_{P,-} \cap M, \lambda_{M,-})$ the representations in Definition* 5.22.

(i) *$\lambda_{P,-}$ and $\lambda_{M,-}$ are irreducible, and $\lambda_- \simeq \mathrm{Ind}(\lambda_{P,-} : J_{P,-}, J_-)$.*

(ii) *Under the identification $J_{P,-} \cap M = \prod_i J(\beta, \mathfrak{A}^{(i)})$, for $1 \le i \le [e/2]$, there is a supercuspidal type $(J(\beta, \mathfrak{A}^{(i)}), \lambda^{(i)})$ in $\mathrm{Aut}_F(V^i)$, and if $t = (e+1)/2 \in \mathbb{Z}$, there is a supercuspidal type $(J_-(\beta, \mathfrak{A}^{(t)}), \lambda_-^{(t)})$ in $U(V^t, h_t)$ such that*

$$\lambda_{M,-} \simeq \lambda^{(1)} \otimes \cdots \otimes \lambda^{(t)},$$

where we understand that $\lambda^{(t)}$ means $\lambda_-^{(t)}$ if e is odd.

(iii) *For $1 \le i, j \le [e/2]$, $\lambda^{(i)} \simeq \lambda^{(j)}$. If the component σ_0 of σ_- is self-dual, then $\lambda^{(i)} \simeq (\lambda^{(i)})^*$, for $1 \le i \le t$.*

Proof By the above argument, we can prove the theorem in the same way as the proof of [5, (7.2.17)]. In particular, for (iii), we can similarly translate properties of κ_- directly to λ_-, if the component σ_0 of σ_- is self-dual. ∎

Corollary 5.24 *Let notation and assumptions be as in Theorem* 5.23. *Let π_i be an irreducible supercuspidal representation of $\mathrm{Aut}_F(V^i)$ which contains $\lambda^{(i)}$, for $1 \le i \le [e/2]$, and when $t = (e+1)/2 \in \mathbb{Z}$, let π_t be an irreducible supercuspidal representation of $U(V^t, h_t)$ which contains $\lambda_-^{(t)}$. We define an irreducible supercuspidal representation π of the Levi subgroup M of G by*

$$\pi = \bigotimes^{[(e+1)/2]} \pi_i$$

Then $(J_{P,-} \cap M, \lambda_{M,-})$ is an $[M, \pi]_M$-type in M.

Proof This follows directly from [5, (6.2.2)] and Theorem 5.23 (see [7, Proposition 1.3]). ∎

Remark 5.25. Let π be an irreducible supercuspidal representation of M as in Corollary 5.24. If the component σ_0 of σ_-, with $\lambda_- = \kappa_- \otimes \sigma_-$, is self-dual, the contragradient representation of π belongs to $[M, \pi]_M$, and this inertial class contains a self-contragradient representation of M. This follows from Theorem 5.23 and statements in [3, 2.2 and Introduction].

6 Hecke Algebras and Types

6.1 Hecke Algebras

In this section, we prove that $(J_{P,-}, \lambda_{P,-})$ is a type in G. To do so, we study the Hecke algebras $\mathcal{H}(G, \lambda_{P,-})$ of $(J_{P,-}, \lambda_{P,-})$.

Suppose that $[\mathfrak{A}, n, 0, \beta]$ is a good simple stratum in A, with $\mathfrak{A} = \mathfrak{A}(\mathfrak{L})$ principal, and (J_-, λ_-) a simple type in G attached to $[\mathfrak{A}, n, 0, \beta]$, with $\lambda_- = \kappa_- \otimes \sigma_-$. Let $E = F[\beta]$, $B = B_\beta$ the A-centralizer of β, and $\mathfrak{B} = \mathfrak{A} \cap B$.

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Proposition 6.1 ([5, (7.2.19)]) *Let $\lambda_{M,-}$ be the representation of $J_{P,-} \cap M$ which is the restriction of $\lambda_{P,-}$ as in Definition 5.22, and $\boldsymbol{W}(\sigma_-)$ be the subgroup of $\boldsymbol{W}(\mathfrak{B})$ defined in Subsection 3.1. Let w be an element of $\boldsymbol{W}(\mathfrak{B})$. Then $I_w(\lambda_{P,-}) = I_w(\lambda_{M,-})$, and if $w \in \boldsymbol{W}(\sigma_-)$, its dimension is equal to one.*

Proof As stated in Subsection 5.6, $\boldsymbol{W}(\mathfrak{B})$ normalizes $J_- \cap M$. Take a representative, $y \in \boldsymbol{N}(E_0) \subset B^\times \cap G$, of w (see 5.1). Clearly $I_y(\lambda_{P,-}) = I_w(\lambda_{P,-}) \subset I_y(\lambda_{M,-})$. We show the converse inclusion. For $\mathrm{GL}(N, D)$ with D a central division F-algebra, we have an Iwahori decomposition of J_P in the proof of [24, Theorem 2.19]. Similarly we obtain

$$J_{P,-} = (J_{P,-} \cap {}^{y}N_\ell)(J_{P,-} \cap M)(J_{P,-} \cap {}^{y}N_u). \tag{6.1}$$

The subgroups $\widetilde{N}_\ell$ and $\widetilde{N}_u$ of $\widetilde{G}$, defined in Subsection 4.1, are denoted by U^- and U respectively in the proof. We have

$$(\widetilde{N}_\ell \widetilde{M} \widetilde{N}_u)^\Gamma = \widetilde{N}_\ell^\Gamma \widetilde{M}^\Gamma \widetilde{N}_u^\Gamma = N_\ell M N_u. \tag{6.2}$$

In the proof of [24, Theorem 2.19], replacing J_P, κ_M and κ_P by $J_{P,-}$, $\lambda_{M,-}$ and $\lambda_{P,-}$ respectively, we imitate the proof to prove $I_y(\lambda_{M,-}) \subset I_y(\lambda_{P,-})$ by using (6.1) and (6.2). Hence the first assertion follows.

Suppose that $w \in \boldsymbol{W}(\sigma_-)$. Then, since by definition $(\sigma_-)^y \simeq \sigma_-$, it follows from Theorem 5.23(iii) that the element y stabilizes $\lambda_{M,-}$ (see the proof of [5, (7.2.19)]). Thus the space $I_y(\lambda_{M,-}) = I_w(\lambda_{M,-})$ has dimension one. ■

Let $P = MN_u$ be a parabolic subgroup of G associated with $[\mathfrak{A}, n, 0, \beta]$, and $(J_{P,-}, \lambda_{P,-})$ the representation obtained from (J_-, λ_-) in Definition 5.22. Let $\mathcal{H}(G, \lambda_-)$ be the Hecke algebra of (J_-, λ_-) (see [5, 4.1]). From Theorem 5.23(i) and [5, (4.1.3)], there is a canonical algebra isomorphism

$$\mathcal{H}(G, \lambda_-) \simeq \mathcal{H}(G, \lambda_{P,-}). \tag{6.3}$$

Proposition 6.2 *The Hecke algebra $\mathcal{H}(G, \lambda_-)$ is spanned by functions with support $J_- w J_-$, $w \in \boldsymbol{W}(\sigma_-)$, as a $\mathbb{C}$-vector space, and the isomorphism of* (6.3) *is support-preserving.*

Proof From Proposition 5.9, the Hecke algebra $\mathcal{H}(G, \lambda_-)$ is spanned by functions with support $J_- w J_-$, $w \in \boldsymbol{W}(\mathfrak{B})$, as a $\mathbb{C}$-vector space. For $w \in \boldsymbol{W}(\mathfrak{B})$, we can show that the dimension of $I_w(\lambda_-)$ is at most one, in a quite similar way to the proof of [5, (5.6.15)]. If w intertwines λ_-, the space $I_w(\lambda_-)$ has one dimension. Thus it follows from [5, (4.1.5)] that w intertwines $\lambda_{P,-}$. Since $I_w(\lambda_{P,-}) = I_w(\lambda_{M,-})$ by Proposition 6.1, it intertwines $\lambda_{M,-}$ as well. Hence, from Theorem 5.23(iii), we see that $w \in \boldsymbol{W}(\sigma_-)$ and that $\mathcal{H}(G, \lambda_-)$ is spanned by functions with support $J_- w J_-$, $w \in \boldsymbol{W}(\sigma_-)$. For $w \in \boldsymbol{W}(\sigma_-)$, again from [5, (4.1.5)] and Proposition 6.1, we see that the spaces $I_w(\lambda_-)$ and $I_w(\lambda_{P,-})$ are both of one dimensional. Thus the algebra isomorphism (6.3) is support-preserving. ■

We may identify $\mathcal{H}(G, \lambda_{P,-})$ with $\mathcal{H}(G, \lambda_-)$ via the isomorphism (6.3). Let $E = F[\beta]$, $B = B_\beta$ the A-centralizer of β, and $\mathfrak{B} = \mathfrak{A} \cap B$. Let $\boldsymbol{D}(\mathfrak{B})$ be the subgroup of $B^\times \cap G$ defined in Subsection 5.1. Let $e = e(\mathcal{L}_{\mathfrak{o}_E})$ and $e' = [e/2]$. We define $\boldsymbol{D}^-(\mathfrak{B})$ to be a submonoid of $\boldsymbol{D}(\mathfrak{B})$ which consists of elements whose eigenvalues are $\varpi_E^{n_1}, \dots, \varpi_E^{n_{e'}}, \varpi_E^{-n_{e'}}, \dots, \varpi_E^{-n_1}$ with $n_1 \geq \cdots \geq n_{e'}$ if e is even, and whose eigenvalues are those, together with 1, if e is odd.

Lemma 6.3 *Let $\lambda_{M,-}$ be the representation of $J_{P,-} \cap M$ as above. Then the Hecke algebra $\mathcal{H}(M, \lambda_{M,-})$ is isomorphic to the Laurent polynomial ring*

$$\mathbb{C}[X_1, \dots, X_{[e/2]}; X_1^{-1}, \dots, X_{[e/2]}^{-1}].$$

Proof From Theorem 5.23, $\lambda_{M,-} \simeq \lambda^{(1)} \otimes \cdots \otimes \lambda^{(t)}$, where $t = [(e+1)/2]$. If $t = (e+1)/2 \in \mathbb{Z}$, $\lambda^{(t)} = \lambda_-^{(t)}$ is a supercuspidal type in $U(V^t, h_t)$. Thus from Theorem 5.21, we have $\mathcal{H}(U(V^t, h_t), \lambda_-^{(t)}) \simeq \mathbb{C}$. However, since $\lambda^{(i)}$, $1 \leq i \leq [e/2]$, is a maximal simple type in $\mathrm{Aut}_F(V^i)$, from [5, (7.6.3)], we have

$$\mathcal{H}(\mathrm{Aut}_F(V^i), \lambda^{(i)}) \simeq \mathbb{C}[X, X^{-1}].$$

Put $e' = [e/2]$. Hence we obtain

$$\begin{aligned}\mathcal{H}(M, \lambda_{M,-}) &\simeq \mathcal{H}(\mathrm{Aut}_F(V^1), \lambda^{(1)}) \otimes \cdots \otimes \mathcal{H}(\mathrm{Aut}_F(V^{e'}), \lambda^{(e')}) \\ &\simeq \mathbb{C}[X_1, X_1^{-1}] \otimes \cdots \otimes \mathbb{C}[X_{e'}, X_{e'}^{-1}] \\ &\simeq \mathbb{C}[X_1, \dots, X_{e'}; X_1^{-1}, \dots, X_{e'}^{-1}].\end{aligned}$$ ■

Proposition 6.4 *There is an injective homomorphism*

$$j_P : \mathcal{H}(M, \lambda_{M,-}) \to \mathcal{H}(G, \lambda_{P,-})$$

such that for $z \in \boldsymbol{D}^-(\mathfrak{B})$ and $\phi \in \mathcal{H}(M, \lambda_{M,-})$ with support $(J_- \cap M)z$, the support of $j_P(\phi)$ is $J_{P,-} z J_{P,-}$, and $j_P(\phi)(z) = \phi(z)$.

Proof Identify $\mathcal{H}(G, \lambda_-) = \mathcal{H}(G, \lambda_{P,-})$ as above. Since $\boldsymbol{D}^-(\mathfrak{B}) \subset \boldsymbol{W}(\sigma_-)$, it follows from Proposition 6.1 that for each $z \in \boldsymbol{D}^-(\mathfrak{B})$, there is a function of $\mathcal{H}(G, \lambda_{P,-})$ supported on $J_{P,-} z J_{P,-}$. Hence the proposition is proved in a quite similar way to the proof of [5, (7.6.2)]. ■

6.2 Types in G

Suppose that (J_-, λ_-), with $\lambda_- = \kappa_- \otimes \sigma_-$, is a simple type in G attached to a good skew simple stratum $[\mathfrak{A}, n, 0, \beta]$, with $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$ principal. Let $P = MN_u$ be a parabolic subgroup G associated with $[\mathfrak{A}, n, 0, \beta]$, and $(J_{P,-}, \lambda_{P,-})$ the natural representation defined by (J_-, λ_-). Then, from Corollary 5.24, there is an irreducible supercuspidal representation π of M, which is of the form $\bigotimes^{e/2} \pi_0$, $\bigotimes^{(e-1)/2} \pi_0 \otimes \pi_1$, according to $e = e(\mathcal{L}_{\mathfrak{o}_E}) \equiv 0,\ 1 \pmod 2$, such that $(J_{P,-} \cap M, \lambda_{M,-})$ is an $[M, \pi]_M$-type in M. Moreover, the representation satisfies the following conditions:

(i) $(J_{P,-}, \lambda_{P,-})$ is a decomposed pair with respect to (M, P), *i.e.*,

$$J_{P,-} = (J_{P,-} \cap N_\ell)(J_- \cap M)(J_{P,-} \cap N_u),$$

and $\lambda_{P,-}$ is trivial on both $J_{P,-} \cap N_\ell$ and $J_{P,-} \cap N_u$.

(ii) $\lambda_{M,-} = \lambda_{P,-}|(J_{P,-} \cap M)$.

Lemma 6.5 *Let notation and assumptions be as above. Then there is an invertible element ξ of $\mathcal{H}(G, \lambda_{P,-})$ supported on the double coset $J_{P,-}z_P J_{P,-}$, where z_P is an element of the center, $Z(M)$, of M, and ξ is a strongly $(P, J_{P,-})$-positive element.*

Proof For an integer j, $1 \le j \le [e/2]$, we put

$$a_j = \mathrm{Diag}(\varpi_E I, \dots, \varpi_E I, I, \dots, I, \varpi_E^{-1} I, \dots, \varpi_E^{-1} I),$$

where $\varpi_E I$ (resp. $\varpi_E^{-1} I$) appears j times. Then these are elements of $\boldsymbol{D}^-(\mathfrak{B})$, and for each an integer i, $1 \le i \le [e/2]$, there is a non-zero function X_i in $\mathcal{H}(M, \lambda_{M,-})$ supported on $(J_{P,-} \cap M)a_i$, as in the proof of [5, (7.6.2)]. This element X_i is the same as that of Lemma 6.3 (see [5, p. 245]) and is invertible in $\mathcal{H}(M, \lambda_{M,-})$. Put $e_0 = e(E|F)$ and $Z_P = X_1^{e_0} X_2^{e_0} \cdots X_{[e/2]}^{e_0}$ in $\mathcal{H}(M, \lambda_{M,-})$. Then the function Z_P is supported on $(J_{P,-} \cap M)z_P$, with $z_P = a_1^{e_0} a_2^{e_0} \cdots a_{[e/2]}^{e_0}$, and it is invertible in $\mathcal{H}(M, \lambda_{M,-})$. It is easy to see $z_P \in Z(M)$. Put $\xi = j_P(Z_P) \in \mathcal{H}(G, \lambda_{P,-})$. Then it follows from Proposition 6.4 that the function ξ is supported on $J_{P,-}z_P J_{P,-}$ and is invertible. ■

Theorem 6.6 *Let $[\mathfrak{A}, n, 0, \beta]$ be a good skew simple stratum in A, with $\mathfrak{A}$ principal, and (J_-, λ_-) a simple type in G attached to $[\mathfrak{A}, n, 0, \beta]$. Let $(J_{P,-}, \lambda_{P,-})$ be the representation defined in Definition* 5.22 *from (J_-, λ_-), and π an irreducible supercuspidal representation of M as in Corollary* 5.24*. Then $(J_{P,-}, \lambda_{P,-})$ is an $[M, \pi]_G$-type in G, and so is (J_-, λ_-).*

Proof From the conditions (i), (ii) and Lemma 6.5, $(J_{P,-}, \lambda_{P,-})$ satisfy the hypotheses of [6, (7.9)]. Thus, (iii) for any smooth irreducible representation $(\mu, \mathcal{V})$ of G, the restriction to $\mathcal{V}^{\lambda_{P,-}}$ of the Jacquet functor r_u is injective. The definition of G-cover, given in [6, (8.1)], is modified so that if the conditions (i), (ii) and (iii) are satisfied for one parabolic subgroup P, then $(J_{P,-}, \lambda_{P,-})$ is a G-cover of $(J_{P,-} \cap M, \lambda_{M,-})$ (see [3, Introduction]). This modification follows from [4]. Since $(J_{P,-} \cap M, \lambda_{M,-})$ is an $[M, \pi]_M$-type in M, the theorem follows from [6, (8.3)]. Moreover, since

$$\lambda_- \simeq \mathrm{Ind}(\lambda_{P,-} : J_{P,-}, J_-)$$

by Theorem 5.23(i), it is easy to see that (J_-, λ_-) is also an $[M, \pi]_G$-type in G [25, 5.3]. ■

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