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Abstract

Colmez has given a recipe to associate a smooth modular representation $\Omega(W)$ of the Borel subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$ to a $\overline{\mathbf{F}}_p$-representation W of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ by using Fontaine's theory of (φ, Γ)-modules. We compute $\Omega(W)$ explicitly and we prove that if W is irreducible and $\dim(W) = 2$, then $\Omega(W)$ is the restriction to the Borel subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$ of the supersingular representation associated to W by Breuil's correspondence.

Contents

Introduction

This article is a contribution to the p-adic Langlands correspondence, and more specifically the 'mod p' correspondence first introduced by Breuil in [Bre03a] which is a bijection between the supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ and the irreducible two-dimensional $\overline{\mathbf{F}}_p$-linear representations of $\mathscr{G}_{\mathbf{Q}_p} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. In [Col07, Col08], Colmez has given a construction of representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ associated to certain p-adic Galois representations and by specializing and extending his functor to the case of $\overline{\mathbf{F}}_p$-representations, we obtain a recipe for constructing a smooth representation $\Omega(W)$ of the Borel subgroup $\mathrm{B} = \mathrm{B}_2(\mathbf{Q}_p)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ starting from the data of an $\overline{\mathbf{F}}_p$-representation W of $\mathscr{G}_{\mathbf{Q}_p}$. In [Ber05], the present author proved that Colmez' construction was compatible with Breuil's mod p correspondence and as a consequence that Colmez' $\varprojlim_{\psi} \mathrm{D}^\sharp(\cdot)$ functor in characteristic p does give Breuil's correspondence (up to semisimplification if W is reducible). The proof of [Ber05] is direct when W is reducible (in which case $\Omega(W)$ is a parabolic induction) but quite indirect when W is absolutely irreducible (in which case $\Omega(W)$ is supersingular) and one aim of this article is to give a direct

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proof in the latter case. A byproduct of the computations of [Ber05] is the fact that the restriction to the Borel subgroup of a supersingular representation is still irreducible. This intriguing fact has since been reproved and generalized by Paškūnas in [Pas07] (see also [Eme08]; another generalization has been worked out by Vignéras in [Vig08]).

In this article, we start by defining some smooth representations of B and we prove that they are irreducible. After that, we define the representations $\Omega(W)$ using Colmez' functor applied to W and finally, we prove that if $\dim(W) \geqslant 2$ and W is irreducible, then the $\Omega(W)$ thus constructed coincide with the representations studied in the first section and that if $\dim(W) = 2$, then they are the restriction to B of the supersingular representations studied by Barthel and Livné in [BL94, BL95] as well as Breuil in [Bre03a].

Let us now give a more precise description of our results. Let E be a finite extension of $\mathbf{F}_p$ which is the field of coefficients of all our representations, and let

$$\mathrm{K} = \begin{pmatrix} \mathbf{Z}_p^\times & \mathbf{Z}_p \\ 0 & \mathbf{Z}_p^\times \end{pmatrix} = \mathrm{B} \cap \mathrm{GL}_2(\mathbf{Z}_p)$$

and let $\mathrm{Z} \simeq \mathbf{Q}_p^\times$ be the center of B. If σ_1 and σ_2 are two smooth characters of $\mathbf{Q}_p^\times$ then $\sigma = \sigma_1 \otimes \sigma_2 : \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \to \sigma_1(a)\sigma_2(d)$ is a smooth character of KZ and we consider the compactly induced representation $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$. Note that the Iwasawa decomposition implies that $\mathrm{B}/\mathrm{KZ} = \mathrm{GL}_2(\mathbf{Q}_p)/\,\mathrm{GL}_2(\mathbf{Z}_p)\mathrm{Z}$ so that $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$ can be seen as a space of 'twisted functions' on the tree of $\mathrm{GL}_2(\mathbf{Q}_p)$.

THEOREM A. *If Π is a smooth irreducible representation of* B *admitting a central character, then there exists $\sigma = \sigma_1 \otimes \sigma_2$ such that Π is a quotient of $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$.*

This result (Theorem 1.2.3) is a direct consequence of the fact that a pro-p-group acting on a smooth E-representation necessarily admits some nontrivial fixed points. Assume now that $\sigma_1(p) = \sigma_2(p)$ and let $\lambda = \sigma_1(p) = \sigma_2(p)$ and let $\mathbf{1}_\sigma$ be the element of $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$ supported on KZ and given there by $\mathbf{1}_\sigma(kz) = \sigma(kz)$. If $n \geqslant 2$ and if $1 \leqslant h \leqslant p^{n-1} - 1$, let $S_n(h, \sigma)$ be the subspace of $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$ generated by the B-translates of

$$(-\lambda^{-1})^n \begin{pmatrix} 1 & 0 \\ 0 & p^n \end{pmatrix} \mathbf{1}_\sigma + \sum_{j=0}^{p^n-1} \binom{j}{h(p-1)} \begin{pmatrix} 1 & -jp^{-n} \\ 0 & 1 \end{pmatrix} \mathbf{1}_\sigma$$

and let $\Pi_n(h, \sigma) = \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma / S_n(h, \sigma)$. We say that h is primitive if there is no $d < n$ dividing n such that h is a multiple of $(p^n - 1)/(p^d - 1)$ (this condition is equivalent to requiring that if we write $h = e_{n-1} \cdots e_1 e_0$ in base p, then the map $i \mapsto e_i$ from $\mathbf{Z}/n\mathbf{Z}$ to $\{0, \ldots, p-1\}$ has no period strictly smaller than n). The main result of § 1.3 is that the $\Pi_n(h, \sigma)$ are irreducible if h is primitive. In § 2, we turn to Galois representations, Fontaine's (φ, Γ)-modules and Colmez' $\Omega(\cdot)$ functor. In particular, we give a careful construction of $\Omega(W)$ and in Theorem 2.2.4, we prove that there exists a character σ such that $\Omega(W)$ is a smooth irreducible quotient of $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$ by a subspace which contains $S_n(h, \sigma)$ where $n = \dim(W)$ and h depends on W. Let ω_n be Serre's fundamental character of level n. For a primitive $1 \leqslant h \leqslant p^n - 2$, let $\mathrm{ind}(\omega_n^h)$ be the unique representation of $\mathcal{G}_{\mathbf{Q}_p}$ whose determinant is ω^h (where $\omega = \omega_1$ is the mod p cyclotomic character) and whose restriction to the inertia subgroup $\mathcal{I}_{\mathbf{Q}_p}$ of $\mathcal{G}_{\mathbf{Q}_p}$ is given by $\omega_n^h \oplus \omega_n^{ph} \oplus \cdots \oplus \omega_n^{p^{n-1}h}$. Every n-dimensional absolutely irreducible E-linear representation W of $\mathcal{G}_{\mathbf{Q}_p}$ is isomorphic to $\mathrm{ind}(\omega_n^h) \otimes \chi$ for some primitive $1 \leqslant h \leqslant p^{n-1} - 1$ and some character χ and our main result is then the following (Theorem 3.1.1).

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Theorem B. *If $n \geqslant 2$ and if $1 \leqslant h \leqslant p^{n-1}-1$ is primitive, then*

$$\Omega(\mathrm{ind}(\omega_n^h) \otimes \chi) \simeq \Pi_n(h, \chi\omega^{h-1} \otimes \chi).$$

After that, we give the connection with Breuil's correspondence. Our main result connecting Colmez' functor with Breuil's correspondence is the following (it is a combination of Theorem 3.1.1 for $n=2$ and Theorem 3.2.6).

Theorem C. *If $1 \leqslant h \leqslant p-1$, then we have*

$$\Omega(\mathrm{ind}(\omega_2^h) \otimes \chi) \simeq \Pi_2(h, \omega^{h-1}\chi \otimes \chi) \simeq \frac{\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathrm{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^{h-1} E^2}{T} \otimes (\chi \circ \det),$$

where the last representation is viewed as a representation of B.

It would have been possible to treat the Galois representations of dimension one in the same way, and therefore to obtain a proof that Colmez' functor gives Breuil's correspondence for reducible representations of dimension two using the methods of this article so that one recovers the corresponding result of [Ber05] without using the stereographic projection of [BL94, BL95]. We have chosen not to include this as it does not add anything conceptually, but it is an instructive exercise for the reader.

Finally, if $h=1$ and $n \geqslant 2$, we can give a more explicit version of Theorem B. We define two B-equivariant operators T_+ and T_- on $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \sigma$ by

$$T_+(\mathbf{1}_\sigma) = \sum_{j=0}^{p-1} \begin{pmatrix} p & j \\ 0 & 1 \end{pmatrix} \mathbf{1}_\sigma \quad \text{and} \quad T_-(\mathbf{1}_\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \mathbf{1}_\sigma$$

so that the Hecke operator is $T = T_+ + T_-$ and Theorem B can be restated as follows.

Theorem D. *We have*

$$\Omega(\mathrm{ind}(\omega_n) \otimes \chi) \simeq \frac{\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(1 \otimes 1)}{T_- + (-1)^n T_+^{n-1}} \otimes (\chi \circ \det).$$

There may be a correspondence between irreducible E-linear representations of dimension n of $\mathcal{G}_{\mathbf{Q}_p}$ and certain objects coming from $\mathrm{GL}_n(\mathbf{Q}_p)$. We hope that Theorem C gives a good place to start looking for this correspondence, along with the ideas of [SV08].

List of notation

Here we give a list of the main notation of the article, in the order in which they appear.

Introduction: $\mathcal{G}_{\mathbf{Q}_p}$; B; E; K; Z; primitive h; $\mathcal{I}_{\mathbf{Q}_p}$; $T_\pm$; T;

§ 1.1: V_n; $v_{k,n}$; $V_{k,n}$; Δ; μ_a;

§ 1.2: $g_{\beta,\delta}$; σ; $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \sigma$; $[g]$; $\alpha(\beta,\delta)$; support; level; n-block; initial n-block; I_1; τ_k;

§ 1.3: $w_{\ell,n}$; λ; $S_n(h,\sigma)$; $\Pi_n(h,\sigma)$; i_k; h_k; B^+;

§ 2.1: $\widetilde{\mathbf{E}}^+$; $\widetilde{\mathbf{E}}$; ε; X; $\mathcal{H}_{\mathbf{Q}_p}$; Γ; $\mathrm{D}(W)$; ω_n; ω; μ_λ; $\mathrm{ind}(\omega_n^h)$;

§ 2.2: ψ; $\Omega(W)$; θ;

§ 3.1: $T_\pm$; T;

§ 3.2: $\mathrm{Sym}^r(E^2)$.

1. Smooth modular representations of $\mathrm{B}_2(\mathbf{Q}_p)$

In this section, we construct a number of representations of B and show that they are irreducible by reasoning directly on the tree of $\mathrm{PGL}_2(\mathbf{Q}_p)$.

1.1 Linear algebra over $\mathbf{F}_p$

The binomial coefficients are defined by the formula $(1+X)^n = \sum_{i \in \mathbf{Z}} \binom{n}{i} X^i$ and we think of them as living in $\mathbf{F}_p$. The following result is due to Lucas.

Lemma 1.1.1. *If a and b are integers and $a = a_s \cdots a_0$ and $b = b_s \cdots b_0$ are their expansions in base p, then*

$$\binom{a}{b} = \binom{a_s}{b_s} \cdots \binom{a_0}{b_0}.$$

Proof. If we write $(1+X)^a = (1+X)^{a_0}(1+X^p)^{a_1} \cdots (1+X^{p^s})^{a_s}$, then the coefficient of X^b on the left is the coefficient of $X^{b_0} X^{pb_1} \cdots X^{p^s b_s}$ on the right. □

Lemma 1.1.2. *If $k, \ell \geqslant 0$ and if*

$$a_{k,\ell} = \sum_{j=0}^{p^n-1} \binom{j}{k}\binom{j}{\ell},$$

then $a_{k,\ell} = 0$ if $k+\ell \leqslant p^n - 2$ and $a_{k,\ell} = (-1)^k$ if $k + \ell = p^n - 1$.

Proof. The number $a_{k,\ell}$ is the coefficient of $X^k Y^\ell$ in the expansion of

$$\sum_{j=0}^{p^n-1} (1+X)^j (1+Y)^j = \frac{(1+X+Y+XY)^{p^n} - 1}{(1+X+Y+XY) - 1} = (X+Y+XY)^{p^n-1}.$$

Each term of this polynomial is of the form $X^a Y^b (XY)^c$ with $a+b+c = p^n - 1$ so that there is no term of total degree $\leqslant p^n - 2$ and the terms of total degree $p^n - 1$ are those for which $c = 0$ and therefore they are the $(-1)^k X^k Y^{p^n-1-k}$. □

Let V_n be the vector space of sequences $(x_0, \ldots, x_{p^n-1})$ with $x_i \in E$. The bilinear map $\langle \cdot, \cdot \rangle : V_n \times V_n \to E$ given by $\langle x, y \rangle = \sum_{j=0}^{p^n-1} x_j y_j$ is a perfect pairing on V_n.

Let $v_{k,n} \in V_n$ be defined by

$$v_{k,n} = \left(\binom{0}{k}, \binom{1}{k}, \ldots, \binom{p^n-1}{k} \right),$$

and let $V_{k,n}$ be the subspace of V_n generated by $v_{0,n}, \ldots, v_{k-1,n}$.

Lemma 1.1.3. *For $0 \leqslant k \leqslant p^n$, the space $V_{k,n}$ is of dimension k and $V_{k,n}^\perp = V_{p^n-k,n}$.*

Proof. Since the first j components of $v_{j,n}$ are zero and the $(j+1)$th is one, the vectors $v_{j,n}$ are linearly independent and $V_{k,n}$ is of dimension k. Lemma 1.1.2 says that $\langle v_{j,n}, v_{\ell,n} \rangle = 0$ if $j + \ell \leqslant p^n - 2$ and this gives us $V_{k,n}^\perp = V_{p^n-k,n}$ by a dimension count. □

In particular, $V_{1,n}$ is the space of constant sequences and $V_{p^n-1,n}$ is the space of zero sum sequences. Note that by Lemma 1.1.1, we have $\binom{j+p^n}{k} = \binom{j}{k}$ if $0 \leqslant k \leqslant p^n - 1$ so that we can safely think of the indices of the $x \in V_n$ as belonging to $\mathbf{Z}/p^n\mathbf{Z}$. Let $\Delta : V_n \to V_n$ be the map defined by $(\Delta x)_j = x_{j-1} - x_j$.

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Lemma 1.1.4. *If $0 \leqslant k+\ell \leqslant p^n$, then Δ^k gives rise to an exact sequence*

$$0 \to V_{k,n} \to V_{\ell+k,n} \xrightarrow{\Delta^k} V_{\ell,n} \to 0,$$

and $\Delta^k(x) \in V_{\ell,n}$ if and only if $x \in V_{\ell+k,n}$.

Proof. There is nothing to prove if $k=0$ and we now assume that $k=1$. It is clear that $\ker(\Delta) = V_{1,n}$ the space of constant sequences, and the formula

$$\binom{j}{m} - \binom{j-1}{m} = \binom{j-1}{m-1}$$

implies that $\Delta(V_{\ell+1,n}) \subset V_{\ell,n}$ so that by counting dimensions we see that there is indeed an exact sequence $0 \to V_{1,n} \to V_{\ell+1,n} \xrightarrow{\Delta} V_{\ell,n} \to 0$. If $\Delta(x) \in V_{\ell,n}$, then this implies that there exists $y \in V_{\ell+1,n}$ such that $\Delta(x) = \Delta(y)$ so that $x \in V_{\ell+1,n} + \ker(\Delta) = V_{\ell+1,n}$. This proves the lemma for $k=1$, and for $k \geqslant 2$ it follows from a straightforward induction. □

Note that Δ is nilpotent of rank p^n and therefore the only subspaces of V_n stable under Δ are the $\ker(\Delta^k) = V_{k,n}$. Since the cyclic shift $(x_j) \mapsto (x_{j-1})$ is equal to $\mathrm{Id} + \Delta$, this also implies that the only subspaces of V_n stable under the cyclic shift are the $V_{k,n}$.

If $a \in \mathbf{Z}_p$, then let $\mu_a : V_n \to V_n$ be the map defined by $\mu_a(x)_j = x_{aj}$.

Lemma 1.1.5. *We have $\mu_a(v_{k,n}) - a^k v_{k,n} \in V_{k,n}$ so that if $x \in V_{k+1,n}$, then $\mu_a(x) \in V_{k+1,n}$.*

Proof. We prove both claims by induction, assuming that it is true for $\ell \leqslant k-1$ (it is immediate if $\ell = 0$ or even $\ell = 1$). Vandermonde's identity gives us

$$\binom{aj}{k} = \binom{aj-a}{k}\binom{a}{0} + \binom{aj-a}{k-1}\binom{a}{1} + \cdots + \binom{aj-a}{0}\binom{a}{k},$$

which shows that $\Delta \circ \mu_a(v_{k,n}) - a^k v_{k-1,n} \in V_{k-1,n}$ by the induction hypothesis and therefore that $\mu_a(v_{k,n}) - a^k v_{k,n} \in V_{k,n}$ by Lemma 1.1.4 which finishes the induction. □

Lemma 1.1.6. *If $x \in V_{k,n}$ and if $0 \leqslant i \leqslant p-1$, then the sequence $y \in V_{n-1}$ given by $y_j = x_{pj+i}$ belongs to $V_{\lfloor (k-1)/p \rfloor + 1, n-1}$.*

Proof. If $\ell \leqslant k-1$ and if we write $\ell = p\lfloor \ell/p \rfloor + \ell_0$ so that $0 \leqslant \ell_0 \leqslant p-1$, then by Lemma 1.1.1, we have

$$\binom{pj+i}{\ell} = \binom{j}{\lfloor \ell/p \rfloor}\binom{i}{\ell_0},$$

which implies the lemma. □

1.2 The twisted tree

We now turn to B/KZ and the smooth representations of B. If $\beta \in \mathbf{Q}_p$ and $\delta \in \mathbf{Z}$, let

$$g_{\beta,\delta} = \begin{pmatrix} 1 & \beta \\ 0 & p^\delta \end{pmatrix}.$$

Let $A = \{\alpha_n p^{-n} + \cdots + \alpha_1 p^{-1}$ where $0 \leqslant \alpha_j \leqslant p-1\}$ so that A is a system of representatives of $\mathbf{Q}_p/\mathbf{Z}_p$.

Lemma 1.2.1. *We have $\mathrm{B} = \coprod_{\beta \in A, \delta \in \mathbf{Z}} g_{\beta,\delta} \cdot \mathrm{KZ}$.*

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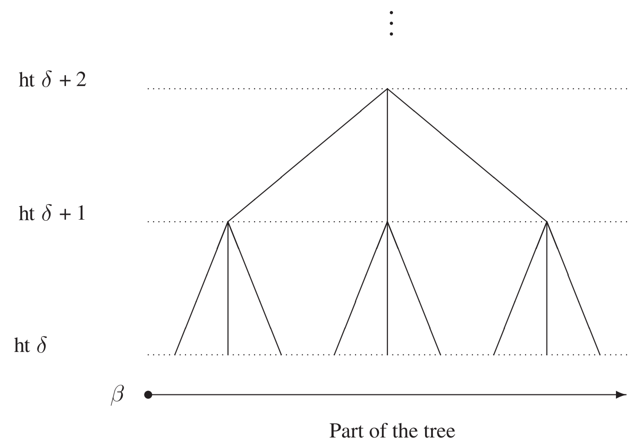

Figure 1. Part of the tree.

Proof. If $\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right) \in \mathrm{B}$, then with obvious notation we have

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a_0 p^\alpha & b \\ 0 & d_0 p^\delta \end{pmatrix} = \begin{pmatrix} 1 & bp^{-\alpha} d_0^{-1} - c \\ 0 & p^{\delta-\alpha} \end{pmatrix} \begin{pmatrix} a_0 & cd_0 \\ 0 & d_0 \end{pmatrix} \begin{pmatrix} p^\alpha & 0 \\ 0 & p^\alpha \end{pmatrix}$$

which tells us that $\mathrm{B} = \bigcup_{\beta\in A, \delta\in\mathbf{Z}} g_{\beta,\delta}\cdot \mathrm{KZ}$ since we can always choose $c \in \mathbf{Z}_p$ such that $bp^{-\alpha}d_0^{-1} - c \in A$. The fact that the union is disjoint is immediate. □

The vertices of the tree of $\mathrm{GL}_2(\mathbf{Q}_p)$ can then be labelled by the $\delta\in\mathbf{Z}$ and the $\beta \in A$ (Figure 1).

If σ_1 and σ_2 are two smooth characters $\sigma_i : \mathbf{Q}_p^\times \to E^\times$, then let $\sigma = \sigma_1 \otimes \sigma_2 : \mathrm{KZ} \to E^\times$ be the character $\sigma : \left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right) \mapsto \sigma_1(a)\sigma_2(d)$ and let $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$ be the set of functions $f : \mathrm{B} \to E$ satisfying $f(kg) = \sigma(k)f(g)$ if $k \in \mathrm{KZ}$ and such that f has compact support modulo Z. If $g \in \mathrm{B}$, denote by $[g]$ the function $[g] : \mathrm{B} \to E$ defined by $[g](h) = \sigma(hg)$ if $h \in \mathrm{KZ}g^{-1}$ and $[g](h) = 0$ otherwise. Every element of $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$ is a finite linear combination of some functions $[g]$. We make $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$ into a representation of B in the usual way: if $g \in \mathrm{B}$, then $(gf)(h) = f(hg)$. In particular, we have $g[h] = [gh]$ in addition to the formula $[gk] = \sigma(k)[g]$ for $k \in \mathrm{KZ}$.

Lemma 1.2.2. *If χ is a smooth character of $\mathbf{Q}_p^\times$, then the map $[g] \mapsto (\chi\circ\det)(g)^{-1}[g]$ extends to a* B*-equivariant isomorphism from $(\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma)\otimes(\chi\circ\det)$ to $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(\sigma_1\chi\otimes\sigma_2\chi)$.*

Proof. Let us write $[\cdot]_\sigma$ and $[\cdot]_{\sigma\chi}$ for the two functions $[\cdot]$ in the two induced representations. We then have $h[g]_\sigma = (\chi\circ\det)(h)[hg]_\sigma$ and

$$(\chi\circ\det)(g)^{-1}h[g]_{\sigma\chi} = (\chi\circ\det)(h)(\chi\circ\det)(hg)^{-1}[hg]_{\sigma\chi}$$

so that the above map is indeed B-equivariant. □

Each $f \in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$ can be written in a unique way as $f = \sum_{\beta,\delta}\alpha(\beta,\delta)[g_{\beta,\delta}]$. The formula

$$\begin{pmatrix} 1 & \beta+\lambda \\ 0 & p^\delta \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & p^\delta \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

and the fact that σ is trivial on $\left(\begin{smallmatrix} 1 & \mathbf{Z}_p \\ 0 & 1\end{smallmatrix}\right)$ imply that we can extend the definition of $\alpha(\beta,\delta)$ to all $\beta\in\mathbf{Q}_p$. We then have the formula $\alpha(\beta,\delta)(\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1\end{smallmatrix}\right)f) = \alpha(\beta-\lambda p^\delta,\delta)(f)$ if $\lambda\in\mathbf{Q}_p$.

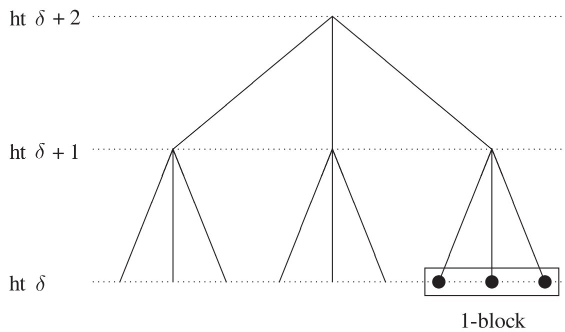

Figure 2. An example of a 1-block.

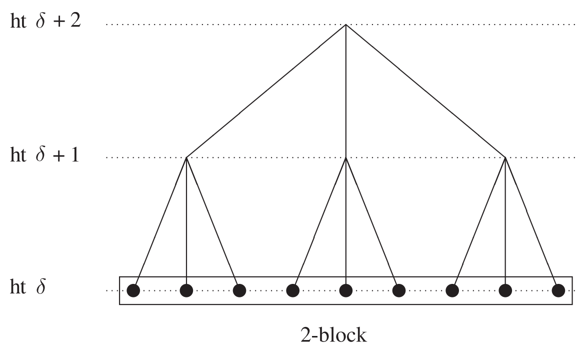

Figure 3. An example of a 2-block.

The *support* of f is the set of $g_{\beta,\delta}$ such that $\alpha(\beta,\delta)\neq 0$. Let us say that the *height* of an element $g_{\beta,\delta}$ is δ. We say that $f\in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$ has support in *levels* $n_1,\dots,n_k$ if all of the elements of its support are of height n_i for some i. If $f\in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$, then we can either raise or lower the support of f using the formula $\left(\begin{smallmatrix}1 & 0\\ 0 & p^{\pm 1}\end{smallmatrix}\right)g_{\beta,\delta}=g_{\beta,\delta\pm 1}$.

If $n\geqslant 0$ let us say that an *n-block* is the set of $g_{\beta-jp^{-n},\delta}$ for $j=0,\dots,p^n-1$ and that the *initial* n-block is the one for which $\beta=0$. We use the same name for the vector of coefficients $\alpha(\beta-jp^{-n},\delta)$ for $j=0,\dots,p^n-1$ so that an n-block is then an element of V_n from § 1.1 (Figures 2 and 3).

In the following, we study some irreducible quotients of $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$ of arithmetic interest, but before we do that, it is worthwhile pointing out that all smooth irreducible representations of B admitting a central character are a quotient of some $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$.

Theorem 1.2.3. *If Π is a smooth irreducible representation of B admitting a central character, then there exists $\sigma=\sigma_1\otimes\sigma_2$ such that Π is a quotient of $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\,\sigma$.*

Proof. The group I_1 defined by

$$\mathrm{I}_1=\begin{pmatrix}1+p\mathbf{Z}_p & \mathbf{Z}_p\\ 0 & 1+p\mathbf{Z}_p\end{pmatrix}$$

is a pro-p-group and hence $\Pi^{\mathrm{I}_1}\neq 0$. Furthermore, I_1 is a normal subgroup of K so that Π^{I_1} is a representation of $\mathrm{K}/\mathrm{I}_1=\mathbf{F}_p^\times\times\mathbf{F}_p^\times$. Since this group is a finite group of order prime to p, we have $\Pi^{\mathrm{I}_1}=\bigoplus_\eta \Pi^{\mathrm{K}=\eta}$ where η runs over the characters of $\mathbf{F}_p^\times\times\mathbf{F}_p^\times$ and since Z acts through a character

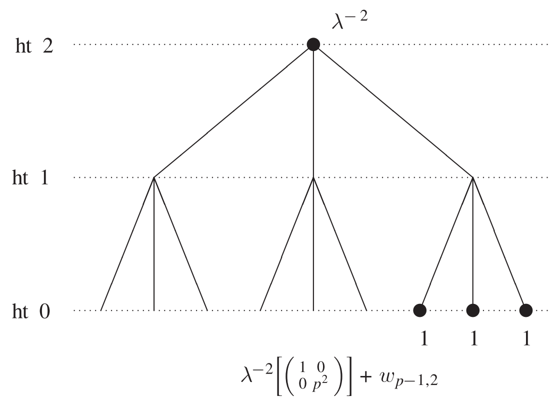

Figure 4. A function on the tree.

by hypothesis, there exists a character $\sigma = \sigma_1 \otimes \sigma_2$ of KZ and $v \in \Pi$ such that $k \cdot v = \sigma(k)v$ for $k \in \mathrm{KZ}$. By Frobenius reciprocity, we obtain a nontrivial map $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \sigma \to \Pi$ and this map is surjective since Π is irreducible. $\square$

Note that σ is not uniquely determined by Π: there are nontrivial intertwinings between some quotients of $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \sigma$ for different σ.

We finish this section with a useful general lemma. Let $\tau_k = \left(\begin{smallmatrix} 1 & -1/p^k \\ 0 & 1 \end{smallmatrix}\right)$ and let Π be any representation of B.

Lemma 1.2.4. *If $v \neq 0 \in \Pi^{\left(\begin{smallmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)}$ and if $k \geqslant 0$, then one of the p^k elements*

$$v_\ell = \sum_{j=0}^{p^k-1} \binom{j}{\ell} \tau_k^j(v), \quad 0 \leqslant \ell \leqslant p^k - 1$$

is nonzero and fixed by τ_k.

Proof. If all p^k elements above were zero, then Lemma 1.1.3 would imply that for any sequence $x = (x_j) \in V_k$ we would have $\sum_{j=0}^{p^k-1} x_j \tau_k^j(v) = 0$ and with $x = (1, 0, \dots, 0)$, we get $v = 0$. Let ℓ be the smallest integer such that $v_\ell \neq 0$. If $\ell = 0$, then $\tau_k(v_0) - v_0 = 0$ since $\tau_k^{p^k} = \tau_0 \in \left(\begin{smallmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ and otherwise $\tau_k(v_\ell) - v_\ell = -v_{\ell-1} = 0$. $\square$

1.3 Some irreducible representations of $\mathbf{B_2(Q}_p\mathbf{)}$

If $n \geqslant 1$ and $0 \leqslant \ell \leqslant p^n - 1$, let $w_{\ell,n} \in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \sigma$ be the element

$$w_{\ell,n} = \sum_{j=0}^{p^n-1} \binom{j}{\ell} \left[\begin{pmatrix} 1 & -jp^{-n} \\ 0 & 1 \end{pmatrix}\right],$$

so that the initial n-block of $w_{\ell,n}$ is $v_{\ell,n}$.

Definition 1.3.1. If $n \geqslant 2$ and if $1 \leqslant h \leqslant p^{n-1} - 1$ and if $\sigma = \sigma_1 \otimes \sigma_2$ is a character of KZ such that $\sigma_1(p) = \sigma_2(p)$, let $\lambda = \sigma_1(p) = \sigma_2(p)$ and let $S_n(h, \sigma)$ be the subspace of $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \sigma$ generated by the translates under the action of B of $(-\lambda^{-1})^n [\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^n \end{smallmatrix}\right)] + w_{h(p-1),n}$ (see Figure 4).

The representations we are interested in are the quotients $\Pi_n(h,\sigma)=\operatorname{ind}_{\mathrm{KZ}}^{\mathrm{B}}\sigma/S_n(h,\sigma)$ and the main result of this section is that they are irreducible if h is primitive. Before we can prove this, we need a number of technical results.

If $f\in\operatorname{ind}_{\mathrm{KZ}}^{\mathrm{B}}\sigma$ and if $0\leqslant i\leqslant n-1$, let

$$f_i=\sum_{\substack{\beta\in A\\ \delta\equiv i \bmod n}}\alpha(\beta,\delta)[g_{\beta,\delta}]$$

so that $f=f_0+f_1+\cdots+f_{n-1}$.

Lemma 1.3.2. *If $f\in\operatorname{ind}_{\mathrm{KZ}}^{\mathrm{B}}\sigma$, then $f\in S_n(h,\sigma)$ if and only if $f_i\in S_n(h,\sigma)$ for all $0\leqslant i\leqslant n-1$.*

Proof. We only need to check that if $f\in S_n(h,\sigma)$, then $f_i\in S_n(h,\sigma)$ and this follows from the fact that $S_n(h,\sigma)$ is generated by elements which have their supports in levels equal modulo n. □

Let $i_{n-1}\ldots i_1i_0$ be the expansion of $h(p-1)$ in base p. Note that $h\leqslant p^{n-1}-1$ implies that $i_{n-1}\leqslant p-2$. Let $h_k=i_{n-k}+pi_{n-k+1}+\cdots+p^{k-1}i_{n-1}$ so that $h_k=ph_{k-1}+i_{n-k}$ and $h_0=0$ and $h_n=h(p-1)$. Recall that the vectors $v_{k,n}$ were defined in § 1.1 and let $\mathrm{B}^+=\coprod_{\beta\in A,\delta\geqslant 0}g_{\beta,\delta}\mathrm{KZ}$.

Lemma 1.3.3. *If the support of $g\in S_n(h,\sigma)$ is in levels $\geqslant 0$, then:*

(i) *g is a linear combination of B^+-translates of $(-\lambda^{-1})^n[\left(\begin{smallmatrix}1&0\\0&p^n\end{smallmatrix}\right)]+w_{h(p-1),n}$;*

(ii) *if $1\leqslant k\leqslant n$, then the k-blocks of level 0 of g are in $V_{h_k+1,k}$.*

Proof. Note first that if $\left(\begin{smallmatrix}a&c\\0&d\end{smallmatrix}\right)\in\mathrm{KZ}$, then

$$\begin{aligned}\begin{pmatrix}a&c\\0&d\end{pmatrix}w_{\ell,n}&=\sum_{j=0}^{p^n-1}\binom{j}{\ell}\begin{pmatrix}a&c\\0&d\end{pmatrix}\left[\begin{pmatrix}1&-jp^{-n}\\0&1\end{pmatrix}\right]\\&=\sum_{j=0}^{p^n-1}\binom{j}{\ell}\left[\begin{pmatrix}1&-jp^{-n}ad^{-1}\\0&1\end{pmatrix}\begin{pmatrix}a&c\\0&d\end{pmatrix}\right]\\&=\sigma_1(a)\sigma_2(d)\sum_{j=0}^{p^n-1}\binom{jda^{-1}}{\ell}\left[\begin{pmatrix}1&-jp^{-n}\\0&1\end{pmatrix}\right],\end{aligned}$$

and note also that the initial n-block of $\left(\begin{smallmatrix}1&jp^{-n}\\0&1\end{smallmatrix}\right)w_{\ell,n}-w_{\ell,n}$ is in $V_{\ell,n}$.

Let us now prove condition (i). Set $\mathrm{B}^0=\{\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)\in\mathrm{B}$ such that $\operatorname{val}_p(a)=\operatorname{val}_p(d)\}$. It is enough to prove that any B^0-linear combination of $\varphi=(-\lambda^{-1})^n[\left(\begin{smallmatrix}1&0\\0&p^n\end{smallmatrix}\right)]+w_{h(p-1),n}$ which is zero in level zero is actually identically zero. If $\sum_{i\in I}\lambda_i\left(\begin{smallmatrix}a_i&b_i\\0&d_i\end{smallmatrix}\right)\cdot\varphi$ is such a combination where we assume for example (using the action of the center) that $d_i=1$, then the terms indexed by i_1 and i_2 contribute to the same n-block in level zero if and only if $b_{i_1}-b_{i_2}\in p^{-n}\mathbf{Z}_p$ and we can therefore assume that

$$\begin{pmatrix}a_i&b_i\\0&d_i\end{pmatrix}\in S=\begin{pmatrix}\mathbf{Z}_p^\times&p^{-n}\mathbf{Z}_p\\0&\mathbf{Z}_p^\times\end{pmatrix}$$

so that we are looking at the initial n-block. The formulas above and Lemma 1.1.5 applied to $da^{-1}\in\mathbf{Z}_p^\times$ show that if $g=\left(\begin{smallmatrix}a&c\\0&d\end{smallmatrix}\right)\in S$, then the initial n-block of $g\cdot\varphi-\sigma_1(a)\sigma_2(d)\varphi$ belongs to $V_{h(p-1),n}$ so that in a linear combination of S-translates of φ, the coefficient of $[\left(\begin{smallmatrix}1&0\\0&p^n\end{smallmatrix}\right)]$ is a nonzero multiple of the coefficient of $w_{h(p-1),n}$; if the latter is zero, then so is the former and our linear combination is identically zero.

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Let us now prove condition (ii). The conclusion of condition (ii) is stable under linear combinations of B^+-translates so by condition (i) we only need to check that if $b \in \mathrm{B}^+$, then the k-blocks of $bw_{h_n,n}$ are in $V_{h_k+1,k}$. If $b = \mathrm{Id}$, then the n-block of $w_{h_n,n}$ is $v_{h_n,n}$ which belongs to $V_{h_n+1,n}$ by definition. If we know that the k-blocks are in $V_{h_k+1,k}$, then the fact that $\lfloor h_k/p \rfloor = h_{k-1}$ and Lemma 1.1.6 imply that the $(k-1)$-blocks are in $V_{h_{k-1}+1,k-1}$, so we are done by induction. Next, the above formula for $\left(\begin{smallmatrix} a & c \\ 0 & d \end{smallmatrix}\right) w_{\ell,n}$ and Lemma 1.1.5 applied to $da^{-1} \in \mathbf{Z}_p^\times$ show that the n-blocks of the $\left(\begin{smallmatrix} a & c \\ 0 & d \end{smallmatrix}\right) w_{h_n,n}$ are contained in $V_{h_n+1,n}$ and we are reduced to the claim above. Finally, $g_{\beta,\delta} \cdot f$ is f moved up by δ and shifted by β and the conclusion of condition (ii) is unchanged under those two operations since the $V_{k,n}$ are stable under the cyclic shift. $\square$

Recall that $\tau_k = \left(\begin{smallmatrix} 1 & -1/p^k \\ 0 & 1 \end{smallmatrix}\right)$ and that $\alpha(\beta,\delta)(\tau_k(f)) = \alpha(\beta + p^{\delta-k},\delta)(f)$ so that the effect of $\tau_k - \mathrm{Id}$ on a k-block y in level zero is to replace it with $\Delta(y)$.

Lemma 1.3.4. *If the support of $f \in S_n(h,\sigma)$ is contained in a single k-block with $0 \leqslant k \leqslant n$, then this k-block is in $V_{h_k,k}$ and all such elements do occur: $w_{\ell,k} \in S_n(h,\sigma)$ for $0 \leqslant \ell \leqslant h_k - 1$.*

Proof. If $k = n$, then the n-block of $\tau_n(w_{h_n,n}) - w_{h_n,n}$ is $v_{h_n-1,n}$ and the set of possible n-blocks is stable under the cyclic shift so we obtain all of $V_{h_n,n}$ but not $V_{h_n+1,n}$ since $\Pi_n(h,\sigma) \neq 0$. If some $v_{\ell,k}$ occurs as the k-block of some f, without loss of generality in level zero, then for all $0 \leqslant m \leqslant p-1$ the $(k+1)$-block of $\sum_{i=0}^{p-1} \binom{i}{m} \tau_{k+1}^i(f)$ is $[\binom{0}{m} v_{\ell,k}, \dots, \binom{p-1}{m} v_{\ell,k}]$ and this is $v_{p\ell+m,k+1}$ since $\binom{j}{\ell}\binom{i}{m} = \binom{pj+i}{p\ell+m}$ by Lemma 1.1.1. In particular, if $v_{h_k,k}$ occurred, then so would $v_{h_{k+1},k+1}$ and we obtain a contradiction. Conversely, assuming inductively that the second assertion of the lemma holds for $k+1$, this tells us that all $[\binom{0}{m} v_{\ell,k}, \dots, \binom{p-1}{m} v_{\ell,k}]$ occur as a $(k+1)$-block for $p\ell + m \leqslant h_{k+1} - 1$ and by taking $m = p-1$ and $\ell \leqslant h_k - 1$ we obtain $v_{\ell,k}$ and we are done by a descending induction on k. $\square$

Let us write as above a $(n+1)$-block as $[b_0, \dots, b_{p-1}]$ where each b_i is a n-block.

Lemma 1.3.5. *If the support of $g \in S_n(h,\sigma)$ is in levels $0, 1, \dots, n-1$, then the $(n+1)$-blocks of level zero of g are of the form*

$$[\mu_0 v_{h_n,n} + x_0, \dots, \mu_{p-1} v_{h_n,n} + x_{p-1}],$$

where $x_i \in V_{h_n,n}$ and $(\mu_0, \dots, \mu_{p-1}) \in V_{h_1+1,1}$.

Proof. By Lemma 1.3.2, we may assume that the support of g is in level zero and Lemma 1.3.3 tells us that the n-blocks of g are in $V_{h_n+1,n}$ so that each of them can be written as $\mu_i v_{h_n,n} + x_i$ where $x_i \in V_{h_n,n}$. By subtracting from g appropriate combinations of translates of the $w_{\ell,n}$ with $0 \leqslant \ell \leqslant h_n - 1$ we obtain a g' such that $x_i = 0$ for all i and by subtracting appropriate combinations of translates of $(-\lambda^{-1})^n [\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^n \end{smallmatrix}\right)] + w_{h_n,n}$ from g' we obtain an element g'' of $S_n(h,\sigma)$ with support in level n and whose 1-blocks are the $-(-\lambda^{-1})^n(\mu_0, \dots, \mu_{p-1})$. Lemma 1.3.3 applied to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/p^n \end{smallmatrix}\right) g''$ gives us $(\mu_0, \dots, \mu_{p-1}) \in V_{h_1+1,1}$. $\square$

Corollary 1.3.6. *If the support of $f \in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \sigma$ is in levels $0, 1, \dots, n-1$ and if $\tau_{n+1}(f) - f \in S_n(h,\sigma)$, then the n-blocks of level zero of f are in $V_{h_n+1,n}$.*

Proof. Lemma 1.3.5 applied to $\tau_{n+1}(f) - f$ tells us that the $(n+1)$-blocks of $\tau_{n+1}(f) - f$ in level zero are of the form $[\mu_0 v_{h_n,n} + x_0, \dots, \mu_{p-1} v_{h_n,n} + x_{p-1}]$ with $x_i \in V_{h_n,n}$ and $(\mu_0, \dots, \mu_{p-1}) \in V_{h_1+1,1}$. If we write $f = \sum_{\beta,\delta} \alpha(\beta,\delta)[g_{\beta,\delta}]$, then the coefficient of $[g_{\beta,0}]$ in $\tau_{n+1}(f) - f$ is $\alpha(\beta + p^{-n-1}, 0) - \alpha(\beta, 0)$ so that the n-blocks of $\tau_{n+1}(f) - f$ are given by (for readability,

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we omit both β and $\delta=0$ from the notation)

$$\begin{array}{cccc}
\alpha\left(\frac{1}{p^{n+1}}\right)-\alpha(0) & \alpha\left(\frac{1}{p^{n+1}}+\frac{1}{p^n}\right)-\alpha\left(\frac{1}{p^n}\right) & \dots & \alpha\left(\frac{1}{p^{n+1}}+\frac{p^n-1}{p^n}\right)-\alpha\left(\frac{p^n-1}{p^n}\right)\\
\alpha\left(\frac{2}{p^{n+1}}\right)-\alpha\left(\frac{1}{p^{n+1}}\right) & \alpha\left(\frac{2}{p^{n+1}}+\frac{1}{p^n}\right)-\alpha\left(\frac{1}{p^{n+1}}+\frac{1}{p^n}\right) & \dots & \alpha\left(\frac{2}{p^{n+1}}+\frac{p^n-1}{p^n}\right)-\alpha\left(\frac{1}{p^{n+1}}+\frac{p^n-1}{p^n}\right)\\
\vdots & \vdots & & \vdots\\
\alpha\left(\frac{p}{p^{n+1}}\right)-\alpha\left(\frac{p-1}{p^{n+1}}\right) & \alpha\left(\frac{p}{p^{n+1}}+\frac{1}{p^n}\right)-\alpha\left(\frac{p-1}{p^{n+1}}+\frac{1}{p^n}\right) & \dots & \alpha\left(\frac{p}{p^{n+1}}+\frac{p^n-1}{p^n}\right)-\alpha\left(\frac{p-1}{p^{n+1}}+\frac{p^n-1}{p^n}\right).
\end{array}$$

Let $y_0,\dots,y_{p-1}$ be the n-blocks of the $(n+1)$-block of f we are considering. By summing the rows of the above array, we obtain (recall that $\alpha(\beta)=\alpha(1+\beta)$)

$$\alpha\left(\beta+\frac{1}{p^n}\right)-\alpha(\beta)\quad \alpha\left(\beta+\frac{2}{p^n}\right)-\alpha\left(\beta+\frac{1}{p^n}\right)\quad\dots\quad \alpha(\beta)-\alpha\left(\beta+\frac{p^n-1}{p^n}\right)$$

which is $\Delta(y_0)$ so that

$$\Delta(y_0)=\sum_{i=0}^{p-1}(\mu_i v_{h_n,n}+x_i)=\sum_{i=0}^{p-1}x_i\in V_{h_n,n}$$

since $\sum_{i=0}^{p-1}\mu_i=0$ because $(\mu_0,\dots,\mu_{p-1})\in V_{h_1+1,1}$ with $h_1+1=i_{n-1}+1\leqslant p-1$ and if $\Delta(y_0)\in V_{h_n,n}$, then $y_0\in V_{h_n+1,n}$ by Lemma 1.1.4. The same result holds for y_j by applying the previous reasoning to $\tau_{n+1}^j(f)$. □

Corollary 1.3.7. *If the support of $f\in\operatorname{ind}_{\mathrm{KZ}}^{\mathrm{B}}\sigma$ is in levels $0,1,\dots,n-1$ and the support in level zero is included in a single n-block and $\tau_n(f)-f\in S_n(h,\sigma)$, then the n-block of f in level zero is in $V_{h_n+1,n}$.*

Proof. Lemma 1.3.5 applied to $g=\tau_n(f)-f$ tells us that the n-block of $\tau_n(f)-f$ is of the form $\mu_0 v_{h_n,n}+x_0$ with $x_0\in V_{h_n,n}$ and $(\mu_0,0,\dots,0)\in V_{h_1+1,1}$ so that $\mu_0=0$ since $h_1+1\leqslant p-1$. If y denotes the n-block of f then the n-block of $\tau_n(f)-f$ is $\Delta(y)$ so that $\Delta(y)\in V_{h_n,n}$ and therefore $y\in V_{h_n+1,n}$ by Lemma 1.1.4. □

If $n\geqslant 1$ and if $1\leqslant h\leqslant p^n-2$, we say that h is primitive if there is no $d<n$ dividing n such that h is a multiple of $(p^n-1)/(p^d-1)$. This condition is equivalent to requiring that if we write $h=e_{n-1}\dots e_1e_0$ in base p, then the map $i\mapsto e_i$ from $\mathbf{Z}/n\mathbf{Z}$ to $\{0,\dots,p-1\}$ has no period strictly smaller than n.

Theorem 1.3.8. *If $n\geqslant 2$ and if $1\leqslant h\leqslant p^{n-1}-1$ is primitive, then $\Pi_n(h,\sigma)$ is irreducible.*

Proof. It is enough to show that if $f\in\operatorname{ind}_{\mathrm{KZ}}^{\mathrm{B}}\sigma$ is such that $\overline{f}\neq 0$ in $\Pi_n(h,\sigma)$, then some linear combination of translates of f is equal to $[\mathrm{Id}]$ mod $S_n(h,\sigma)$.

Suppose that the support of f is in levels $\geqslant a$. Since $(-\lambda^{-1})^n[\left(\begin{smallmatrix}1&0\\0&p^n\end{smallmatrix}\right)]+w_{h_n,n}$ is an element whose support is one element of height n and a n-block of height zero, by subtracting suitable linear combinations of translates of this from f we may assume that the support of f is in levels a, $a+1,\dots,a+n-1$; multiplying f by some power of $\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)$ we may then assume that the support of f is in levels $0,1,\dots,n-1$. In particular, we have $f\in(\operatorname{ind}_{\mathrm{KZ}}^{\mathrm{B}}\sigma)^{\left(\begin{smallmatrix}1&\mathbf{Z}_p\\0&1\end{smallmatrix}\right)}$. Let $s_0,s_1,\dots,s_{n-1}\gg 0$ be such that the support of f is included in the initial s_0-block in level zero, the initial s_1-block in level one, …, the initial s_{n-1}-block in level $n-1$.

Lemma 1.2.4 applied with $k=n+1$ shows that we may replace f by one of the $\sum_{j=0}^{p^{n+1}-1}\binom{j}{\ell}\tau_{n+1}^j(f)$ so that $\tau_{n+1}(f)-f\in S_n(h,\sigma)$. The support of this new f is included in the initial $\max(s_j, n+1-j)$-block in level j for $0 \leqslant j \leqslant n-1$. Corollary 1.3.6 then shows that there exists $g \in S_n(h,\sigma)$ which is a linear combination of $\left(\begin{smallmatrix}1 & \mathbf{Q}_p \\ 0 & 1\end{smallmatrix}\right)$-translates of $(-\lambda^{-1})^n[\left(\begin{smallmatrix}1 & 0 \\ 0 & p^n\end{smallmatrix}\right)] + w_{h_n,n}$ and of the $w_{\ell,n}$ for $0 \leqslant \ell \leqslant h_n - 1$ such that the n-blocks of f in level zero are the same as the n-blocks of g in level zero. We can then replace f by $\left(\begin{smallmatrix}1 & 0 \\ 0 & 1/p\end{smallmatrix}\right)(f-g)$ and the support of this new f is included in the initial $\max(s_{j+1}, n-j)$-block in level j for $0 \leqslant j \leqslant n-2$ and in the initial $\max(s_0 - n, 1)$-block in level $n-1$ if $j = n-1$. By iterating the procedure of this paragraph, we can reduce the width of the support of f until $s_j = n-j$ for $0 \leqslant j \leqslant n-1$.

The modified f coming from the previous paragraph satisfies $\tau_n(f) - f \in S_n(h,\sigma)$ and its support is included in the initial $(n-j)$-block in level j for $0 \leqslant j \leqslant n-1$. Corollary 1.3.7 then shows that there exists $g \in S_n(h,\sigma)$ which is a linear combination of $\left(\begin{smallmatrix}1 & \mathbf{Q}_p \\ 0 & 1\end{smallmatrix}\right)$-translates of $(-\lambda^{-1})^n[\left(\begin{smallmatrix}1 & 0 \\ 0 & p^n\end{smallmatrix}\right)] + w_{h_n,n}$ and of the $w_{\ell,n}$ for $0 \leqslant \ell \leqslant h_n - 1$ such that the n-block of g in level zero is the same as the n-block of f in level zero. We can then replace f by $\left(\begin{smallmatrix}1 & 0 \\ 0 & 1/p\end{smallmatrix}\right)(f-g)$ and the support of this new f is included in the initial $(n-j-1)$-block in level j for $0 \leqslant j \leqslant n-1$.

The modified f coming from the previous paragraph satisfies $\tau_{n-1}(f) - f \in S_n(h,\sigma)$ and its support is included in the initial $(n-j-1)$-block in level j for $0 \leqslant j \leqslant n-1$ and the k-block x_k of f in level $n-k-1$ is in $V_{h_k+1,k}$ by applying Lemmas 1.3.2, 1.3.4 and 1.1.4. By Lemma 1.3.4, we can subtract elements of $V_{h_k,k}$ from x_k without changing the class of f in $\Pi_n(h,\sigma)$ so we can assume that each x_k is a (possibly zero) multiple of $v_{h_k,k}$. If $0 \leqslant m \leqslant p-1$, let U_m be the operator defined by $U_m(f) = \sum_{i=0}^{p-1} \binom{i}{m} \tau_n^i(f)$ as in the proof of Lemma 1.3.4. At level $n-1-k$ it has the effect of turning $v_{h_k,k}$ into $v_{h_{k+1}+m-i_{n-k-1},k+1}$ since $h_{k+1} = ph_k + i_{n-k-1}$ and $\binom{j}{\ell}\binom{i}{m} = \binom{pj+i}{p\ell+m}$. If we choose m such that $m - i_{n-k-1} \leqslant 0$ and $m - i_{n-k-1} = 0$ for at least one value of k, then $U_m(f)$ is made up of $(k+1)$-blocks in level $n-k-1$ and we can get rid of all of those for which $m - i_{n-k-1} \leqslant -1$. This allows us to lower the number of nonzero blocks of f unless $m = i_{n-k-1}$ for all the corresponding nonzero blocks. In this case we lower f by one level and if there is a block in level zero we send it to level n before lowering f by subtracting an appropriate multiple of $(-\lambda^{-1})^n[\left(\begin{smallmatrix}1 & 0 \\ 0 & p^n\end{smallmatrix}\right)] + w_{h_n,n}$. By iterating this procedure (replacing f by $U_m(f)$ and lowering a possibly modified f), we can reduce the number of nonzero blocks of f until our procedure starts cycling.

If this is the case then there exists some d dividing n such that at some point f has nonzero blocks exactly in levels $n-1-\ell d$ for $0 \leqslant \ell \leqslant (n/d)-1$ and the map $r \mapsto i_r$ is then also periodic of period d. If $d = n$, then we are done. If $d < n$, then we claim that h_d is not divisible by $p-1$. Indeed, we have $h(p-1) = h_d(p^n - 1)/(p^d - 1)$ since $r \mapsto i_r$ is periodic of period d, and if $p-1$ divides h_d, then h is not primitive. If $a \in \mathbf{Z}_p^\times$ is such that $\overline{a}$ is a generator of $\mathbf{F}_p^\times$, then $\mu_a(v_{\ell,k}) - a^\ell v_{\ell,k} \in V_{\ell,k}$ by Lemma 1.1.5. This implies that $\sigma_2(a^{-1})\left(\begin{smallmatrix}1 & 0 \\ 0 & a\end{smallmatrix}\right)f - f$ has at least one fewer block (the top one) and is nonzero (the block of level $n-1-d$ is not in $S_n(h,\sigma)$), so that we can iterate again our procedure of the previous paragraph (replacing f by $U_m(f)$ and lowering a possibly modified f) until $d = n$ so that f becomes equivalent to an element supported on only one point. $\square$

Remark 1.3.9. We have $\Pi_n(h,\sigma) \otimes (\chi \circ \det) \simeq \Pi_n(h, \sigma_1\chi \otimes \sigma_2\chi)$ by Lemma 1.2.2.

2. Galois representations and (φ, Γ)-modules

In this section, we construct the (φ,Γ)-modules associated to the absolutely irreducible E-linear representations of $\mathcal{G}_{\mathbf{Q}_p}$ and then apply Colmez's functor to them in order to obtain a smooth irreducible representation of B.

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2.1 Construction of (φ,Γ)-modules

Let $\mathbf{C}_p$ be the completion of $\overline{\mathbf{Q}}_p$ and let $\widetilde{\mathbf{E}}^+ = \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbf{C}_p}$ be the ring defined by Fontaine (see for example [Fon94, § 1.2]). Recall that if $x, y \in \widetilde{\mathbf{E}}^+$, then

$$(xy)^{(i)} = x^{(i)}y^{(i)} \quad \text{and} \quad (x+y)^{(i)} = \lim_{j\to\infty}(x^{(i+j)} + y^{(i+j)})^{p^j}$$

and that $\widetilde{\mathbf{E}}^+$ is endowed with the valuation $\mathrm{val}_{\mathbf{E}}$ defined by $\mathrm{val}_{\mathbf{E}}(y) = \mathrm{val}_p(y^{(0)})$. If we choose once and for all a compatible system $\{\zeta_{p^n}\}_{n\geqslant 0}$ of p^nth roots of one, then $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \widetilde{\mathbf{E}}^+$ and we set $X = \varepsilon - 1$ and $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}^+[1/X]$ so that by [Win83, § 4.3], $\widetilde{\mathbf{E}}$ is an algebraically closed field of characteristic p, which contains $\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}}$ as a dense subfield. Given the construction of $\widetilde{\mathbf{E}}$ from $\mathbf{C}_p$, we see that it is endowed with a continuous action of $\mathcal{G}_{\mathbf{Q}_p}$. We have, for instance, $g(X) = (1+X)^{\chi_{\mathrm{cycl}}(g)} - 1$ if $g \in \mathcal{G}_{\mathbf{Q}_p}$ so that $\mathcal{H}_{\mathbf{Q}_p} = \ker \chi_{\mathrm{cycl}}$ acts trivially on $\mathbf{F}_p(\!(X)\!)$ and we obtain a map $\mathcal{H}_{\mathbf{Q}_p} \to \mathrm{Gal}(\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}}/\mathbf{F}_p(\!(X)\!))$ which is an isomorphism (this follows from the theory of the 'field of norms' of [FW79], see for example [Fon90, Theorem 3.1.6]). We also obtain an action of $\Gamma = \mathcal{G}_{\mathbf{Q}_p}/\mathcal{H}_{\mathbf{Q}_p}$ on $\mathbf{F}_p(\!(X)\!)$.

If W is an $\mathbf{F}_p$-linear representation of $\mathcal{G}_{\mathbf{Q}_p}$, then the $\mathbf{F}_p(\!(X)\!)$-vector space $\mathrm{D}(W) = (\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}} \otimes_{\mathbf{F}_p} W)^{\mathcal{H}_{\mathbf{Q}_p}}$ inherits the Frobenius φ of $\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}}$ and the residual action of Γ.

Definition 2.1.1. A (φ,Γ)-*module* over $\mathbf{F}_p(\!(X)\!)$ is a finite-dimensional $\mathbf{F}_p(\!(X)\!)$-vector space endowed with a semilinear Frobenius φ such that $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathbf{F}_p(\!(X)\!))$ and a continuous and semilinear action of Γ commuting with φ.

We see that $\mathrm{D}(W)$ is then a (φ,Γ)-module over $\mathbf{F}_p(\!(X)\!)$. If E is a finite extension of $\mathbf{F}_p$, we endow it with the trivial φ and the trivial action of Γ so that we may talk about (φ,Γ)-modules over $E(\!(X)\!) = E \otimes_{\mathbf{F}_p} \mathbf{F}_p(\!(X)\!)$ and we then have the following result which is proved in [Fon90, § 1.2] and whose proof we recall for the convenience of the reader.

Theorem 2.1.2. *The functor $W \mapsto \mathrm{D}(W)$ gives an equivalence of categories between the category of E-representations of $\mathcal{G}_{\mathbf{Q}_p}$ and the category of (φ,Γ)-modules over $E(\!(X)\!)$.*

Sketch of proof. Given the isomorphism $\mathcal{H}_{\mathbf{Q}_p} \simeq \mathrm{Gal}(\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}}/\mathbf{F}_p(\!(X)\!))$, Hilbert's theorem 90 tells us that $\mathrm{H}^1_{\mathrm{discrete}}(\mathcal{H}_{\mathbf{Q}_p}, \mathrm{GL}_d(\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}})) = \{1\}$ if $d \geqslant 1$ so that if W is an $\mathbf{F}_p$-linear representation of $\mathcal{H}_{\mathbf{Q}_p}$, then

$$\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}} \otimes_{\mathbf{F}_p} W \simeq (\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}})^{\dim(W)}$$

as representations of $\mathcal{H}_{\mathbf{Q}_p}$ so that the $\mathbf{F}_p(\!(X)\!)$-vector space $\mathrm{D}(W) = (\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}} \otimes_{\mathbf{F}_p} W)^{\mathcal{H}_{\mathbf{Q}_p}}$ is of dimension $\dim(W)$ and $W = (\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}} \otimes_{\mathbf{F}_p(\!(X)\!)} \mathrm{D}(W))^{\varphi=1}$.

If D is a (φ,Γ)-module over $\mathbf{F}_p(\!(X)\!)$, then let $W(\mathrm{D}) = (\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}} \otimes_{\mathbf{F}_p(\!(X)\!)} \mathrm{D})^{\varphi=1}$. If we choose a basis of D and if $\mathrm{Mat}(\varphi) = (p_{ij})_{1\leqslant i,j\leqslant \dim(\mathrm{D})}$ in that basis, then the algebra

$$A = \mathbf{F}_p(\!(X)\!)[X_1, \ldots, X_{\dim(\mathrm{D})}] \bigg/ \bigg(X_j^p - \sum_i p_{ij}X_i\bigg)_{1\leqslant j\leqslant \dim(\mathrm{D})}$$

is an étale $\mathbf{F}_p(\!(X)\!)$-algebra of rank $p^{\dim(\mathrm{D})}$ and $W(\mathrm{D}) = \mathrm{Hom}_{\mathbf{F}_p(\!(X)\!)\text{-algebra}}(A, \mathbf{F}_p(\!(X)\!)^{\mathrm{sep}})$ so that $W(\mathrm{D})$ is an $\mathbf{F}_p$-vector space of dimension $\dim(\mathrm{D})$.

It is then easy to check that the functors $W \mapsto \mathrm{D}(W)$ and $\mathrm{D} \mapsto W(\mathrm{D})$ are the inverse of each other. Finally, if $E \neq \mathbf{F}_p$, then one can consider an E-representation as an $\mathbf{F}_p$-representation with an E-linear structure and likewise for (φ,Γ)-modules, so that the equivalence carries over. □

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We now compute the (φ,Γ)-modules associated to certain Galois representations. If n is an integer greater than or equal to one, choose $\pi_n \in \overline{\mathbf{Q}}_p$ such that $\pi_n^{p^n-1}=-p$. The fundamental character of level n defined in [Ser72, § 1.7], $\omega_n : \mathcal{I}_{\mathbf{Q}_p} \to \overline{\mathbf{F}}_p^\times$ is given by $\omega_n(g)=\overline{g(\pi_n)/\pi_n} \in \overline{\mathbf{F}}_p^\times$ for $g \in \mathcal{I}_{\mathbf{Q}_p}$. This definition does not depend on the choice of π_n and shows that ω_n extends to a character $\mathcal{G}_{\mathbf{Q}_{p^n}} \to \mathbf{F}_{p^n}^\times$. With this definition, ω_n is actually the reduction mod p of the Lubin–Tate character associated to the uniformizer p of the field $\mathbf{Q}_{p^n}$.

In order to describe the (φ,Γ)-modules associated to irreducible mod p representations, we need to give a 'characteristic p' construction of ω_n. Let $\omega=\omega_1$ be the mod p cyclotomic character and let $Y \in \mathbf{F}_p(\!(X)\!)^{\mathrm{sep}}$ be an element such that $Y^{(p^n-1)/(p-1)}=X$. If $g \in \mathcal{G}_{\mathbf{Q}_p}$, then $f_g(X)=\omega(g)X/g(X)$ depends only on the image of g in Γ. Since $f_g(X) \in 1+X\mathbf{F}_p[\![X]\!]$, the formula $f_g^s(X)$ makes sense if $s \in \mathbf{Z}_p$.

Lemma 2.1.3. *If $g \in \mathcal{G}_{\mathbf{Q}_{p^n}}$, then $g(Y)=Y\omega_n^p(g)f_g^{-(p-1)/(p^n-1)}(X)$.*

Proof. Recall that $X \in \widetilde{\mathbf{E}}^+ = \varprojlim \mathcal{O}_{\mathbf{C}_p}$ is equal to $\varepsilon-1$ where $\varepsilon=(\zeta_{p^j})_{j\geqslant 0}$ and where $\{\zeta_{p^j}\}_{j\geqslant 0}$ is a compatible sequence. If $j \geqslant 1$, pick $\pi_{n,j} \in \mathcal{O}_{\mathbf{C}_p}$ such that

$$\pi_{n,j}^{(p^n-1)/(p-1)}=\zeta_{p^j}-1.$$

If $g \in \mathcal{G}_{\mathbf{Q}_{p^n}}$, then $g(\zeta_{p^j}-1)=[\omega(g)](\zeta_{p^j}-1)f_g^{-1}(\zeta_{p^j}-1)$ where we also write $f_g(X)$ for $[\omega(g)]X/((1+X)^{\chi_{\mathrm{cycl}}(g)}-1) \in 1+X\mathbf{Z}_p[\![X]\!]$ and so there exists $\omega_{n,j}(g) \in \mathbf{F}_{p^n}^\times$ such that

$$\frac{g(\pi_{n,j})}{\pi_{n,j}}=[\omega_{n,j}(g)]f_g^{-(p-1)/(p^n-1)}(\zeta_{p^j}-1),$$

where $[\cdot]$ is the Teichmüller lift from $\mathbf{F}_{p^n}^\times$ to $\mathbf{Q}_{p^n}^\times$. The map $g \mapsto \omega_{n,j}(g)$ is a character of $\mathcal{G}_{\mathbf{Q}_{p^n}}$ which does not depend on the choice of $\pi_{n,j}$. In addition, we have

$$\begin{cases}(\zeta_{p^{j+1}}-1)^p=(\zeta_{p^j}-1)\cdot(1+\mathrm{O}(p^{1/p})) & \text{if } j\geqslant 1,\\ (\zeta_p-1)^{p-1}=-p\cdot(1+\mathrm{O}(p^{1/p})),\end{cases}$$

so that $\omega_{n,j+1}^p=\omega_{n,j}$ if $j\geqslant 1$ and $\omega_{n,1}=\omega_n$. This also tells us that we may choose the $\pi_{n,j}$ so that $\pi_{n,j+1}^p/\pi_{n,j} \in 1+p^{1/p}\mathcal{O}_{\mathbf{C}_p}$. If we write $Y=(y^{(i)}) \in \varprojlim \mathcal{O}_{\mathbf{C}_p}$, then we have $y^{(i)}=\lim_{j\to+\infty}\pi_{n,i+j}^{p^j}$ since the $\pi_{n,j}$ are compatible in the sense that $\pi_{n,j+1}^p/\pi_{n,j} \in 1+p^{1/p}\mathcal{O}_{\mathbf{C}_p}$ so that if $g \in \mathcal{G}_{\mathbf{Q}_{p^n}}$, then

$$\frac{g(y^{(i)})}{y^{(i)}}=[\omega_{n,i}(g)]\cdot\lim_{j\to+\infty}(f_g^{-(p-1)/(p^n-1)}(\zeta_{p^{i+j}}-1))^{p^j},$$

and therefore we have $g(Y)=Y\omega_n^p(g)f_g^{-((p-1)/(p^n-1))}(X)$ in $\widetilde{\mathbf{E}}$. □

If $1\leqslant h\leqslant p^n-2$ is primitive, the characters $\omega_n^h,\omega_n^{ph},\dots,\omega_n^{p^{n-1}h}$ of $\mathcal{I}_{\mathbf{Q}_p}$ are pairwise distinct. Let μ_λ be the unramified character sending the arithmetic Frobenius to λ^{-1} (so that later when we normalize class field theory to send the geometric Frobenius to p, then $\mu_\lambda(p)=\lambda$).

Lemma 2.1.4. *Every absolutely irreducible n-dimensional E-linear representation of $\mathcal{G}_{\mathbf{Q}_p}$ is isomorphic (after possibly enlarging E) to $(\mathrm{ind}_{\mathcal{G}_{\mathbf{Q}_{p^n}}}^{\mathcal{G}_{\mathbf{Q}_p}}\omega_n^h)\otimes\mu_\lambda$ for some primitive $1\leqslant h\leqslant p^n-2$ and some $\lambda\in E^\times$.*

Proof. If W is such a representation, then by [Ser72, § 1.6], we may extend E so that $W|_{\mathcal{I}_{\mathbf{Q}_p}}$ splits as a direct sum of n tame characters and since W is irreducible, these characters are

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transitively permuted by Frobenius so that they are of level n and there exists a primitive h such that $W=\bigoplus_{i=0}^{n-1} W_i$ where $\mathcal{I}_{\mathbf{Q}_p}$ acts on W_i by $\omega_n^{p^i h}$. Since ω_n extends to $\mathcal{G}_{\mathbf{Q}_{p^n}}$ each W_i is stable under $\mathcal{G}_{\mathbf{Q}_{p^n}}$ which then acts on it by $\omega_n^{p^i h}\chi_i$ where χ_i is an unramified character of $\mathcal{G}_{\mathbf{Q}_{p^n}}$. The lemma then follows from Frobenius reciprocity. □

If $\lambda\in\overline{\mathbf{F}}_p^\times$ is such that $\lambda^n\in\mathbf{F}_p^\times$, let $W_\lambda=\{\alpha\in\overline{\mathbf{F}}_p$ such that $\alpha^{p^n}=\lambda^{-n}\alpha\}$ so that W_λ is a $\mathbf{F}_{p^n}$-vector space of dimension one and hence a $\mathbf{F}_p$-vector space of dimension n. By composing the map $\mathrm{Gal}(\mathbf{Q}_p^{\mathrm{nr}}(\pi_n)/\mathbf{Q}_p)\xrightarrow{\sim}\mathbf{F}_{p^n}^\times\rtimes\hat{\mathbf{Z}}$ with the map $\mathbf{F}_{p^n}^\times\rtimes\hat{\mathbf{Z}}\to\mathrm{End}_{\mathbf{F}_p}(W_\lambda)$ given by $(x,0)\mapsto m_x^h$ (where m_x is the multiplication by x map) and by $(1,1)\mapsto(\alpha\mapsto\alpha^p)$ we obtain an n-dimensional $\mathbf{F}_p$-linear representation of $\mathcal{G}_{\mathbf{Q}_p}$ which is isomorphic to $(\mathrm{ind}_{\mathcal{G}_{\mathbf{Q}_{p^n}}}^{\mathcal{G}_{\mathbf{Q}_p}}\omega_n^h)\otimes\mu_\lambda$ after extending scalars and whose determinant is $\omega^h\mu_{-1}^{n-1}\mu_\lambda^n$ so that if $\lambda^n=(-1)^{n-1}$, then the determinant is ω^h and we call $\mathrm{ind}(\omega_n^h)$ the representation thus constructed; it is then uniquely determined by the two conditions $\det\mathrm{ind}(\omega_n^h)=\omega^h$ and $\mathrm{ind}(\omega_n^h)|_{\mathcal{I}_{\mathbf{Q}_p}}=\bigoplus_{i=0}^{n-1}\omega_n^{p^i h}$ since $(\mathrm{ind}_{\mathcal{G}_{\mathbf{Q}_{p^n}}}^{\mathcal{G}_{\mathbf{Q}_p}}\omega_n^h)\otimes\mu_{\lambda_1}=(\mathrm{ind}_{\mathcal{G}_{\mathbf{Q}_{p^n}}}^{\mathcal{G}_{\mathbf{Q}_p}}\omega_n^h)\otimes\mu_{\lambda_2}$ if and only if we have $\lambda_1^n=\lambda_2^n$.

Corollary 2.1.5. *Every absolutely irreducible n-dimensional E-linear representation of $\mathcal{G}_{\mathbf{Q}_p}$ is isomorphic to $\mathrm{ind}(\omega_n^h)\otimes\mu_\lambda$ for some primitive $1\leqslant h\leqslant p^n-2$ and some $\lambda\in\overline{\mathbf{F}}_p^\times$ such that $\lambda^n\in E^\times$.*

Theorem 2.1.6. *The (φ,Γ)-module $\mathrm{D}(\mathrm{ind}(\omega_n^h))$ is defined over $\mathbf{F}_p(\!(X)\!)$ and admits a basis $e_0,\dots,e_{n-1}$ in which $\gamma(e_j)=f_\gamma(X)^{hp^j(p-1)/(p^n-1)}e_j$ if $\gamma\in\Gamma$ and $\varphi(e_j)=e_{j+1}$ for $0\leqslant j\leqslant n-2$ and $\varphi(e_{n-1})=(-1)^{n-1}X^{-h(p-1)}e_0$.*

Proof. Let W be the $\mathbf{F}_p$-representation of $\mathcal{G}_{\mathbf{Q}_p}$ associated to the (φ,Γ)-module described in the theorem. If $f=X^h e_0\wedge\cdots\wedge e_{n-1}$, then $\varphi(f)=f$ and $\gamma(f)=\omega(\gamma)^h f$ so that the determinant of W is indeed ω^h and therefore we only need to show that the restriction of $\mathbf{F}_{p^n}\otimes_{\mathbf{F}_p}W$ to $\mathcal{I}_{\mathbf{Q}_p}$ is $\omega_n^h\oplus\omega_n^{ph}\oplus\cdots\oplus\omega_n^{p^{n-1}h}$. To clarify things, let us write $\mathbf{F}_{p^n}^\natural$ for $\mathbf{F}_{p^n}$ when it occurs as a coefficient field, so that φ is trivial on $\mathbf{F}_{p^n}^\natural$.

If we write $\mathbf{F}_{p^n}^\natural\otimes_{\mathbf{F}_p}\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}}$ as $\prod_{k=0}^{n-1}\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}}$ via the map $x\otimes y\mapsto(\sigma^k(x)y)$ where σ is the absolute Frobenius on $\mathbf{F}_{p^n}^\natural$, then given $(x_0,\dots,x_{n-1})\in\prod_{k=0}^{n-1}\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}}$, we have

$$\begin{aligned}\varphi((x_0,\dots,x_{n-1}))&=(\varphi(x_{n-1}),\varphi(x_0),\dots,\varphi(x_{n-2}))\\ g((x_0,\dots,x_{n-1}))&=(g(x_0),\dots,g(x_{n-1})),\end{aligned}$$

if $g\in\mathcal{G}_{\mathbf{Q}_{p^n}}$ (but not if $g\in\mathcal{G}_{\mathbf{Q}_p}$). Choose some $\alpha\in\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}}$ such that $\alpha^{p^n-1}=(-1)^{n-1}$ and define

$$\begin{aligned}v_0&=(\alpha Y^h,0,\dots,0)\cdot e_0+(0,\alpha^pY^{ph},\dots,0)\cdot e_1+\cdots(0,\dots,0,\alpha^{p^{n-1}}Y^{p^{n-1}h})\cdot e_{n-1}\\ v_1&=(0,\alpha Y^h,\dots,0)\cdot e_0+(0,0,\alpha^pY^{ph},\dots,0)\cdot e_1+\cdots(\alpha^{p^{n-1}}Y^{p^{n-1}h},0,\dots,0)\cdot e_{n-1}\\ &\vdots\\ v_{n-1}&=(0,\dots,0,\alpha Y^h)\cdot e_0+(\alpha^pY^{ph},0,\dots,0)\cdot e_1+\cdots(0,\dots,0,\alpha^{p^{n-1}}Y^{p^{n-1}h},0)\cdot e_{n-1}.\end{aligned}$$

The vectors $v_0,\dots,v_{n-1}$ give a basis of $\mathbf{F}_{p^n}^\natural\otimes_{\mathbf{F}_p}(\mathbf{F}_p(\!(X)\!)^{\mathrm{sep}}\otimes_{\mathbf{F}_p(\!(X)\!)}\mathrm{D}(W))$ and the formulas for the action of φ imply that $\varphi(v_j)=v_j$ so that $v_j\in\mathbf{F}_{p^n}^\natural\otimes_{\mathbf{F}_p}W$. The formulas for the action of Γ and Lemma 2.1.3 imply that $g(v_j)=\omega_n^{hp^{1-j}}v_j$ if $g\in\mathcal{I}_{\mathbf{Q}_p}$ which finishes the proof. □

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2.2 From Galois to Borel

If $\alpha(X) \in E(\!(X)\!)$, then we can write

$$\alpha(X) = \sum_{j=0}^{p-1} (1+X)^j \alpha_j(X^p)$$

in a unique way, and we define a map $\psi : E(\!(X)\!) \to E(\!(X)\!)$ by the formula $\psi(\alpha)(X) = \alpha_0(X)$. A direct computation shows that if $0 \leqslant r \leqslant p-1$, then $\psi(X^{pm+r}) = (-1)^r X^m$. If D is a (φ, Γ)-module over $E(\!(X)\!)$ and if $y \in \mathrm{D}$, then likewise we can write $y = \sum_{j=0}^{p-1} (1+X)^j \varphi(y_j)$ and we set $\psi(y) = y_0$. The operator ψ thus defined commutes with the action of Γ and satisfies $\psi(\alpha(X)\varphi(y)) = \psi(\alpha)(X) y$ and $\psi(\alpha(X^p) y) = \alpha(X)\psi(y)$.

If $W = \mathrm{ind}(\omega_n^h) \otimes \chi$ with $\chi = \omega^s \mu_\lambda$ where from now on $\lambda \in E^\times$, then Theorem 2.1.6 above implies that the (φ, Γ)-module $\mathrm{D}(W)$ is defined on $E(\!(X)\!)$ and admits a basis $e_0, \dots, e_{n-1}$ in which $\gamma(e_j) = \omega^s(\gamma) f_\gamma(X)^{hp^j(p-1)/(p^n-1)} e_j$ if $\gamma \in \Gamma$ and $\varphi(e_j) = \lambda e_{j+1}$ for $0 \leqslant j \leqslant n-2$ and $\varphi(e_{n-1}) = (-1)^{n-1} \lambda X^{-h(p-1)} e_0$. Since $\omega_n^{(p^n-1)/(p-1)} = \omega$, we can always modify h (and χ accordingly) in order to have $1 \leqslant h \leqslant (p^n-1)/(p-1) - 1$ so that $h(p-1) \leqslant p^n - 2$. Recall that $i_{n-1} \dots i_1 i_0$ is the expansion of $h(p-1)$ in base p and that $h_k = i_{n-k} + p i_{n-k+1} + \dots + p^{k-1} i_{n-1}$ so that $h_0 = 0$ and $h_n = h(p-1)$.

LEMMA 2.2.1. *If $f_j = X^{h_j} e_j$ and $\alpha(X) \in E(\!(X)\!)$, then we have*

$$\psi(\alpha(X) f_j) = \begin{cases} \lambda^{-1} \psi(\alpha(X) X^{i_{n-j}}) f_{j-1} & \text{if } j \geqslant 1, \\ \lambda^{-1} (-1)^{n-1} \psi(\alpha(X) X^{i_0}) f_{n-1} & \text{if } j = 0. \end{cases}$$

Proof. If $j \geqslant 1$, then we can write $\alpha(X) f_j = \lambda^{-1} \alpha(X) X^{h_j} \varphi(e_{j-1})$ and since $h_j = p h_{j-1} + i_{n-j}$, we have

$$\psi(\alpha(X) f_j) = \lambda^{-1} X^{h_{j-1}} \psi(\alpha(X) X^{i_{n-j}}) e_{j-1} = \lambda^{-1} \psi(\alpha(X) X^{i_{n-j}}) f_{j-1}.$$

If $j = 0$, then $\alpha(X) f_0 = \alpha(X) e_0 = \alpha(X)(-1)^{n-1} \lambda^{-1} X^{h(p-1)} \varphi(e_{n-1})$ so that

$$\psi(\alpha(X) f_0) = \lambda^{-1} (-1)^{n-1} X^{h_{n-1}} \psi(\alpha(X) X^{i_0}) e_{n-1} = \lambda^{-1} (-1)^{n-1} \psi(\alpha(X) X^{i_0}) f_{n-1}$$

which finishes the proof. □

COROLLARY 2.2.2. *The $E[\![X]\!]$-module $\mathrm{D}^\sharp(W) = \bigoplus_{j=0}^{n-1} E[\![X]\!] \cdot f_j$ is stable under ψ and the map $\psi : \mathrm{D}^\sharp(W) \to \mathrm{D}^\sharp(W)$ is surjective.*

Proof. Lemma 2.2.1 implies that $\mathrm{D}^\sharp(W)$ is stable under ψ. Furthermore, the formula $\psi(X^{pm+r}) = (-1)^r X^m$ for $0 \leqslant r \leqslant p-1$ implies that the map $\alpha_j(X) \mapsto \psi(\alpha_j(X) X^{i_{n-j}})$ is surjective for $j \geqslant 1$, as well as the map $\alpha_0(X) \mapsto \psi(\alpha_0(X) X^{i_0})$, which implies that $\psi : \mathrm{D}^\sharp(W) \to \mathrm{D}^\sharp(W)$ is surjective. □

A quick computation shows that if $y \in \mathrm{D}^\sharp(W)$, then $\psi^n(X^{-1} y) \in \mathrm{D}^\sharp(W)$ so that our $\mathrm{D}^\sharp(W)$ coincides with the lattice defined by Colmez in [Col07, Proposition II.4.2(iv)]. We now define Colmez's functor (see [Col07, § III]):

$$\varprojlim_{\psi} \mathrm{D}^\sharp(W) = \{y = (y_0, y_1, \dots) \text{ with } y_i \in \mathrm{D}^\sharp(W) \text{ such that } \psi(y_{i+1}) = y_i \text{ for all } i \geqslant 0\},$$

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and we endow this space with an action of B (using the same normalization as in [Ber05] which differs by a twist from the normalization of [Col07])

$$\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \cdot y\right)_i = (\omega^{h-1}\chi^2)^{-1}(x) y_i;$$
$$\left(\begin{pmatrix} 1 & 0 \\ 0 & p^j \end{pmatrix} \cdot y\right)_i = y_{i-j} = \psi^j(y_i);$$
$$\left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \cdot y\right)_i = \gamma_{a^{-1}}(y_i) \quad \text{where } \gamma_{a^{-1}} \in \Gamma \text{ is such that } \chi_{\mathrm{cycl}}(\gamma_{a^{-1}}) = a^{-1} \in \mathbf{Z}_p^\times;$$
$$\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot y\right)_i = \psi^j((1+X)^{p^{i+j}z} y_{i+j}) \quad \text{for } i+j \geqslant -\mathrm{val}(z).$$

We then define $\Omega(W) = (\varprojlim_\psi \mathrm{D}^\sharp(W))^*$ so that $\Omega(W)$ is a smooth representation (see § 2.3 for a proof of this) of B whose central character is $\omega^{h-1}\chi^2$. Denote by θ_0 the linear form on $\mathrm{D}^\sharp(W)$ given by

$$\theta_0 : \alpha_0(X) f_0 + \cdots + \alpha_{n-1}(X) f_{n-1} \mapsto \alpha_0(0).$$

If $y = (y_0, y_1, \ldots)$, then we define $\theta \in \Omega(W)$ to be the linear form $\theta : y \mapsto \theta_0(y_0)$.

Lemma 2.2.3. *If* $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{KZ}$, *then* $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \cdot \theta = \omega^{h-1}(a)\chi(ad)\theta$.

Proof. We have

$$\begin{aligned}
\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \theta\right)(y) &= \theta\left(\begin{pmatrix} a^{-1} & -ba^{-1}d^{-1} \\ 0 & d^{-1} \end{pmatrix} \cdot y\right) \\
&= \theta\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & -bd^{-1} \\ 0 & ad^{-1} \end{pmatrix} \cdot y\right) \\
&= (\omega^{h-1}\chi^2)(a)\omega^s(a^{-1}d)\theta(y) \\
&= \omega^{h-1}(a)\chi(ad)\theta(y),
\end{aligned}$$

since $\mu_\lambda(a) = \mu_\lambda(d)$ because $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{KZ}$ so that $\chi(a) = \chi(d)\omega^s(ad^{-1})$. □

For $0 \leqslant k \leqslant n$, recall that $h_k = i_{n-k} + p i_{n-k+1} + \cdots + p^{k-1} i_{n-1}$ so that $h_n = h(p-1)$.

Theorem 2.2.4. *The linear form* θ *is killed by*

$$(-1)^{n-1}\lambda^n \cdot \mathrm{Id} - \sum_{j=0}^{p^n-1} \binom{j}{h(p-1)} \begin{pmatrix} p^n & -j \\ 0 & 1 \end{pmatrix}.$$

Proof. Using the definition of the action of B on $\varprojlim_\psi \mathrm{D}^\sharp(W)$, we obtain

$$\begin{aligned}
&\left((-1)^{n-1}\lambda^n \cdot \theta - \sum_{j=0}^{p^n-1} \binom{j}{h(p-1)} \begin{pmatrix} p^n & -j \\ 0 & 1 \end{pmatrix} \theta\right)(y) \\
&= (-1)^{n-1}\lambda^n \cdot \theta_0(y_0) - \lambda^{2n} \cdot \theta_0 \circ \psi^n\left(\sum_{j=0}^{p^n-1} \binom{j}{h(p-1)} (1+X)^j y_0\right),
\end{aligned}$$

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and this is equal to zero for obvious reasons if $y_0 = \alpha_i(X) f_i$ with $i \neq 0$ so that we now assume that $y_0 = \alpha_0(X) f_0$. Lemma 1.1.2 implies that

$$\sum_{j=0}^{p^n-1} \binom{j}{h(p-1)} (1+X)^j \in X^{p^n-h_n-1} + X^{p^n-h_n} E[\![X]\!],$$

and the fact that $p^\ell - h_\ell + i_{n-\ell} = p(p^{\ell-1} - h_{\ell-1})$ for $1 \leqslant \ell \leqslant n$ together with the formulas of Lemma 2.2.1 and the fact that $\psi(X^{pm+r}) = (-1)^r X^m$ then imply that

$$\psi^n\biggl(\sum_{j=0}^{p^n-1} \binom{j}{h(p-1)} (1+X)^j \alpha_0(X) f_0\biggr) \equiv (-1)^{n-1} \lambda^{-n} \alpha_0(X) f_0 \mod X\mathrm{D}^\sharp(W),$$

which proves our claim. $\square$

2.3 Profinite representations and smooth representations

In this section, we prove that $\Omega(W)$ is a smooth irreducible representation of B if $\dim(W) \geqslant 2$. In order to do so, we recall a few results concerning profinite representations and their dual. Let G be a topological group and let X be a profinite E-linear representation of G where E is as before a finite extension of $\mathbf{F}_p$. Let X^* be the dual of X, that is the set of continuous linear forms on X.

LEMMA 2.3.1. *The representation X^* is a smooth representation of G.*

Proof. If $f \in X^*$, then the map $(g, x) \mapsto f(gx - x)$ is a continuous map $G \times X \to E$ and its kernel is therefore open in $G \times X$ so that there exists an open subgroup K of G and an open subspace Y of X such that $f(ky - y) = 0$ whenever $k \in K$ and $y \in Y$. Since X is compact, Y is of finite codimension in X and we can write $X = Y \oplus \bigoplus_{i=1}^s Ex_i$. For each i there is an open subgroup K_i of G such that $f(kx_i - x_i) = 0$ if $k \in K_i$ and this implies that if $H = K \cap \bigcap_{i=1}^s K_i$, then $f(hx - x) = 0$ for any $x \in X$ so that $f \in (X^*)^H$ with H an open subgroup of G. $\square$

LEMMA 2.3.2. *If X is topologically irreducible, then X^* is irreducible.*

Proof. If $X = \varprojlim_{i \in I} X_i$ where each X_i is a finite-dimensional E-vector space, then a linear form on X is continuous if and only if it factors through some X_i and hence $X^* = \varinjlim_{i \in I} X_i^*$ so that $(X^*)^* = (\varinjlim_{i \in I} X_i^*)^* = \varprojlim_{i \in I} X_i = X$. If Λ is a G-invariant subspace of X^*, then $\ker(\Lambda) = \bigcap_{f \in \Lambda} \ker(f)$ is a G-invariant closed subspace of X which is therefore either equal to X or to $\{0\}$. If it is equal to X, then obviously $\Lambda = \{0\}$ and if it is equal to $\{0\}$, then the fact that $(X^*)^* = X$ implies that no nonzero linear form on X^* is zero on Λ so that $\Lambda = X^*$. $\square$

The representation $\varprojlim_\psi \mathrm{D}^\sharp(W)$ is a profinite representation of B since $\mathrm{D}^\sharp(W) \simeq E[\![X]\!]^{\dim(W)}$ and we have the following result (see also [Ber05, Proposition 1.2.3]).

PROPOSITION 2.3.3. *The representation $\Omega(W) = (\varprojlim_\psi \mathrm{D}^\sharp(W))^*$ is a smooth irreducible representation of* B *if* $\dim(W) \geqslant 2$.

Proof. Lemma 2.3.2 shows that it is enough to prove that $\varprojlim_\psi \mathrm{D}^\sharp(W)$ is a topologically irreducible representation of B, and [Col07, Lemma III.3.6] asserts that any closed B-invariant subspace of $\varprojlim_\psi \mathrm{D}^\sharp(W)$ is of the form $\varprojlim_\psi M$ where M is a sub-$E[\![X]\!]$-module of $\mathrm{D}^\sharp(W)$ stable under ψ and Γ and such that $\psi : M \to M$ is surjective. Since $\mathrm{D}(W)$ is irreducible, M is a lattice by [Col07, Proposition II.3.5] applied to $E(\!(X)\!) \otimes_{E[\![X]\!]} M$ and [Col07, Proposition II.4.2(iv)] implies that such an M contains $X\cdot \mathrm{D}^\sharp(W)$ and the formulas of Lemma 2.2.1 imply that $\psi(Xf_j)\in E^\times\cdot f_{j-1}$ if $i_{n-j}\neq p-1$. Since $h(p-1)\neq p^n-1$, at least one of the i_{n-j} is $\neq p-1$ so that M contains one f_j and hence all of them by repeatedly applying ψ. □

3. Breuil's correspondence for mod p representations

In this section, we show that the representations constructed in § 1 are the same as those arising from Colmez's functor applied to n-dimensional absolutely irreducible representations of $\mathcal{G}_{\mathbf{Q}_p}$. We also show that if $n=2$, then these representations are the restriction to B of the supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ predicted by Breuil.

3.1 The isomorphism in dimension n

By Corollary 2.1.5, every absolutely irreducible n-dimensional E-linear representation W of $\mathcal{G}_{\mathbf{Q}_p}$ is isomorphic (after possibly enlarging E) to $\mathrm{ind}(\omega_n^h)\otimes\chi$ with $1\leqslant h\leqslant p^n-2$ primitive and $\chi:\mathcal{G}_{\mathbf{Q}_p}\to E^\times$ a character. Furthermore, $\omega_n^{(p^n-1)/(p-1)}=\omega$ so we can change h and χ in order to have $1\leqslant h\leqslant (p^n-1)/(p-1)-1$ which implies that at least one of the n digits of h in base p is zero. The intertwining $\mathrm{ind}(\omega_n^h)\simeq\mathrm{ind}(\omega_n^{ph})$ implies that we can make a cyclic permutation of the digits of h without changing $\mathrm{ind}(\omega_n^h)$ and if we arrange for the leading digit to be zero, then $1\leqslant h\leqslant p^{n-1}-1$.

Theorem 3.1.1. *If $W=\mathrm{ind}(\omega_n^h)\otimes\chi$ with $n\geqslant 2$ and $1\leqslant h\leqslant p^{n-1}-1$ primitive, then $\Omega(W)\simeq \Pi_n(h,\sigma)$ with $\sigma=\chi\omega^{h-1}\otimes\chi$.*

Proof. By Lemma 2.2.3 and Frobenius reciprocity, $\Omega(W)$ is a quotient of $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\sigma$ with $\sigma=\chi\omega^{h-1}\otimes\chi$, the map being given by $\sum_{\beta,\delta}\alpha(\beta,\delta)[g_{\beta,\delta}]\mapsto\sum_{\beta,\delta}\alpha(\beta,\delta)g_{\beta,\delta}\cdot\theta$. This map is surjective (since it is nonzero and $\Omega(W)$ is irreducible by Proposition 2.3.3) and bearing in mind that $\left(\begin{smallmatrix}p^n&0\\0&p^n\end{smallmatrix}\right)$ acts by λ^{2n}, Theorem 2.2.4 implies that its kernel contains $(-\lambda^{-1})^n[\left(\begin{smallmatrix}1&0\\0&p^n\end{smallmatrix}\right)]+w_{h(p-1),n}$ and hence $S_n(h,\sigma)$, so that we obtain a nontrivial map $\Pi_n(h,\sigma)\to\Omega(W)$. Since $\Pi_n(h,\sigma)$ is irreducible by Theorem 1.3.8, this map is an isomorphism. □

Note that we can define two B-equivariant operators T_+ and T_- on $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}\sigma$ by

$$T_+([g])=\sum_{j=0}^{p-1}\left[g\begin{pmatrix}p&j\\0&1\end{pmatrix}\right]\quad\text{and}\quad T_-([g])=\left[g\begin{pmatrix}1&0\\0&p\end{pmatrix}\right],$$

so that the 'usual' Hecke operator is $T=T_++T_-$ (see Figures 5 and 6). It is easy to see that Theorem 3.1.1 applied with $h=1$ simply says that

$$\Omega(\mathrm{ind}(\omega_n)\otimes\chi)\simeq\frac{\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(1\otimes 1)}{T_-+(-1)^nT_+^{n-1}}\otimes(\chi\circ\det).$$

3.2 Supersingular representations restricted to $\mathrm{B}_2(\mathbf{Q}_p)$

We now explain how to relate the representations $\Pi_2(h,\sigma)$ to the supersingular representations of [BL94, BL95, Bre03a]. Recall that if $r\geqslant 0$, then $\mathrm{Sym}^r E^2$ is the space of polynomials in x and y which are homogeneous of degree r with coefficients in E, endowed with the action of $\mathrm{GL}_2(\mathbf{Z}_p)$ factoring through $\mathrm{GL}_2(\mathbf{F}_p)$ given by $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)P(x,y)=P(ax+cy,bx+dy)$ and that we extend the action of $\mathrm{GL}_2(\mathbf{Z}_p)$ to an action of $\mathrm{GL}_2(\mathbf{Z}_p)\mathrm{Z}$ by $\left(\begin{smallmatrix}p&0\\0&p\end{smallmatrix}\right)P(x,y)=P(x,y)$. We now assume that $0\leqslant r\leqslant p-1$.

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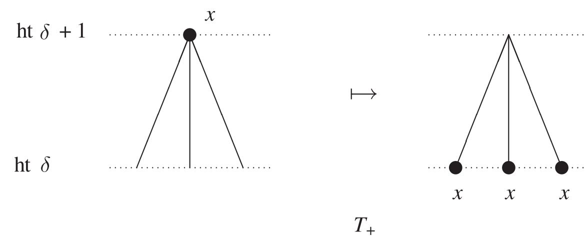

Figure 5. The operator T_+.

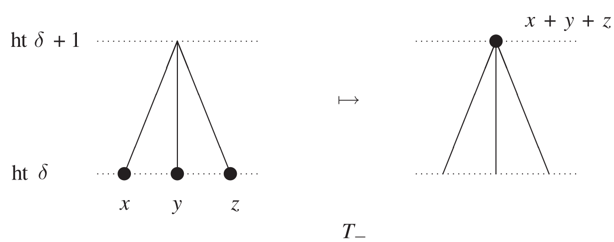

Figure 6. The operator T_-.

Lemma 3.2.1. *The 'restriction to* B*' map*

$$\mathrm{res}_{\mathrm{B}} : \mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathrm{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r E^2 \to \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \mathrm{Sym}^r E^2$$

is an isomorphism.

Proof. This follows from the Iwasawa decomposition $\mathrm{GL}_2(\mathbf{Q}_p) = \mathrm{B} \cdot \mathrm{GL}_2(\mathbf{Z}_p)$. □

Let T be the Hecke operator defined in [BL94, BL95]. Let $[g, v] \in \mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathrm{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r E^2$ be the element defined by $[g, v](h) = \mathrm{Sym}^r(hg)(v)$ if $hg \in \mathrm{GL}_2(\mathbf{Z}_p)\mathrm{Z}$ and $[g, v](h) = 0$ otherwise, so that $h[g, v] = [hg, v]$ and $[gk, v] = [g, \mathrm{Sym}^r(k)v]$ if $k \in \mathrm{GL}_2(\mathbf{Z}_p)\mathrm{Z}$.

Lemma 3.2.2. *We have*

$$T([1, x^{r-i}y^i]) = \begin{cases} \displaystyle\sum_{j=0}^{p-1} \begin{pmatrix} p & j \\ 0 & 1 \end{pmatrix} [1, (-j)^i x^r] & \text{if } i \leqslant r-1; \\ \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} [1, y^r] + \displaystyle\sum_{j=0}^{p-1} \begin{pmatrix} p & j \\ 0 & 1 \end{pmatrix} [1, (-j)^r x^r] & \text{if } i = r. \end{cases}$$

Proof. See [Bre03b, § 2.2]. □

The group KZ acts on $x^r \in \mathrm{Sym}^r E^2$ by $\omega^r \otimes 1$ so that we obtain a nontrivial injective map $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \omega^r \otimes 1 \to \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \mathrm{Sym}^r E^2$.

Proposition 3.2.3. *The map*

$$\frac{\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(\omega^r \otimes 1)}{T(\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \mathrm{Sym}^r E^2) \cap \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(\omega^r \otimes 1)} \to \frac{\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \mathrm{Sym}^r E^2}{T(\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \mathrm{Sym}^r E^2)}$$

is an isomorphism.

Proof. The map above is injective by construction, and the representation to the right is generated by the B-translates of $[1, y^r]$ since the $\left(\begin{smallmatrix}1 & \mathbf{Z}_p\\ 0 & 1\end{smallmatrix}\right)$-translates of y^r generate $\mathrm{Sym}^r E^2$. Lemma 3.2.2 applied with $i=r$ shows that $[1, y^r] \in T(\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \mathrm{Sym}^r E^2) + \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(\omega^r \otimes 1)$ so that the map is surjective. □

Lemma 3.2.4. *If $r \geqslant 1$, then $T(\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \mathrm{Sym}^r E^2) \cap \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(\omega^r \otimes 1)$ is generated by the* B*-translates of*

$$\begin{cases} T([1, x^{r-i}y^i]) & \text{for } 0 \leqslant i \leqslant r-1, \\ T\left(\displaystyle\sum_{i=0}^{p-1} \lambda_i \left[\begin{pmatrix} p & i \\ 0 & 1\end{pmatrix}, y^r\right]\right) & \text{where } (\lambda_0, \ldots, \lambda_{p-1}) \in V_{r,1}^{\perp}. \end{cases}$$

Proof. Lemma 3.2.2 above implies that $T([1, x^{r-i}y^i]) \in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(\omega^r \otimes 1)$ if $i \leqslant r-1$ and hence likewise for the B-translates of those vectors. We therefore only need to determine when a vector of the form $T(\sum_\alpha [b_\alpha, \lambda_\alpha y^r])$ belongs to $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(\omega^r \otimes 1)$. If v is a vector $v = \sum_{\beta,\delta} \sum_{i=0}^{p-1} \lambda_{\beta,\delta,i} [g_{p^{-1}\beta + p^{-1}i,\delta}, y^r]$ (note that $A = \coprod_{i=0}^{p-1} p^{-1}A + p^{-1}i$), then we have

$$T(v) = \sum_{\beta,\delta} g_{\beta,\delta+1} \cdot T\left(\lambda_{\beta,\delta,0}\left[\begin{pmatrix}1 & 0\cdot p^{-1}\\ 0 & p^{-1}\end{pmatrix}, y^r\right] + \cdots + \lambda_{\beta,\delta,p-1}\left[\begin{pmatrix}1 & (p-1)\cdot p^{-1}\\ 0 & p^{-1}\end{pmatrix}, y^r\right]\right),$$

so that by Lemma 3.2.2, the set of vectors v such that $T(v) \in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(\omega^r \otimes 1)$ is generated by the B-translates of the $v_\lambda = \sum_{i=0}^{p-1} \lambda_i [\left(\begin{smallmatrix}1 & p^{-1}i\\ 0 & p^{-1}\end{smallmatrix}\right), y^r]$ such that $T(v_\lambda) \in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(\omega^r \otimes 1)$. Lemma 3.2.2 shows that this is the case if and only if

$$\sum_{i=0}^{p-1} \lambda_i \left[\begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}, y^r\right] \in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(\omega^r \otimes 1)$$

and so if and only if $\sum_{i=0}^{p-1} \lambda_i (ix+y)^r \in E \cdot x^r$ which is equivalent to $(\lambda_0, \ldots, \lambda_{p-1}) \in V_{r,1}^{\perp}$ since the vector space generated by the sequences $(0^\ell, 1^\ell, \ldots, (p-1)^\ell)$ for $0 \leqslant \ell \leqslant r-1$ is $V_{r,1}$ (here $0^0 = 1$). Finally, we multiply the resulting v_λ by $\left(\begin{smallmatrix}p & 0\\ 0 & p\end{smallmatrix}\right)$. □

Lemma 3.2.5. *If $r = 0$, then $T(\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(1 \otimes 1))$ is generated by the* B*-translates of*

$$\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}[1,1] + \sum_{j=0}^{p-1} \begin{pmatrix}p & j\\ 0 & 1\end{pmatrix}[1,1]$$

and if $r \geqslant 1$, then $T(\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \mathrm{Sym}^r E^2) \cap \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(\omega^r \otimes 1)$ is generated by the B*-translates of*

$$\sum_{j=0}^{p-1} \lambda_j \begin{pmatrix}p & j\\ 0 & 1\end{pmatrix}[1, x^r],$$

for $(\lambda_0, \ldots, \lambda_{p-1}) \in V_{r,1}$ and of

$$\sum_{i=0}^{p-1} \mu_i i^r [1, x^r] + \sum_{i=0}^{p-1} \mu_i \begin{pmatrix}p & i\\ 0 & 1\end{pmatrix} \sum_{j=0}^{p-1} (-j)^r \begin{pmatrix}p & j\\ 0 & 1\end{pmatrix}[1, x^r],$$

where $(\mu_0, \ldots, \mu_{p-1}) \in V_{r,1}^{\perp}$.

Proof. Since $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(1 \otimes 1)$ is generated by the B-translates of $[1,1]$, the space $T(\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}}(1 \otimes 1))$ is generated by the B-translates of

$$T([1,1]) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} [1,1] + \sum_{j=0}^{p-1} \begin{pmatrix} p & j \\ 0 & 1 \end{pmatrix} [1,1]$$

which proves the first part.

If $r \geqslant 1$, then Lemma 3.2.2 tells us that

$$T([1, x^{r-i}y^i]) = \sum_{j=0}^{p-1} \begin{pmatrix} p & j \\ 0 & 1 \end{pmatrix} [1, (-j)^i x^r]$$

for $i \leqslant r-1$ and that

$$T\left(\sum_{i=0}^{p-1} \mu_i \left[\begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}, y^r\right]\right) = \sum_{i=0}^{p-1} \mu_i \left[\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, y^r\right] + \sum_{i=0}^{p-1} \mu_i \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} \sum_{j=0}^{p-1} (-j)^r \begin{pmatrix} p & j \\ 0 & 1 \end{pmatrix} [1, x^r].$$

The condition $(\mu_0, \ldots, \mu_{p-1}) \in V_{r,1}^{\perp}$ implies that $\sum_{i=0}^{p-1} \mu_i [(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}), y^r] = \sum_{i=0}^{p-1} \mu_i i^r [1, x^r]$ and we are done by Lemma 3.2.4. □

Theorem 3.2.6. *If $1 \leqslant h \leqslant p-1$, then we have an isomorphism of representations of* B

$$\Pi_2(h, \sigma) \simeq \frac{\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathrm{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^{h-1} E^2}{T(\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathrm{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^{h-1} E^2)} \otimes (\chi \circ \det).$$

Proof. First of all, we have

$$(\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathrm{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^{h-1} E^2)/T \simeq (\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \mathrm{Sym}^{h-1} E^2)/T$$

by Lemma 3.2.1, so we work with the latter space. We can twist both sides by the inverse of $\chi \circ \det$ so that $\sigma = \omega^{h-1} \otimes 1$ by Remark 1.3.9. Given Proposition 3.2.3, all we need to check is that if

$$T(h,\sigma) = T(\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \mathrm{Sym}^{h-1} E^2) \cap \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{B}} \sigma,$$

then $T(h,\sigma)$ contains $S_2(h,\sigma)$. The space generated by the vectors $(\lambda_0, \ldots, \lambda_{p-1}) \in V_{h-1,1}$ and by $(0^{h-1}, 1^{h-1}, \ldots, (p-1)^{h-1})$ is $V_{h,1}$ so that by Lemma 3.2.5, $T(h,\sigma)$ contains all of the elements

$$\sum_{i=0}^{p-1} \mu_i i^{h-1} [\mathrm{Id}] + \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \mu_i \nu_j \left[\begin{pmatrix} p^2 & pj+i \\ 0 & 1 \end{pmatrix}\right],$$

with $\mu \in V_{p-h+1,1}$ and $\nu \in (-1)^{h-1}(h-1)! v_{h-1,1} + V_{h-1,1}$. If we take $\mu_i = \binom{-i}{p-h}$ and $\nu_j = (h-1)!\binom{-j-1}{h-1}$, then the fact that

$$\binom{-i}{p-h}\binom{-j-1}{h-1} = \binom{-pj-i}{p(h-1)+p-h} = \binom{-pj-i}{h(p-1)}$$

shows that $T(h,\sigma)$ contains $S_2(h,\sigma)$. □

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References

BL94 L. Barthel and R. Livné, *Irreducible modular representations of* GL_2 *of a local field*, Duke Math. J. **75** (1994), 261–292.

BL95 L. Barthel and R. Livné, *Modular representations of* GL_2 *of a local field: the ordinary, unramified case*, J. Number Theory **55** (1995), 1–27.

Ber05 L. Berger, *Représentations modulaires de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *et représentations galoisiennes de dimension* 2, Astérisque (2005), to appear.

Bre03a C. Breuil, *Sur quelques représentations modulaires et p-adiques de* $\mathrm{GL}_2(\mathbf{Q}_p)$*. I*, Compositio Math. **138** (2003), 165–188.

Bre03b C. Breuil, *Sur quelques représentations modulaires et p-adiques de* $\mathrm{GL}_2(\mathbf{Q}_p)$*. II*, J. Inst. Math. Jussieu **2** (2003), 23–58.

Col07 P. Colmez, (φ,Γ)*-modules et représentations du mirabolique de* $\mathrm{GL}_2(\mathbf{Q}_p)$, Astérisque (2007), to appear.

Col08 P. Colmez, *Représentations de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *et* (φ,Γ)*-modules*, Astérisque (2008), to appear.

Eme08 M. Emerton, *On a class of coherent rings, with applications to the smooth representation theory of* $\mathrm{GL}_2(\mathbf{Q}_p)$ *in characteristic p*, Preprint (2008).

Fon90 J.-M. Fontaine, *Représentations p-adiques des corps locaux. I*, in *The Grothendieck Festschrift, Vol. II*, Progress in Mathematics, vol. 87 (Birkhäuser, Boston, MA, 1990), 249–309.

Fon94 J.-M. Fontaine, *Le corps des périodes p-adiques*, Astérisque (1994), 59–111, With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988).

FW79 J.-M. Fontaine and J.-P. Wintenberger, *Le corps des normes de certaines extensions algébriques de corps locaux*, C. R. Acad. Sci. Paris Sér. A–B **288** (1979), A367–A370.

Pas07 V. Paškūnas, *On the restriction of representations of* $\mathrm{GL}_2(F)$ *to a Borel subgroup*, Compositio Math. **143** (2007), 1533–1544.

SV08 P. Schneider and M.-F. Vignéras, *A functor from smooth o-torsion representations to* (φ,Γ)*-modules*, Preprint (2008).

Ser72 J.-P. Serre, *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), 259–331.

Vig08 M.-F. Vignéras, *Série principale modulo p de groupes réductifs p-adiques*, Geom. Funct. Anal. **17** (2008), 2090–2112.

Win83 J.-P. Wintenberger, *Le corps des normes de certaines extensions infinies de corps locaux; applications*, Ann. Sci. École Norm. Sup. (4) **16** (1983), 59–89.

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