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T-SPACES BY THE GOTTLIEB GROUPS AND DUALITY

MOO HA WOO and YEON SOO YOON

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Abstract

It is shown that all the generalized Whitehead products vanish in X and all the components of $X^{\Sigma A}$ have the same homotopy type when X is a T-space. It is also shown that any T-space is a G-space. The dual spaces of T-spaces are introduced and studied.

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1. Introduction

A based map $f : A \to X$ is called *cyclic* [13] if there exists a map $F : X \times A \to X$ such that $Fj \sim \nabla(1 \vee f)$, where $j : X \vee A \to X \times A$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. The *Gottlieb set*, denoted $G(A, X)$, is the set of all homotopy classes of cyclic maps from A to X. The concept of cyclic maps was first introduced and studied by Gottlieb [5] and Varadarajan [13]. Gottlieb [6] introduced and studied the evaluation subgroups $G_m(X)$ of $\pi_m(X)$. $G_m(X)$ is defined to be the set of all homotopy classes of cyclic maps from S^m to X. A space X satisfying $G_m(X) = \pi_m(X)$ for all m is called a G-space. It is known [6] that any H-space is a G-space. On the other hand, a based map $f : X \to A$ is called *cocyclic* [13] if there exists a map $\phi : X \to X \vee A$ such that $j\phi \sim (1 \times f)\Delta$, where $j : X \vee A \to X \times A$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. The *dual Gottlieb set*, denoted $DG(X, A)$, is the set of all homotopy classes of cocyclic maps from X to A. Haslam [7] introduced and studied the coevaluation subgroups $G^m(X)$ of $H^m(X)$. $G^m(X)$ is defined to be the set of all homotopy classes of cocyclic maps from X to $K(\mathbb{Z}, m)$. A space X satisfying $G^m(X) = H^m(X)$ for all m is called a G'-space. It is

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known [7] that any co-H-space is a G'-space.

The purpose of this paper is to examine a class of spaces intermediate between H-spaces and G-spaces and to dualize. In 1987, Aguade introduced and studied T-spaces in [2]. A space X is called a *T-space* if the fibration $\Omega X \to X^{S^1} \to X$ is fiber homotopy equivalent to the trivial fibration $\Omega X \to X \times \Omega X \to X$, where X^{S^1} is the free loop space of X. In section 2, we characterize T-spaces by the Gottlieb groups and use them to show that any H-space is a T-space and any T-space is a G-space. In 1981, Aguade showed [1] that, in the category of spheres, only S^1, S^3 and S^7 are T-spaces. In fact, we can easily show that H-spaces, T-spaces and G-spaces are equivalent in the category of spheres. Moreover, there are many T-spaces which are not H-spaces. We show that X is a T-space if and only if X^{S^1} is a T-space when X is a simply-connected space. In Section 3, we introduce a co-T-space as a dual space of a T-space and characterize co-T-spaces by the dual Gottlieb groups, and use it to show that any co-H-space is a co-T-space and any co-T-space is a G'-space. Moreover, we obtain that for any space A, the group $[X, \Omega A]$ is abelian when X is a co-T-space. There is a G'-space but not a co-T-space. We show that if X is a co-T-space, then all cup products vanish in X. Most results in Section 2 are dualized. Throughout this paper, space means a space of the homotopy type of a connected locally finite CW complex. We assume also that spaces have non-degenerate base points. All maps shall mean continuous functions. All homotopies and maps are to respect base points (except for the case of the mapping space X^A). The base point as well as the constant map will be denoted by *. For simplicity, we use the same symbol for a map and its homotopy class. Also, we denote by $[X, Y]$ the set of homotopy classes of pointed maps $X \to Y$. The identity map of space will be denoted by 1 when it is clear from the context. The diagonal map $\Delta : X \to X \times X$ is given by $\Delta(x) = (x, x)$ for each $x \in X$, the folding map $\nabla : X \vee X \to X$ is given by$\nabla(x, *) = \nabla(*, x) = x$ for each $x \in X$. ΣX denote the reduced suspension of X and ΩX denote the based loop space of X. The adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$ will be denoted by τ. The symbols e and e' denote $\tau^{-1}(1_{\Omega X})$ and $\tau(1_{\Sigma X})$ respectively.

2. T-spaces and cyclic maps

In this section we characterize T-spaces by the Gottlieb groups and use them to study the relationships between H-spaces, T-spaces and G-spaces. We also study some properties of T-spaces. It is known [13] that if A is a co-H-group, then $G(A, X)$ is an abelian subgroup of $[A, X]$. On the other hand, $G(A, X)$ is the evaluation subgroup given by Gottlieb [5,6], when A is a sphere. It is clear that if $f : A \to X$ is a cyclic map and $\theta : B \to A$ is an arbitrary map, then $f\theta : B \to X$ is a cyclic map. It is also easily obtained that an H-space may be characterized by the Gottlieb set as

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follows;

PROPOSITION 2.1 [10]. *The following are equivalent:*

(1) *X is an H-space;*
(2) *1_X is cyclic;*
(3) *$G(A, X) = [A, X]$ for any space A.*

Aguade showed [2] that X is a T-space if and only if X is a T_1-space. In fact, X being a T_1-space means that $e : \Sigma\Omega X \to X$ is cyclic. Thus we obtain the next theorem which shows that a T-space may be characterized by the Gottlieb group.

THEOREM 2.2. *The following are equivalent:*

(1) *X is a T-space;*
(2) *$e : \Sigma\Omega X \to X$ is cyclic;*
(3) *$G(\Sigma A, X) = [\Sigma A, X]$ for any space A.*

PROOF. The assertion (1) if and only if (2) follows from Proposition 4.1 in [2]. (2) implies (3): Let $f : \Sigma A \to X$ be a map. Then we have, from the fact that $f = e\Sigma\tau(f) : \Sigma A \to X$ and e is cyclic, that $f : \Sigma A \to X$ is cyclic. (3) implies (2): Take $A = \Omega X$. Then $e : \Sigma\Omega X \to X$ is cyclic.

Let $X^{\Sigma A}$ be the space of maps from ΣA to X with the compact-open topology. For a based map $f : \Sigma A \to X$, let $X_f^{\Sigma A}$ be the path component of $X^{\Sigma A}$ containing f. Let $(X_f^{\Sigma A})_0$ denote the space of base point preserving maps in $X_f^{\Sigma A}$. In general, the components of $X^{\Sigma A}$ almost never have the same homotopy type. However, it is well known that if X is an H-space, then $X_f^{S^p}$ and $X_g^{S^p}$ have the same homotopy type for arbitrary f and g in $\pi_p(X)$. Clearly the evaluation map $\omega : X_f^{\Sigma A} \to X$ is a fibration with fiber $(X_f^{\Sigma A})_0$. The second author showed [16] that $f : \Sigma A \to X$ is cyclic if and only if $X_f^{\Sigma A}$ is fiber homotopy equivalent to $X_*^{\Sigma A}$. From the above fact and Theorem 2.2, we can get the following corollary.

COROLLARY 2.3. *If X is a T-space, then $X_f^{\Sigma A}$ and $X_g^{\Sigma A}$ have the same homotopy type for arbitrary f and g in $[\Sigma A, X]$.*

It is known [2] that if X is a T-space, then all Whitehead products vanish in X. The following corollary says that a T-space has a more powerful property.

COROLLARY 2.4. *If X is a T-space, then all the generalized Whitehead products vanish in X.*

PROOF. Suppose X is a T-space. Let $f : \Sigma A \to X$ and $g : \Sigma B \to X$ be arbitrary maps. From Theorem 2.2, $g : \Sigma B \to X$ is cyclic. Thus there is a map $G : X \times \Sigma B \to$

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X such that $Gj = \nabla(1 \vee g)$, where $j : X \vee \Sigma B \to X \times \Sigma B$ is the inclusion. Let $H = G(f \times 1) : \Sigma A \times \Sigma B \to X$. Then $Hj' = G(f \times 1)j' = Gj(f \vee 1) = \nabla(f \vee g)$, where $j' : \Sigma A \vee \Sigma B \to \Sigma A \times \Sigma B$ is the inclusion. This proves the corollary.

A based map $f : A \to X$ is called *weakly cyclic* [15] if for any sphere S^n and any map $\theta : S^n \to A$, $f\theta : S^n \to X$ is cyclic. In fact, $f : A \to X$ is weakly cyclic if and only if $f_*(\pi_n(A)) \subset G_n(X)$ for all n. We showed [15] that any cyclic map is a weakly cyclic map, but the converse does not hold. Also, it is clear that if $f : A \to X$ is a weakly cyclic map and $\theta : B \to A$ is an arbitrary map, then $f\theta : B \to X$ is weakly cyclic.

THEOREM 2.5. *The following are equivalent:*

(1) *X is a G-space;*
(2) *$e : \Sigma\Omega X \to X$ is weakly cyclic;*
(3) *$G(S^n, X) = [S^n, X]$ for any sphere S^n.*

PROOF. (1) implies (2): Since X is a G-space, the identity map 1_X of X is weakly cyclic. Thus we have that $e = 1_X e : \Sigma\Omega X \to X$ is weakly cyclic. (2) implies (3): Let $f : S^n \to X$ be a map. We may assume that $S^n = \Sigma S^{n-1}$. It is clear that if $h : A \to B$ is a homotopy equivalence, then $h^* : G(B, X) \to G(A, X)$ is a one-to-one correspondence. Since $f = e\Sigma\tau(f) : \Sigma S^{n-1} \to X$ and $e : \Sigma\Omega X \to X$ is weakly cyclic, we have, from the definition of weakly cyclic, that $f : S^n \to X$ is cyclic. (3) implies (1): This follows from the definition of a G-space.

From the above Proposition 2.1, Theorems 2.2 and 2.5, we get the relationships between H-spaces, T-spaces and G-spaces as follows.

COROLLARY 2.6. *Any H-space is a T-space and any T-space is a G-space.*

We will obtain many examples of T-spaces which are not H-spaces from Corollaries 2.13 and 2.14. But it remains open whether or not there are any G-spaces that are not T-spaces. However, it is known [1] that, in the category of spheres, only S^1, S^3 and S^7 are T-spaces. This result can be easily obtained from the following proposition.

PROPOSITION 2.7. *Let X be a co-H-space. Then X is an H-space if and only if X is a T-space.*

PROOF. It is sufficient to show that if X is a T-space and co-H-space, then X is an H-space. Since X is a co-H-space, there is a map $s : X \to \Sigma\Omega X$ such that $es \sim 1_X$. Since $e : \Sigma\Omega X \to X$ is cyclic, 1_X is cyclic and X is an H-space. This proves the proposition.

COROLLARY 2.8. *H-spaces, T-spaces and G-spaces are equivalent in the category of spheres.*

PROOF. Suppose S^n is a G-space. Since $\pi_n(S^n) = G_n(S^n)$, 1_{S^n} is cyclic and S^n is an H-space. Thus we know, from Corollary 2.6, that H-spaces, T-spaces and G-spaces are equivalent in the category of spheres.

PROPOSITION 2.9. *Let X be a T-space. If there is a map $r : X \to Y$ such that $ri \sim 1_Y$, where $i : Y \to X$, then Y is a T-space.*

PROOF. Since $e_X : \Sigma\Omega X \to X$ is cyclic, we have that $ie_Y = e_X\Sigma\Omega i : \Sigma\Omega Y \to X$ is cyclic. It is known [13] that if $g : X \to Y$ is a map which has a right homotopy inverse and $f : A \to X$ is cyclic, then $gf : A \to Y$ is cyclic. Thus we have that $e_Y \sim r(ie_Y) : \Sigma\Omega Y \to Y$ is cyclic.

COROLLARY 2.10. *A retract of a T-space is a T-space.*

THEOREM 2.11. *$X \times Y$ is a T-space if and only if X and Y are T-spaces.*

PROOF. Suppose $X \times Y$ is a T-space. Then we have, from Corollary 2.10, that X and Y are T-spaces. On the other hand, let X and Y be T-spaces. We show that $G(\Sigma A, X \times Y) = [\Sigma A, X \times Y]$ for any space A. Let $f : \Sigma A \to X \times Y$ be a map. Since X and Y are T-spaces, $p_1 f : \Sigma A \to X$ and $p_2 f : \Sigma A \to Y$ are cyclic maps, where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are projections. It is known [10] that if $f_1 : A_1 \to X_1$ and $f_2 : A_2 \to X_2$ are cyclic, then so is $f_1 \times f_2 : A_1 \times A_2 \to X_1 \times X_2$. Thus $f = (p_1 f \times p_2 f)\Delta : \Sigma A \to X \times Y$ is cyclic. This proves the theorem.

THEOREM 2.12. *Let X be a simply connected space. Then X is a T-space if and only if X^{S^1} is a T-space.*

PROOF. Suppose X is a T-space. Since X is simply connected, ΩX is connected. Thus we know, from the definition of a T-space and Theorem 2.11, that X^{S^1} is a connected space and T-space. On the other hand, suppose X^{S^1} is a T-space. Consider the inclusion $i : X \to X^{S^1}$, $i(x) = \hat{x}$, where $\hat{x}(t) = x$ for all $t \in S^1$. Then the evaluation map $p : X^{S^1} \to X$ is a retraction of X^{S^1} to X. Thus we have, from Corollary 2.10, that X is a T-space.

Consider the two stage Postnikov system

$$K(\mathbb{Z}_2, 2n-1) \to \mathbb{E}_n \to K(\mathbb{Z}, 2)$$

with k-invariant $\alpha^n \in H^{2n}(K(\mathbb{Z}, 2); \mathbb{Z}_2), n \geq 2$, where α is the generator of $H^2(K(\mathbb{Z}, 2); \mathbb{Z}_2)$. Then the following facts were known by Haslam (See [7, Theorem 1.2.10]).

(1) $\mathbb{E}_n$ is an H-space if and only if $n = 2^k$;

(2) $\mathbb{E}_n$ is a G-space if and only if n is even.

COROLLARY 2.13. *$\mathbb{E}_n^{S^1}$ is a T-space if and only if n is even.*

PROOF. Since $K(\mathbb{Z}_2, 2n-1) \to \mathbb{E}_n \to K(\mathbb{Z}, 2)$ is the principal fibration induced by $\alpha^n : K(\mathbb{Z}, 2) \to K(\mathbb{Z}_2, 2n)$ for $n \geq 2$, $\mathbb{E}_n$ is a simply-connected space. From Theorem 2.12 and Corollary 2.6, and the above fact (2), it is sufficient to show that if n is even, $\mathbb{E}_n$ is a T-space. Let $m^* : H^*(K(\mathbb{Z}, 2); \mathbb{Z}_2) \to H^*(K(\mathbb{Z}, 2); \mathbb{Z}_2) \otimes H^*(K(\mathbb{Z}, 2); \mathbb{Z}_2)$ be the homomorphism induced by the multiplication $m : K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$. On the other hand, $K(\mathbb{Z}, 2)$ has a unique T-structure $r : K(\mathbb{Z}, 2)^{S^1} \to \Omega K(\mathbb{Z}, 2)$ which comes from the H-structure m of $K(\mathbb{Z}, 2)$ (see [2]). Let $\omega : H^i(K(\mathbb{Z}, 2); \mathbb{Z}_2) \to H^{i-1}(\Omega K(\mathbb{Z}, 2); \mathbb{Z}_2)$ be the suspension and $h : K(\mathbb{Z}, 2) \times \Omega K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)^{S^1}$ be a map given by $h(x, \eta)(t) = m(x, \eta(t))$, where $\eta \in \Omega K(\mathbb{Z}, 2)$. Then $(rh)^*(y) = 1 \otimes y$ for all $y \in H^*(\Omega K(\mathbb{Z}, 2); \mathbb{Z}_2)$ and $m^*(\alpha^n) = \sum_{i=0}^{n} nCi(\alpha^{n-i} \otimes \alpha^i)$, where the binomial coefficient nCi is taken mod 2. Since $nC1 \equiv 0 \bmod 2$, $m^*(\alpha^n) = \alpha^n \otimes 1 + 1 \otimes \alpha^n + \sum_{i=2}^{n-2} nCi(\alpha^{n-i} \otimes \alpha^i)$. Since $\omega(\alpha^i) \in H^{2i-1}(\Omega K(\mathbb{Z}, 2); \mathbb{Z}_2) = 0$ for all $i \geq 2$, we know, from Proposition 2.2 [2], that $\alpha^n : K(\mathbb{Z}, 2) \to K(\mathbb{Z}_2, 2n)$ is a T-map. Thus we know, from Proposition 2.1 [2], that $\mathbb{E}_n$ is a T-space.

COROLLARY 2.14. *$\mathbb{E}_n^{S^1}$ is an H-space if and only if $n = 2^k$.*

PROOF. Let $\mathbb{E}_n^{S^1}$ be an H-space. Then the evaluation map $p : \mathbb{E}_n^{S^1} \to \mathbb{E}_n$ is a retraction of $\mathbb{E}_n^{S^1}$ to $\mathbb{E}_n$. It is well known that a retract of an H-space is an H-space. Thus we obtain, from the Haslam's result above, that $n = 2^k$. Suppose $n = 2^k$. Since $\mathbb{E}_n$ is a T-space, $\mathbb{E}_n^{S^1}$ is homotopy equivalent to $\mathbb{E}_n \times \Omega\mathbb{E}_n$. It is also well-known that if X and Y are H-spaces, then $X \times Y$ is an H-space. Thus we know that $\mathbb{E}_n^{S^1}$ is an H-space.

Thus we know, in the proof of Corollary 2.13, that T-spaces are equivalent to G-spaces in the class $\{\mathbb{E}_n \mid n \geq 2\}$ of $\mathbb{E}_n$, and $\mathbb{E}_{10}$ is an example of T-space which is not an H-space. Moreover, we may get many new G-spaces $\mathbb{E}_n^{S^1}$ which are not H-spaces.

THEOREM 2.15. *Let Z be a homotopy equivalent to $X \times Y$ for some space Y. Then X is a T-space if and only if there are maps $r : Z \to X$ and $i : X \to Z$ such that $ri \sim 1_X$ and $ie : \Sigma\Omega X \to Z$ is cyclic.*

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PROOF. Suppose X is a T-space. Let $f : Z \to X \times Y, g : X \times Y \to Z$ be maps such that $gf \sim 1_Z$ and $fg \sim 1_{X\times Y}$. Let $r = p_1 f : Z \to X$ and $i = gi_1 : X \to Z$, where $p_1 : X \times Y \to X$ is the projection and $i_1 : X \to X \times Y$ is the inclusion. Then $ri = p_1 fgi_1 \sim p_1 i_1 = 1_X$. Now we show that $ie : \Sigma\Omega X \to Z$ is cyclic. Since X is a T-space,there is a map $E : \Sigma\Omega X \times X \to X$ such that $Ej = \nabla(e \vee 1)$. Consider the map $F = g(E \times 1)(1 \times f) : \Sigma\Omega X \times Z \to Z$. Then $Fj' \sim \nabla(ie \vee 1_Z)$. Thus $ie : \Sigma\Omega X \to Z$ is cyclic. On the other hand, suppose there is a map $r : Z \to X$ which has a right homotopy inverse $i : X \to Z$ and $ie : \Sigma\Omega X \to Z$ is cyclic. Then we know that $e \sim r(ie) : \Sigma\Omega X \to X$ is cyclic. Thus X is a T-space.

3. co-T-spaces and cocyclic maps

In this section we introduce a co-T-space which is the dual of a T-space, characterize co-T-spaces by the dual Gottlieb groups, and use them to study the relationships between co-H-spaces, co-T-spaces and G'-spaces. We study some properties of co-T-spaces. We also define a weakly cocyclic map, and characterize G'-spaces by these maps. It is known [12] that if A is an H-group, then $DG(X, A)$ is an abelian subgroup of $[X, A]$. On the other hand, $DG(X, A)$ is the coevaluation subgroup given by Haslam [7], when A is a $K(\mathbb{Z}, n)$. It is also known [13] that if $f : X \to A$ is a cocyclic map and $\theta : A \to B$ is an arbitrary map, then $\theta f : X \to B$ is a cocyclic map. It is easily shown that a co-H-space may be characterized by the dual Gottlieb set as follows.

PROPOSITION 3.1 [12]. *The following are equivalent:*

(1) *X is a co-H-space;*
(2) *1_X is cocyclic;*
(3) *$DG(X, A) = [X, A]$ for any space A.*

DEFINITION 3.2. A space X is called a *co-T-space* if $e' : X \to \Omega\Sigma X$ is cocyclic.

We show that a co-T-space may be characterized by the dual Gottlieb group as follows;

THEOREM 3.3. *The following are equivalent:*

(1) *X is a co-T-space;*
(2) *$e' : X \to \Omega\Sigma X$ is cocyclic;*
(3) *$DG(X, \Omega A) = [X, \Omega A]$ for any space A.*

PROOF. (1) if and only if (2). This follows from the definition of co-T-space. (2) implies (3). Let $f : X \to \Omega A$ be a map. From the fact $f = \Omega\tau^{-1}(f)e' : X \to \Omega A$

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and e' is cocyclic, we know that $f : X \to \Omega A$ is a cocyclic. (3) implies (2). Take $A = \Sigma X$. Then $e' : X \to \Omega\Sigma X$ is cocyclic.

It is known [12] that $DG(X, \Omega A)$ is an abelian subgroup of $[X, \Omega A]$ for any space A. Thus we know, from Theorem 3.3, that if X is a co-T-space, then $[X, \Omega A]$ is abelian for any space A. Also, Lim showed [12] that $DG(X, \Omega A) \subset DW(X, \Omega A) \subset [X, \Omega A]$, where $DW(X, \Omega A) = \{\alpha \in [X, \Omega A] \mid [\alpha, \beta]' = 0$ for all $\beta \in [X, \Omega B]$ for all $B\}$, and $[\,,\,]'$ is the dual generalized Whitehead product [3]. Thus we obtain the following corollary.

COROLLARY 3.4. *If X is a co-T-space, then all the dual generalized Whitehead products vanish in X.*

Also, it is known [9] that ΣX is homotopy commutative if and only if $[e', e']' = 0$. Thus we know, from Corollary 3.4, that if X is a co-T-space, then ΣX is homotopy commutative.

EXAMPLE 3.5. Haslam showed [7] that the real projective space $\mathbb{R}P^2$ is a G'-space but not co-H-space. In fact, he showed that $DG(\mathbb{R}P^2, K(\mathbb{Z}_2, 1)) \neq [\mathbb{R}P^2, K(\mathbb{Z}_2, 1)]$. Thus we know, from the fact $DG(\mathbb{R}P^2, \Omega K(\mathbb{Z}_2, 2)) \neq [\mathbb{R}P^2, \Omega K(\mathbb{Z}_2, 2)]$ and Theorem 3.3, that $\mathbb{R}P^2$ is not a co-T-space.

We can identify $H^m(X; \pi)$ with $[X, K(\pi, m)]$, and define the coevaluation subgroup $G^m(X; \pi)$ of $H^m(X; \pi)$ to be the set of all homotopy classes of cocyclic maps from X to $K(\pi, m)$. The group $G^m(X) = G^m(X; \mathbb{Z})$ is the dual to the evaluation subgroup $G_m(X)$ of $\pi_m(X)$ considered in [6].

COROLLARY 3.6. *If X is a co-T-space, then $G^m(X; \pi) = H^m(X; \pi)$ for all m and π.*

PROOF. Let $h : K(\pi, m) \to \Omega K(\pi, m+1)$ be a homotopy equivalence and let $f : X \to K(\pi, m)$ be a representative of an element of $H^m(X; \pi)$. Then we have, from Theorem 3.3, that $hf : X \to \Omega K(\pi, m+1)$ is cocyclic. Since $h : K(\pi, m) \to \Omega K(\pi, m+1)$ is a homotopy equivalence, $f : X \to K(\pi, m)$ is cocyclic. This proves the corollary.

Let R be a ring and set $P^m(X; R) = \{\alpha \in H^m(X; R) | \beta \cup \alpha = 0$ for all $\beta \in \tilde{H}^*(X; R)\}$, where $\beta \cup \alpha$ denotes the cup product of β and α. It is known [7] that $G^m(X; R) \subseteq P^m(X; R)$ for all m and R. Thus we have the following corollary.

COROLLARY 3.7. *If X is a co-T-space, then all cup products vanish in X.*

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DEFINITION 3.8. A based map $f : X \to A$ is called *weakly cocyclic* if for any $n \geq 1$ and any map $\theta : A \to K(\mathbb{Z}, n)$, $\theta f : X \to K(\mathbb{Z}, n)$ is cocyclic. In fact $f : X \to A$ is weakly cocyclic if and only if $f^*(H^n(A)) \subset G^n(X)$ for all $n \geq 1$. The set of all homotopy classes of weakly cocyclic maps from X to A is denoted by $WDG(X, A)$.

Any cocyclic map is a weakly cocyclic map, but the converse does not hold. It follows from Proposition 3.1 and Example 3.5 that the identity map of $\mathbb{R}P^2$ is not cocyclic. But we easily know, from the fact that $\mathbb{R}P^2$ is a G'-space, that the identity map of $\mathbb{R}P^2$ is weakly cocyclic. It is clear that if $f : X \to A$ is an weakly cocyclic map and $\theta : A \to B$ is an arbitrary map, then $\theta f : X \to B$ is weakly cocyclic.

LEMMA 3.9. *Let X be any space. If A is homotopy equivalent to a $K(\mathbb{Z}, n)$, then $f : X \to A$ is cocyclic if and only if $f : X \to A$ is weakly cocyclic.*

PROOF. Clearly any cocyclic map is a weakly cocyclic map. Now suppose that $f : X \to A$ is weakly cocyclic. Since A is homotopy equivalent to $K(\mathbb{Z}, n)$, there exist maps $k : A \to K(\mathbb{Z}, n)$ and $h : K(\mathbb{Z}, n) \to A$ such that $hk \sim 1$. Since $f : X \to A$ is weakly cocyclic, $kf : X \to K(\mathbb{Z}, n)$ is cocyclic. Thus we have that $f \sim h(kf) : X \to A$ is cocyclic. This proves the lemma.

THEOREM 3.10. *The following are equivalent:*

(1) *X is a G'-space;*
(2) *$e' : X \to \Omega\Sigma X$ is weakly cocyclic;*
(3) *$DG(X, \Omega K(\mathbb{Z}, n+1)) = [X, \Omega K(\mathbb{Z}, n+1)]$ for all n.*

PROOF. (1) implies (2): Since X is a G'-space, the identity map 1 of X is weakly cocyclic. Thus we have that $e' = e'1_X : X \to \Omega\Sigma X$ is weakly cocyclic. (2) implies (3): Let $f : X \to \Omega K(\mathbb{Z}, n+1)$ be a map. Then we have, from the fact that $f = \Omega\tau^{-1}(f)e' : X \to \Omega K(\mathbb{Z}, n+1)$ and e' is weakly cocyclic, that $f : X \to \Omega K(\mathbb{Z}, n+1)$ is weakly cocyclic. Let $h : K(\mathbb{Z}, n) \to \Omega K(\mathbb{Z}, n+1)$ be a homotopy equivalence. Since $h_* : WDG(X, A) \to WDG(X, B)$ is a one-to-one correspondence, there is a weakly cocyclic map $g : X \to K(\mathbb{Z}, n)$ such that $hg \sim f$. Thus we have, from Lemma 3.9, that $g : X \to K(\mathbb{Z}, n)$ is cocyclic. Since $f \sim hg$, $f : X \to \Omega K(\mathbb{Z}, n+1)$ is cocyclic. (3) implies (1): Since there is a homotopy equivalence $h : K(\mathbb{Z}, n) \to \Omega K(\mathbb{Z}, n+1)$, $h_* : [X, K(\mathbb{Z}, n)] \to [X, \Omega K(\mathbb{Z}, n+1)]$ is a one-to-one correspondence. Thus it follows from the fact that $h_* : DG(X, A) \to DG(X, B)$ is a one-to-one correspondence and the definition of G'-space.

From the above Proposition 3.1, Theorems 3.3 and 3.10, we obtain the relationships of co-H-spaces, co-T-spaces and G'-spaces as follows.

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COROLLARY 3.11. *Any co-H-space is a co-T-space, and any co-T-space is a G'-space.*

It is well-known [13] that if $f : X \to A$ is cocyclic and $i : Y \to X$ has a left homotopy inverse, then $fi : Y \to A$ is cocyclic.

PROPOSITION 3.12. *Let X be a co-T-space. If there is a map $r : X \to Y$ which has a right homotopy inverse $i : Y \to X$, then Y is a co-T-space.*

PROOF. Since $e'_X : X \to \Omega\Sigma X$ is cocyclic and $e'_Y r = (\Omega\Sigma r)e'_X$, we have that $e'_Y r : X \to \Omega\Sigma Y$ is cocyclic. Since $i : Y \to X$ has a left homotopy inverse r, we have $e'_Y \sim (e'_Y r)i : Y \to \Omega\Sigma Y$ is cocyclic and Y is a co-T-space. This proves the proposition.

COROLLARY 3.13. *A retract of a co-T-space is a co-T-space.*

THEOREM 3.14. *$X \vee Y$ is a co-T-space if and only if X and Y are co-T-spaces.*

PROOF. Suppose $X \vee Y$ is a co-T-space. Let $r_1 = p_1 j : X \vee Y \to X$ and $r_2 = p_2 j : X \vee Y \to Y$, where $j : X \vee Y \to X \times Y$ is the inclusion, $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are natural projections. Then $r_1 i_1 = 1_X$ and $r_2 i_2 = 1_Y$, where $i_1(x) = (x, *), i_2(y) = (*, y)$. It is known [12] that if $f : X \to A$ and $g : Y \to B$ are cocyclic maps, then $f \vee g : X \vee Y \to A \vee B$ is cocyclic. Thus we have that X and Y are co-T-spaces. On the other hand, suppose that X and Y are co-T-spaces. Let $h : \Sigma(X \vee Y) \to \Sigma X \vee \Sigma Y$ be the natural homeomorphism and $k : \Omega\Sigma X \vee \Omega\Sigma Y \to \Omega(\Sigma X \vee \Sigma Y)$ be given by $k(\omega, *) = \Omega i_1(\omega), k(*, \eta) = \Omega i_2(\eta)$, where $i_1 : \Sigma X \to \Sigma X \vee \Sigma Y$ is given by $i_1(\langle x, t\rangle) = (\langle x, t\rangle, *)$ and $i_2 : \Sigma Y \to \Sigma X \vee \Sigma Y$ is given by $i_2(\langle y, t\rangle) = (*, \langle y, t\rangle)$. Then we have the following commutative diagram

$$\begin{tikzcd}
X \vee Y \arrow[r, "e'"] \arrow[d, "e'_X \vee e'_Y"] & \Omega\Sigma(X \vee Y) \arrow[d, "\Omega h"] \\
\Omega\Sigma X \vee \Omega\Sigma Y \arrow[r, "k"] & \Omega(\Sigma X \vee \Sigma Y).
\end{tikzcd}$$

Since $e'_X \vee e'_Y$ is cocyclic, $\Omega h e' = k(e'_X \vee e'_Y)$ is cocyclic. Since Ωh is homeomorphism, $e' : X \vee Y \to \Omega\Sigma(X \vee Y)$ is cocyclic. This proves the theorem.

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Department of Mathematics Education
Korea University
136-075, Seoul
Korea

Department of Mathematics
Hannam University
300-791, Taejon
Korea

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