ON THE NUMBER OF ORDINARY LINES DETERMINED BY n POINTS

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1. Introduction. More than sixty years ago, Sylvester **(13)** proposed the following problem: Let n given points have the property that the straight line joining any two of them passes through a third point of the set. Must the n points all lie on one line?

An alleged solution (not by Sylvester) advanced at the time proved to be fallacious and the problem remained unsolved until about 1933 when it was revived by Erdös **(7)** and others. Gallai (see **5**), Robinson (see **12**), Steinberg (see **4**, p. 30), Kelly (see **3**) and Lang **(11)** produced solutions of varying characters, the first affine, the second likewise affine (after dualizing), the third projective, and the fourth and fifth Euclidean. The answer is that in real projective space the points must indeed be on a line. Simple examples show that such is not the case in the complex projective plane **(3)**. The answer is also negative in finite projective geometries, where each line contains the same number of points. The property is very strongly dependent on the axioms of order.

The problem may be formulated in more general terms. Let P be a set of n points in real projective space and S the set of *connecting* lines which join these points. Call a line of S *ordinary* if it contains exactly two points of P. If S contains more than one line, show that it contains at least one ordinary line and determine lower bounds for the number of such lines. It is clear that the number of ordinary lines is invariant under a suitable central projection and so the question need only be settled in the real projective plane. The remainder of this investigation will be in the real projective plane, with P a set of n non-collinear points and S the set of their connecting lines. Let m denote the number of lines which are ordinary. Dirac **(6)** showed that $m \geqslant 3$ and Motzkin **(12)** showed that the order of magnitude of m is at least $\sqrt{n}$. It is the purpose of this note to show that $m \geqslant 3n/7$ and that, in a certain sense, this is a best possible bound.

2. Definitions, notation and preliminary theorems. A generic point of P is denoted by p and a generic line of S by s. Subscripts distinguish particular points and lines.

It is a known and easily established fact that a set of two or more lines in the plane which do not form a pencil effect a subdivision of the plane into two or more regions (see **(14)** for relevant definitions). With this in mind it is

Received May 21, 1957.

apparent that, except in the cases to be noted presently, the lines of S not passing through p dissect the plane into polygonal regions. In the event that the $n - 1$ points of P distinct from p are on a line, no division is effected. If exactly $n - 1$ of the points, including p, are on a line, then the division is into $n - 2$ angular regions, that is, regions bounded by two lines. In all other cases the division is into polygonal regions (bounded by at least three edges).

The point p is, of course, in the interior of one of these regions, which is called the *residence* of p, and p is said to *reside* in the region. The lines of S containing the edges of the residence are *neighbours* of p.

A set of n lines in the plane exactly $n - 1$ of which are concurrent is a *near-pencil*. This configuration is slightly exceptional in this study. We observe that if $n - 1$ points of P lie on one line, then S is a near-pencil.

THEOREM 2.1. *If a point p has precisely one neighbour, then S is a near-pencil.*

Proof. In this case the neighbour of p is the only line of S which does not pass through p; on this neighbour lie the remaining $n - 1$ points of P.

THEOREM 2.2. *If a point p has precisely two neighbours, then S is a near-pencil.*

Proof. In this case the lines of S which do not pass through p form a pencil; for otherwise they would form a proper dissection of the plane and p would have at least three neighbours. Let q be the vertex of this pencil. Let s_i, s_j be any two of the lines through q; let p_i, p_j, both different from q, be points on s_i, s_j respectively. The connecting line through p_i and p_j does not pass through q; hence, it must pass through p. It follows that there is only one line of S which passes through p; the remaining lines pass through q, which is necessarily a point of P.

THEOREM 2.3. *If S is not a near-pencil then each point of P has at least three neighbours.*

Proof. If a point of P has only one or two neighbours, then, by Theorems 2.1 and 2.2, S is a near-pencil.

3. Ordinary lines. The number of ordinary lines passing through p is the *order* of p. The number of neighbours of p which are ordinary lines is the *rank* of p. The order plus the rank is the *index*.

THEOREM 3.1. *If the order of p is zero then every neighbour of p is an ordinary line.*

Proof. Suppose, to the contrary, that the neighbour s of p passes through three points of P, say p_1, p_2, p_3. Let x be a point on s which lies on the boundary of the residence of p. Suppose the notation so chosen that $p_1x//p_2p_3$, that is, p_1 and x separate p_2 and p_3. Since p is of order zero, the connecting line through p and p_1 passes through a third point of P, say p_4. The lines p_2p_4 and p_3p_4

intersect both the segments determined by p and x on the line through them. Hence, x cannot be a point of p's residence, and we have a contradiction which proves the theorem.

It is now apparent that the residence of a point of order zero is one of the polygons into which the plane is dissected by the m ordinary lines. Furthermore, it is clear that a polygon in this dissection cannot contain in its interior two points of order zero. Since the m ordinary lines pass through at most $2m$ points of P and dissect the plane into at most $\binom{m}{2} + 1$ polygons, it follows that

$$\binom{m}{2} + 1 + 2m \geqslant n.$$

This is Motzkin's proof of

THEOREM 3.2.

$$\binom{m+2}{2} \geqslant n.$$

Note that

$$\binom{m+2}{2} < \tfrac{1}{2}(m+2)^2$$

so that the theorem shows that $m > \sqrt{[2n]} - 2$.

THEOREM 3.3. *The index of each point of P which is not of order two is at least three.*

Proof. First observe that the theorem is true when S is a near-pencil and dismiss this case from further consideration.

Case 1. The order of p is zero. Since S is not a near-pencil, P has at least three neighbours; by Theorem 3.1 they are all ordinary lines.

Case 2. The order of p is one. Let p_1 be the second point on the ordinary line through p. The proof of Theorem 3.1 shows that if a neighbour of p is not ordinary, then it passes through p_1. Since three neighbours of p cannot have a common point, it follows that if p has more than three neighbours then at least two of them are ordinary. On the other hand, if p has precisely three neighbours then two of them must be ordinary. For, in this case, if s_1 and s_2 are two non-ordinary neighbours of p then both pass through p_1, which is therefore a vertex of the triangular residence of p. If x, a boundary point of the residence of p, is on s_1, and p_1, p_2, p_3 three points on s_1 with the notation so chosen that $p_1p_2//xp_3$ then (as in the proof of Theorem 3.1) pp_3 is a second ordinary line through p. This contradiction shows that if p has precisely three neighbours, then at most one of them is non-ordinary.

Case 3. The order of p is at least three. Then the index of p is at least three.

THEOREM 3.4. *If a line s of S is a neighbour of three points p_1, p_2, p_3, then the points of P which lie on s are on the connecting lines determined by p_1, p_2, p_3.*

Proof. Clearly, three points which have a common neighbour cannot be collinear. Let the points of intersection of s with the line $p_i p_j$ be x_k (i, j, k a permutation of 1, 2, 3). Suppose p, a point different from x_1, x_2, x_3, lies on s in the segment $x_i x_j / x_k$, that is, $x_i x_j // p x_k$. Then, because of the lines pp_i and pp_j, s cannot be a neighbour of p_k. Thus, p must coincide with one of the points x_1, x_2, x_3.

COROLLARY 3.4. *A line of S is a neighbour of at most four points.*

Remark. It is easy to show that if s is a neighbour of exactly four points of P, then s joins two diagonal points of the complete quadrangle determined by the four points. Furthermore, s is then ordinary.

THEOREM 3.5. *If I_i is the index of point p_i, then*

$$m \geqslant \frac{1}{6} \sum_{i=1}^{n} I_i.$$

Proof. We count the number of ordinary lines by observing the index of each point of P. In this counting, a particular ordinary line may be counted at most six times, four times as a neighbour (Corollary 3.4) and twice because it passes through a point.

THEOREM 3.6. $m \geqslant 3n/7$.

Proof. Let k be the number of points of order two. Clearly

$$m \geqslant k.$$

By Theorems 3.3 and 3.5

$$m \geqslant \frac{3(n-k)+2k}{6}.$$

Eliminating k from these inequalities, we obtain the desired result.

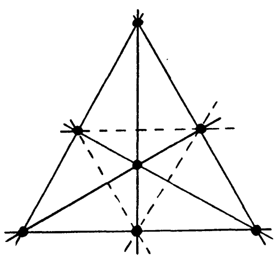

FIGURE 3.1

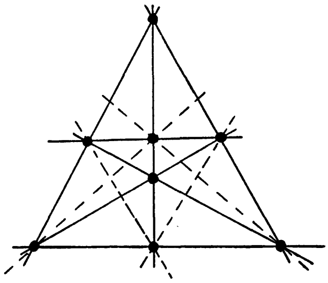

FIGURE 3.2

For $n = 7, 8$ the theorem shows that $m \geqslant 3, 4$. Figures 3.1 and 3.2 exhibit configuration of 7, 8 points with $m = 3, 4$ respectively (the ordinary lines are "broken"). In this sense Theorem 3.6 is best possible. However, for large n it would seem to us that the configuration with fewest possible lines is probably near the near-pencil arrangement. If this be so, then, for large n, m should be at least $n - 1$. Thus, a reasonable conjecture **(6)** is $m \geqslant \frac{1}{2}n$ for $n > 7$; the present method does not seem to allow us to draw this conclusion.

4. Connecting lines. In this section we derive an interesting inequality which we use to establish a bound on the number of connecting lines.

The dual of P is a set of n lines, $\bar{P}$, not a pencil; the dual of S is $\bar{S}$, the set of points of intersection of these lines.

In the dissection of the plane by the lines of $\bar{P}$, F_i denotes the number of polygons each having exactly i edges, and V_i denotes the number of vertices each incident with exactly i edges. V, E, and F denote the total number of vertices, edges and faces respectively.

Clearly $V_i = 0$ for all odd i,

4.10 $$V = V_4 + V_6 + V_8 + \dots$$

and

4.11 $$F = F_3 + F_4 + F_5 + \dots.$$

Since each edge has two vertices and belongs to two polygons

4.20 $$E = 2V_4 + 3V_6 + 4V_8 + \dots$$

and

4.21 $$2E = 3F_3 + 4F_4 + 5F_5 + \dots.$$

Adding 4.20 and 4.21 yields

4.22 $$3E = 2V_4 + 3V_6 + 4V_8 + \dots + 3F_3 + 4F_4 + 5F_5 + \dots.$$

By Euler's theorem,

4.3 $$V - E + F = 1.$$

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Replacing V, F and E in 4.3 by their values 4.10, 4.11 and 4.22 respectively yields, after simplification,

4.4 $$V_4 = 3 + V_8 + 2V_{10} + 3V_{12} + \ldots + F_4 + 2F_5 + 3F_6 + \ldots .$$

Call a line of S which passes through precisely i points of P an *i-line* and let t_i denote the number of i-lines, for example, an ordinary line is a 2-line and $m = t_2$. Clearly, the dual of V_{2i} is t_i; hence, by dualizing 4.4 we establish

4.5 $$m = t_2 \geqslant 3 + t_4 + 2t_5 + 3t_6 + \ldots .$$

We have immediately Dirac's result **(6)**, $m \geqslant 3$.

Inequality 4.5 may be used to prove that, if n is even, $m > (n + 11)/6$. In fact, the assumption that there is a set P of n points with n even and $m \leqslant (n + 11)/6$ leads us to a contradiction. For, in such a case, we see from 4.5 that the number of points of P each of which is incident with a k-line for some $k \neq 3$ is at most

$$\begin{aligned} 2t_2 + 4t_4 + 5t_5 + \ldots &\leqslant 2t_2 + 4(t_4 + 2t_5 + 3t_6 + \ldots) \\ &\leqslant 2t_2 + 4(t_2 - 3) = 6m - 12 \leqslant n - 1. \end{aligned}$$

Thus, there is at least one point of P incident solely with 3-lines; clearly n must be odd, and we have a contradiction

Call a point of P which is incident with exactly k connecting lines a k-point and let v_k denote the number of k-points. Clearly

4.60 $$\sum_{k=2} v_k = n;$$

also

4.61 $$\sum_{k=2} kt_k = \sum_{k=2} kv_k,$$

for in both sums a k-line is counted k times. From 4.5 we have

$$3t_2 + 3t_3 + 3t_4 + \ldots \geqslant 3 + 2t_2 + 3t_3 + 4t_4 + \ldots$$

which together with 4.61 yields

4.62 $$3t \geqslant 3 + \sum_{k=2} kt_k = 3 + \sum_{k=2} kv_k,$$

where

$$t = \sum t_k$$

denotes the number of connecting lines.

At the same time that Erdös **(7)** reproposed Sylvester's problem he also posed the following one: Show that S contains at least n lines. This was answered successfully by Steinberg (see **7**) as well as de Bruijn and Erdös **(5)** and Hanani (**9**; **10**). The configuration with S a near-pencil shows that this is a best possible result.

Erdös **(8)** conjectured that if n is large enough and at most $n - 2$ points are collinear then $2n - 4$ lines are determined. If we insist that at most $n - 3$

points be collinear, then approximately $3n$ lines should be determined. In general, if at most $n - k$ points are collinear and n is large (with respect to k) we would expect the minimum number of lines to be of order kn. This is the substance of the following theorem, a corollary of which establishes the truth of his conjecture.

THEOREM 4.1. *If at most $n - k$ points of P are collinear and*

4.70 $$n \geqslant \tfrac{1}{2}\{3(3k-2)^2 + 3k - 1\}$$

then

$$t \geqslant kn - \tfrac{1}{2}(3k+2)(k-1).$$

The proof will follow that of

LEMMA 4.1. *If exactly $n - r$ points of P are on a line, and*

$$n \geqslant \frac{3r}{2} \geqslant 3$$

then

$$t \geqslant rn - \tfrac{1}{2}(3r+2)(r-1).$$

Proof. Suppose the $n - r$ points $p_{r+1}, p_{r+2}, \ldots, p_n$ lies on the line s, and the r points $p_1, p_2, \ldots, p_r$ do not lie on s. Two lines $p_a p_b$ and $p_c p_d$ $(1 \leqslant a, c \leqslant r;\ r+1 \leqslant b, d \leqslant n)$ are certainly distinct if $b \neq d$. Hence, among the $r(n-r)$ connecting lines $p_i p_j$ $(i = 1, 2, \ldots, r;\ j = r+1, r+2, \ldots, n)$ at least

$$r(n-r) - \tfrac{1}{2}r(r-1)$$

are distinct. Counting the line s we have

$$t \geqslant 1 + r(n-r) - \tfrac{1}{2}r(r-1) = rn - \tfrac{1}{2}(3r+2)(r-1).$$

We now proceed to the proof of Theorem 4.1 and consider two cases.

Case 1.

$$\sum_{i=2}^{3k-1} v_i \geqslant 2.$$

In this case there are two points, say p_1 and p_2, each of which lies on at most $3k - 1$ connecting lines. Let s be the line through p_1 and p_2. The connecting lines through p_1 and p_2 other than s intersect in at most $(3k-2)^2$ points. Hence, s contains at least $n - (3k-2)^2$ points. Suppose it contains $n - x$ points, where

$$k \leqslant x \leqslant (3k-2)^2.$$

Inequalities 4.70 and 4.71 insure that $n \geqslant \frac{3}{2}x$; hence, by the lemma, at least

$$xn - \tfrac{1}{2}(3x+2)(x-1)$$

connecting lines are determined. Using 4.70 and 4.71 we have

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$$n \geqslant \tfrac{1}{2}\{3(x + k) - 1\}$$

or

$$n(x - k) \geqslant \tfrac{1}{2}(3x^2 - 3k^2 - x + k)$$

or

$$t \geqslant xn - \tfrac{1}{2}(3x + 2)(x - 1) \geqslant kn - \tfrac{1}{2}(3k + 2)(k - 1).$$

Case 2.

$$\sum_{i=2}^{3k-1} v_i \leqslant 1.$$

From 4.62 we have

$$3t \geqslant 3 + 2 + 3k(n - 1)$$

or

$$t \geqslant kn - k + \frac{5}{3} > kn - \tfrac{1}{2}(3k + 2)(k - 1).$$

This completes the proof of Theorem 4.1.

Taking $k = 2$, we have

COROLLARY 4.1. *If at most $n - 2$ points are collinear and $n \geqslant 27$, then at least $2n - 4$ connecting lines are determined.*

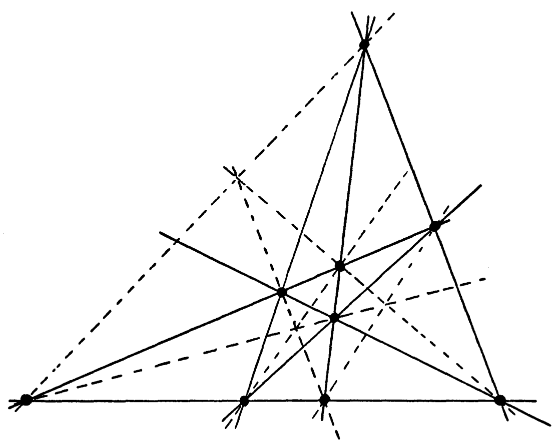

FIGURE 4.1

Figures 3.1, 3.2, and 4.1 show configurations with $n = 7, 8, 9$ and $t = 9, 11, 13$ respectively. On the other hand, using the above methods a somewhat detailed analysis leads to the conclusion that these are best possible, that is, if no $n - 1$ points are collinear and $n = 7, 8, 9$, then $t \geqslant 2n - 5$. Similarly, a detailed analysis shows that if $n = 10$ and no 9 points are collinear, then

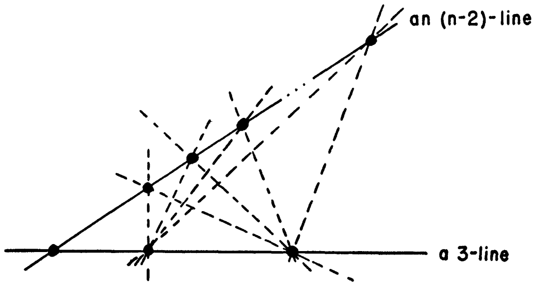

FIGURE 4.2

$t \geqslant 16$. Figure 4.2 shows a configuration with arbitrary n and $t = 2n - 4$. Thus, it seems very likely that, if at most $n - 2$ points are collinear, then $t \geqslant 2n - 5$ ($n = 7, 8, 9$) and $t \geqslant 2n - 4$ ($n > 9$) and that for each n there is a configuration for which equality holds.

The related problem of finding configurations for which t_3 is as large as possible is considered by Ball (**1**, pp. 105-6).

5. Zonohedra. In this section we point out a connection between the configurations we have been studying and the convex solids (in Euclidean space) known as zonohedra.

A *zonohedron* is a convex polyhedron whose faces all possess central symmetry (**2**, pp. 27-30). These properties insure that the solid has central symmetry. Each edge of a zonohedron determines a zone of faces in which each face has two sides equal and parallel to the given edge. If the edges occur in n different directions, there are n zones.

Let us call a set of n concurrent lines (in Euclidean space) a star. Then we may say that every zonohedron determines a star having one line parallel to each of the n directions in which the edges occur.

To every pair of faces of the zonohedra there corresponds the connecting plane (through the vertex of the star and parallel to the pair of faces) which contains the lines of the star parallel to the edges of the faces. The projection of the star and its connecting planes onto the projective plane at infinity is precisely a configuration of n non-collinear points and their connecting lines. Thus, a pair of parallel $2k$-gons on a zonohedron corresponds to a k-line of S. Theorem 3.6 now shows that:

Every zonohedron with n zones has at least $3n/7$ *pairs of parallelogram faces.*

https://doi.org/10.4153/CJM-1958-024-6 Published online by Cambridge University Press

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