The Polar Properties of the Plane Trinodal Quartic Curve

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§ 1. *Introduction.*

Clebsch[1] was the first to investigate the properties of the plane quartic curve. The object of the present paper is to study the properties of the plane trinodal quartic curve.

We shall take the three nodes as triangle of reference. The equation of the quartic then assumes the form

$$fy^2 z^2 + gz^2 x^2 + hx^2 y^2 + 2xyz\,(px + qy + rz) = 0. \tag{1}$$

We shall write the equation of the polar cubic of $P \equiv (x', y', z')$, in the form

$$a_2 x^2 y + a_3 x^2 z + b_3 y^2 z + b_1 y^2 x + c_1 z^2 x + c_2 z^2 y + 2kxyz = 0. \tag{2}$$

It is evident that the coefficients of (2) are linear in the coordinates of P.

The locus of P, moving so that its polar cubic shall be equianharmonic, is the quartic S_4, where

$$\begin{aligned} S_4 \equiv {} & (k^2 - b_1 c_1 - c_2 a_2 - a_3 b_3)^2 \\ & + 3\,\{k\,(a_2 b_3 c_1 + a_3 b_1 c_2) - (c_2 a_2 a_3 b_3 + a_3 b_3 b_1 c_1 + b_1 c_1 c_2 a_2)\} = 0. \end{aligned} \tag{3}$$

The locus of P moving so that its polar cubic shall be harmonic is a sextic curve, T_6, whose equation is:—

$$\begin{aligned} T_6 \equiv {} & 8\,(k^2 - b_1 c_1 - c_2 a_2 - a_3 b_3)^3 + 27\,(a_2 b_3 c_1 + a_3 b_1 c_2)^2 \\ & + 36\,(k^2 - b_1 c_1 - c_2 a_2 - a_3 b_3)\,\{k\,(a_2 b_3 c_1 + a_3 b_1 c_2) - (c_2 a_2 a_3 b_3 + a_3 b_3 b_1 c_1 + b_1 c_1 c_2 a_2)\} = 0. \end{aligned} \tag{4}$$

The sextic curve T_6 is not general in character and we shall prove that certain conics exist which touch it at six points. In the first part of the paper we shall define these conics and discuss their sex-tangent properties. In the second part we shall consider other sextic curves which are touched by these same conics at six points.

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[1] *Journal für Math.*, **59** (1861), 125–145.

§ 2. *Definitions.*

The locus of P moving so that A, B, C is an apolar triad with respect to the polar cubic of P is the line

$$F_{123} \equiv k = 0.$$

Let the polar cubic of P cut BC in a point L, other than B or C. Then the locus of P moving so that A and L shall be apolar points with respect to the Hessian of the polar cubic of P (*i.e.* so that the polar line of A with respect to the Hessian of the polar cubic of P passes through L, and conversely) is a conic U, where

$$U \equiv k^2 - b_1 c_1 - c_2 a_2 - a_3 b_3 = 0.$$

The symmetry of this result shows that the conic can be similarly obtained with respect to the other two vertices of the triangle of reference. If the tangent at A to the polar cubic of P passes through L, then the locus of P is the conic Σ_1, where

$$\Sigma_1 \equiv 3\,(c_2 a_2 - a_3 b_3) = 0.$$

Suppose the polar cubic of P be such that B and C are apolar points with respect to the Hessian of the polar cubic. The locus of P is then the conic

$$U_{23} \equiv k^2 + 2b_1 c_1 - c_2 a_2 - a_3 b_3 = 0.$$

The conics U_{31} and U_{12} are similarly defined.

We shall now define several cubic curves which will be required.

The locus of P moving so that A, B, C is an apolar triad with respect to the Hessian of the polar cubic of P is the cubic Γ_{123}, where

$$\Gamma_{123} \equiv 2k\,(k^2 - b_1 c_1 - c_2 a_2 - a_3 b_3) + 3\,(a_2 b_3 c_1 + a_3 b_1 c_2) = 0.$$

If the tangents at B and C to the polar cubic of P meet on the cubic again the locus of P is the cubic Γ_{23} whose equation is

$$\Gamma_{23} \equiv (a_2 b_3 c_1 + a_3 b_1 c_2) - 2kb_1 c_1.$$

We define in a similar way the cubics Γ_{31} and Γ_{12}.

Finally we define the following cubics

$$\Phi \equiv a_2 b_3 c_1 + a_3 b_1 c_2$$
$$\Psi \equiv a_2 b_3 c_1 - a_3 b_1 c_2.$$

Φ is the locus of P moving so that the tangents at A, B, C to the polar cubic of P are concurrent. The cubic Ψ is the locus of points whose polar cubics touch the same conic at the vertices of the triangle of reference.

§ 3. *The Sex-tangent Conics of T_6.*

In virtue of the previous definitions we find that the equation to the harmonic polar locus, T_6, can be written in the following ways:

$$T_6 \equiv 4U(3S_4 - U^2) + 27\Psi^2 \tag{5}$$

$$T_6 \equiv -4U_{23}U_{31}U_{12} + 3\Gamma_{123}^2 \tag{6}$$

$$T_6 \equiv -4U_{23}U_{31}U_{12} + 6kU_{23}(\Gamma_{123} + 3\Gamma_{23}) + 27\Gamma_{23}^2. \tag{7}$$

From the first of these equations we see that U is a sex-tangent conic of T_6 (*i.e.* a conic touching T_6 at six points) and that the cubic curve Ψ passes through the six points of contact. Furthermore the quartic curve $Q_4 \equiv 3S_4 - U^2$, passes through the twelve remaining points of intersection of T_6 and Ψ, and touches T_6 at each of these twelve points. Q_4 is a quartic touching S_4 at the eight points where S_4 is met by the conic U.

In virtue of equation (6), U_{23}, U_{31}, U_{12} are all sex-tangent conics of the harmonic polar locus T_6, and the cubic Γ_{123} passes through the six points of contact of each conic with T_6. These eighteen points of contact are the complete points of intersection of the cubic Γ_{123} and the sextic T_6.

In addition to showing that U_{23} is a sex-tangent conic of T_6, equation (7) shows that Γ_{23} passes through the points of contact of U_{23} and T_6. Also we see that the quartic Q'_4 where

$$Q'_4 \equiv -4U_{31}U_{12} + 6k(\Gamma_{123} + 3\Gamma_{23}) = 0$$

passes through the twelve remaining points of intersection of the harmonic polar locus T_6 and Γ_{23}, and touches T_6 at each of these twelve points. The quartic Q'_4 passes through the four points in which the line k cuts the conics U_{31}, U_{12} and also through the twelve points in which the cubic $(\Gamma_{123} + 3\Gamma_{23})$ cuts these conics. The cubic $(\Gamma_{123} + 3\Gamma_{23})$ passes through the six points of contact of U_{23} with T_6.

Consider the following relations which can be obtained from the definitions in § 2:

$$\Gamma_{123} \equiv 3\Gamma_{23} + 2kU_{23} \equiv \text{etc.} \tag{8}$$

$$\Gamma_{123} \equiv 3\Phi + 2kU \tag{9}$$

$$\Gamma_{23} - \Gamma_{31} \equiv 2k\Sigma_3 \tag{10}$$

$$\Gamma_{23} - \Phi \equiv -2kb_1c_1 \tag{11}$$

$$U_{23} - U \equiv 3b_1c_1 \tag{12}$$

$$U_{23} - U_{31} \equiv \Sigma_3 \tag{13}$$

$$\Sigma_1 + \Sigma_2 + \Sigma_3 \equiv 0. \tag{14}$$

Equation (8) shows that the cubics Γ_{123}, Γ_{23} cut in six points on the conic U_{23} (these are the six points in which U_{23} touches T_6) and in three points on the line k, *i.e.* F_{123}. Similarly from (9) the cubics Γ_{123}, Φ cut in six points on U and in three points on the line F_{123}. Equations (10) and (11) have similar interpretations. Thus the cubics Γ_{123}, Γ_{23}, Γ_{31}, Γ_{12}, Φ all cut the line F_{123} in the same three points.

From (12) and (13) we see that the lines b_1, c_1 pass through the four points of intersection of U_{23} and U while the conic Σ_3 passes through the intersections of U_{23} and U_{31}. The conics Σ_1, Σ_2, Σ_3 belong to a pencil.

§ 4. In the rest of the paper we shall consider sextic curves which are touched by the conics U, U_{23}, U_{31}, U_{12}. The following theorem will be required:—

If the conic G_2 is sex-tangent to G_6 then it is also a sex tangent conic of $(G_6 + \lambda G_4 G_2)$ where G_4 is any quartic.

We shall apply this to the case where G_2 is one of the conics U, U_{23}, U_{31}, U_{12}, where G_6 is the harmonic polar locus T_6 and where G_4 is the equianharmonic polar locus S_4.

Let us consider first the following sextic curves through the twenty-four points of intersection of T_6 and S_4:

$$T_6 - \lambda \quad U \quad S_4 = 0 \tag{15}$$

$$T_6 - \lambda_{23} U_{23} S_4 = 0 \tag{16}$$

$$T_6 - \lambda_{31} U_{31} S_4 = 0 \tag{17}$$

$$T_6 - \lambda_{12} U_{12} S_4 = 0. \tag{18}$$

Suppose these equations can be written as follows:

$$T_6 - \lambda \quad U \quad S_4 \equiv U \quad Q \quad + 3C^2 \equiv U \quad Q' \quad + 3C'^2 \equiv 0 \tag{19}$$

$$T_6 - \lambda_{23} U_{23} S_4 \equiv U_{23} Q_{23} + 3C_{23}^2 \equiv U_{23} Q'^2_{23} + 3C'^2_{23} \equiv 0 \tag{20}$$

$$T_6 - \lambda_{31} U_{31} S_4 \equiv U_{31} Q_{31} + 3C_{31}^2 \equiv U_{31} Q'_{31} + 3C'^2_{31} \equiv 0 \tag{21}$$

$$T_6 - \lambda_{12} U_{12} S_4 \equiv U_{12} Q_{12} + 3C_{12}^2 \equiv U_{12} Q'_{12} + 3C'^2_{12} \equiv 0 \tag{22}$$

where the Q's are quartics and the C's are cubics.

Then in virtue of (19), U is a sex-tangent conic of $(T_6 - \lambda U S_4)$ and the cubics C and C' pass through the six points of contact. The quartic Q touches $(T_6 - \lambda U S_4)$ at the remaining twelve points in which C cuts $(T_6 - \lambda U S_4)$. Also Q' touches $(T_6 - \lambda U S_4)$ at the remaining twelve points in which C' cuts $(T_6 - \lambda U S_4)$.

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Thus the cubics C and C' cut in nine points, six of which are the points where U touches $(T_6 - \lambda U S_4)$. Hence the remaining three lie on a line l (say). Thus we have $C - C' \equiv plU$, where p is constant. Using this it can be shown from (19) that

$$Q - Q' \equiv 3plU(C + C'). \tag{23}$$

From (23) we see that the quartics Q, Q' cut in four points on the line l and in twelve points on the cubic $(C + C')$. The cubic $(C + C')$ passes through the points in which U touches $(T_6 - \lambda U S_4)$.

In a similar way it can be shown that

$$C_{23} - C'_{23} \equiv ql_{23} U_{23} \tag{24}$$
$$C_{31} - C'_{31} \equiv rl_{31} U_{31} \tag{25}$$
$$C_{12} - C'_{12} \equiv sl_{12} U_{12} \tag{26}$$

and

$$Q'_{23} - Q_{23} \equiv 3ql_{23}(C_{23} + C'_{23}) \tag{27}$$
$$Q'_{31} - Q_{31} \equiv 3rl_{31}(C_{31} + C'_{31}) \tag{28}$$
$$Q'_{12} - Q_{12} \equiv 3sl_{12}(C_{12} + C'_{12}), \tag{29}$$

where q, r, s are constants and l_{23}, l_{31}, l_{12} represent straight lines.

The above is general in character and applies to all quartics, cubics and conics which satisfy the relations (15), (16), (17), (18). The following are two particular cases worthy of consideration.

$$\text{A}\left\{\begin{array}{ll} Q \equiv (12 - \lambda) S_4 - 4U^2, & C \equiv 3\Psi \\ Q_{23} \equiv -4U_{31} U_{12} - \lambda_{23} S_4, & C_{23} \equiv \Gamma_{123} \\ Q_{31} \equiv -4U_{12} U_{23} - \lambda_{31} S_4, & C_{31} \equiv \Gamma_{123} \\ Q_{12} \equiv -4U_{23} U_{31} - \lambda_{12} S_4, & C_{12} \equiv \Gamma_{123} \end{array}\right\} \tag{30}$$

$$\text{B}\left\{\begin{array}{ll} Q' \equiv (12 - \lambda) S_4 - 4U^2 + 6k(6\Psi - 2kU), & C' \equiv 3\Psi - 2kU \\ Q'_{23} \equiv -4U_{31} U_{12} - \lambda_{23} S_4 + 6k(\Gamma_{123} + \Gamma_{23}), & C'_{23} \equiv 3\Gamma_{23} \\ Q'_{31} \equiv -4U_{12} U_{23} - \lambda_{31} S_4 + 6k(\Gamma_{123} + \Gamma_{31}), & C'_{31} \equiv 3\Gamma_{31} \\ Q'_{12} \equiv -4U_{23} U_{31} - \lambda_{12} S_4 + 6k(\Gamma_{123} + \Gamma_{12}), & C'_{12} \equiv 3\Gamma_{12} \end{array}\right\} \tag{31}$$

In these cases we get

$$\left.\begin{array}{l} C - C' \equiv 2kU \\ C_{23} - C'_{23} \equiv 2kU_{23} \\ C_{31} - C'_{31} \equiv 2kU_{31} \\ C_{12} - C'_{12} \equiv 2kU_{12} \end{array}\right\} \tag{32}$$

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and
$$\left.\begin{aligned} Q\ \ - Q' &\equiv 6k\,(6\,\Psi - 2kU) \\ Q'_{23} - Q_{23} &\equiv 6k\,(\Gamma_{123} + 3\,\Gamma_{23}) \\ Q'_{31} - Q_{31} &\equiv 6k\,(\Gamma_{123} + 3\,\Gamma_{23}) \\ Q'_{12} - Q_{12} &\equiv 6k\,(\Gamma_{123} + 3\,\Gamma_{31}) \end{aligned}\right\}. \tag{33}$$

Thus the quartics Q_{23}, Q'_{23} cut in four points on the line k and in twelve points on the cubic $(\Gamma_{123} + 3\,\Gamma_{23})$. The latter cubic cuts both Γ_{123} and Γ_{23} in the six points where U_{23} touches $(T_6 - \lambda U_{23} S_4)$ and in three points on the line k. The cubics $(\Gamma_{123} + 3\,\Gamma_{23})$, $(\Gamma_{123} + 3\,\Gamma_{31})$, $(\Gamma_{123} + 3\,\Gamma_{12})$ all cut the line k in the same three points.

§ 5. Finally we discuss the case where the λ's are equal, *i.e.* the sextics
$$\left.\begin{aligned} J\ \ &\equiv T_6 - \lambda U\ \ S_4 = 0 \\ J_{23} &\equiv T_6 - \lambda U_{23} S_4 = 0 \\ J_{31} &\equiv T_6 - \lambda U_{31} S_4 = 0 \\ J_{12} &\equiv T_6 - \lambda U_{12} S_4 = 0 \end{aligned}\right\}. \tag{34}$$

From these equations we find that
$$J\ \ - J_{23} \equiv 3\lambda b_1 c_1 S_4 \tag{35}$$
$$J_{31} - J_{23} \equiv \ \lambda\,\Sigma_3 S_4. \tag{36}$$

Thus the sextics J, J_{23} cut in the twenty-four points of intersection of T_6 and S_4 and in six points on each of the lines b_1 and c_1. Also J_{23}, J_{31} cut in twenty-four points on T_6 and S_4 and in twelve points on the conic Σ_3.

From equations (30) with the λ's equal we have
$$Q_{31} - Q_{12} \equiv 4U_{23}\,\Sigma_1$$
$$Q_{12} - Q_{23} \equiv 4U_{31}\,\Sigma_2$$
$$Q_{23} - Q_{31} \equiv 4U_{12}\,\Sigma_3,$$

from which we see that the quartics Q_{31}, Q_{12} cut in eight points on each of the conics U_{23}, Σ_1.

Taking $\lambda = 0$ in the above we get the theory of the sex-tangent conics of the harmonic polar locus, T_6. The properties of these were dealt with in the first part of this paper.

Taking $\lambda = 8$ we get four important sextic curves connected with the trinodal quartic. They may be defined as follows.

Suppose the polar cubic of P with respect to the trinodal quartic meets BC in a point L other than B or C. Then the condition that A and L shall be apolar points with respect to that member of the Hessian pencil which is apolar to the Cayleyan of the polar cubic of P, is $T_6 - 8US_4 = 0$. The symmetrical nature of this result shows that it could be obtained from any vertex and the point where the polar cubic meets the opposite side, other than at the nodes.

The condition that B and C shall be apolar points with respect to that member of the Hessian pencil which is apolar to the Cayleyan of the polar cubic is $T_6 - 8U_{23}S_4 = 0$. The sextic $(T_6 - 8US_4)$ passes through the nodes of the trinodal quartic and has U as a sex-tangent conic with Ψ passing through the points of contact. The quartic $(S_4 - U^2)$ touches $(T_6 - 8US_4)$ at the remaining twelve points in which Ψ cuts $(T_6 - 8US_4)$. $(S_4 - U^2)$ also passes through the nodes of the trinodal quartic and touches S_4 at the eight points in which U cuts S_4.

In a similar way U_{23} is a sex-tangent conic of $(T_6 - 8U_{23}S_4)$ with Γ_{123} passing through the six points of contact. At the remaining twelve points of intersection of $(T_6 - 8U_{23}S_4)$ and Γ_{123}, $(T_6 - 8U_{23}S_4)$ is touched by the quartic Q_{23} where

$$Q_{23} \equiv -4U_{31}U_{12} - 8S_4 = 0.$$

Taking $\lambda = 8$ in equations (31) we can show that Γ_{23} passes through the points of contact of U_{23} with $(T_6 - 8U_{23}S_4)$.

Throughout this paper we have dealt with the conic U_{23}. It should be noted that similar properties exist in the cases of the conics U_{31} and U_{12}.

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