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ON THE LOCAL-INDICABILITY COHEN–LYNDON THEOREM

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Abstract. For a group H and a subset X of H, we let ${}^{H}X$ denote the set $\{hxh^{-1} \mid h \in H,\ x \in X\}$, and when X is a free-generating set of H, we say that the set ${}^{H}X$ is a *Whitehead subset of* H. For a group F and an element r of F, we say that r is *Cohen–Lyndon aspherical in* F if ${}^{F}\{r\}$ is a Whitehead subset of the subgroup of F that is generated by ${}^{F}\{r\}$. In 1963, Cohen and Lyndon (D. E. Cohen and R. C. Lyndon, Free bases for normal subgroups of free groups, *Trans. Amer. Math. Soc.* **108** (1963), 526–537) independently showed that in each free group each non-trivial element is Cohen–Lyndon aspherical. Their proof used the celebrated induction method devised by Magnus in 1930 to study one-relator groups. In 1987, Edjvet and Howie (M. Edjvet and J. Howie, A Cohen–Lyndon theorem for free products of locally indicable groups, *J. Pure Appl. Algebra* **45** (1987), 41–44) showed that if A and B are locally indicable groups, then each cyclically reduced element of $A*B$ that does not lie in $A\cup B$ is Cohen–Lyndon aspherical in $A*B$. Their proof used the original Cohen–Lyndon theorem. Using Bass–Serre theory, the original Cohen–Lyndon theorem and the Edjvet–Howie theorem, one can deduce the local-indicability Cohen–Lyndon theorem: if F is a locally indicable group and T is an F-tree with trivial edge stabilisers, then each element of F that fixes no vertex of T is Cohen–Lyndon aspherical in F. Conversely, by Bass–Serre theory, the original Cohen–Lyndon theorem and the Edjvet–Howie theorem are immediate consequences of the local-indicability Cohen–Lyndon theorem. In this paper we give a detailed review of a Bass–Serre theoretical form of Howie induction and arrange the arguments of Edjvet and Howie into a Howie-inductive proof of the local-indicability Cohen–Lyndon theorem that uses neither Magnus induction nor the original Cohen–Lyndon theorem. We conclude with a review of some standard applications of Cohen–Lyndon asphericity.

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1. Introduction.

NOTATION 1.1. Let F be a multiplicative group, fixed throughout the paper.

The disjoint union of two sets X and Y will be denoted by $X \vee Y$.

By a *transversal for* a (left or right) F-action on a set X we mean a subset of X that contains exactly one element of each F-orbit of X; by the axiom of choice, transversals always exist.

For elements x and y of F, we write $\overline{x} := x^{-1}$, ${}^{x}y := xy\overline{x}$, $[x, y] := xy\overline{x}\,\overline{y}$ and $\mathbf{C}_F(x) := \{f \in F \mid {}^{f}x = x\}$. For any subgroup H of F, we say that x and y are H-*conjugate* if there exists some $h \in H$ such that ${}^{h}x = y$. Conjugation actions will always be left actions in this paper.

If R and X are subsets of F, we let $\langle\, R \,\rangle$ denote the subgroup of F generated by R, and we write ${}^{X}R := \{{}^{x}r \mid r \in R, x \in X\}$ and $F/\langle\!\langle R \rangle\!\rangle := F/\langle\, {}^{F}R \,\rangle$. If R consists of a single element r, we simply write $\langle\, r \,\rangle$, ${}^{X}r$ and $F/\langle\!\langle r \rangle\!\rangle$, respectively. We say that X is a *free-generating set of* F if the induced group homomorphism $\langle\, X \mid \,\rangle \to F$ is bijective.

A subset Y of F is said to be a *Whitehead subset of* F if there exists some free-generating set X of F such that ${}^{F}X = Y$, that is, Y is closed under the F-conjugation action and some transversal for the F-conjugation action on Y is a free-generating set of F.

Borrowing terminology from [**3**], we say that an element r of F is *Cohen–Lyndon aspherical in* F if ${}^{F}r$ is a Whitehead subset of $\langle\, {}^{F}r \,\rangle$. Thus, r is Cohen–Lyndon aspherical in F if and only if some transversal for the $\langle\, {}^{F}r \,\rangle$-conjugation action on ${}^{F}r$ is a free-generating set of $\langle\, {}^{F}r \,\rangle$. Thus, r is Cohen–Lyndon aspherical in F if and only if there exists some subset X of F such that X is a transversal for the $(\langle\, {}^{F}r \,\rangle \mathbf{C}_F(r))$-action on F by multiplication on the right and, moreover, ${}^{X}r$ is a free-generating set of $\langle\, {}^{F}r \,\rangle$. Here $X \neq \emptyset$ and $r \neq 1$, and the map $X \to {}^{X}r$, $x \mapsto {}^{x}r$ is bijective.

We say that a subset R of F is *Cohen–Lyndon aspherical in* F if no two distinct elements of R are F-conjugate and ${}^{F}R$ is a Whitehead subset of $\langle\, {}^{F}R \,\rangle$. Thus, R is Cohen–Lyndon aspherical in F if and only if no two distinct elements of R are F-conjugate and some transversal for the $\langle\, {}^{F}R \,\rangle$-conjugation action on ${}^{F}R$ is a free-generating set of $\langle\, {}^{F}R \,\rangle$. Thus, R is Cohen–Lyndon aspherical in F if and only if no two distinct elements of R are F-conjugate and there exists some family $(\, X_r \mid r \in R)$ of subsets of F with the properties that, for each $r \in R$, X_r is a transversal for the $(\langle\, {}^{F}R \,\rangle \mathbf{C}_F(r))$-action on F by multiplication on the right and, moreover, $\bigcup\limits_{r \in R}({}^{X_r}r)$ is a free-generating set of $\langle\, {}^{F}R \,\rangle$.

A group is said to be *indicable* if it is trivial or has some quotient that is infinite and cyclic. A group is said to be *locally indicable* if all its finitely generated subgroups are indicable. For example, all free groups are locally indicable.

Cohen and Lyndon [**4**, Theorem 4.1] proved the following.

The original Cohen–Lyndon theorem. *If F is a free group, then each non-trivial element of F is Cohen–Lyndon aspherical in F.* □

Both the proof in [**4**] and its simplification by Karrass and Solitar [**13**, Theorem 2] use the famous induction method that was devised by Magnus in 1930 to study one-relator groups.

Howie [**8**, **9**, **11**] developed a powerful induction technique that amounts to the following: being given certain information about a locally indicable group; choosing an appropriate finitely generated subgroup that contains the given information; choosing an appropriate normal subgroup of the finitely generated subgroup for which the quotient group is infinite and cyclic; translating the given information into information about the normal subgroup; repeating this cycle as often as possible. This simple procedure has many applications. Magnus induction has a formally similar format but requires a free-generating set and more careful choices at each step. We shall give a self-contained review of a Bass–Serre theoretical form of Howie induction.

In [**6**, Theorem 1.1], Edjvet and Howie used the original Cohen–Lyndon theorem and Karrass–Solitar reduction to prove the following result.

The Edjvet–Howie theorem. *If A and B are locally indicable groups, then each cyclically reduced element of $A*B$ that does not lie in $A \cup B$ is Cohen–Lyndon aspherical in $A*B$.* □

In many situations, Howie induction has proved to be more powerful and more direct than Magnus induction; see, for example, Corollary 4.5. We found it unnatural that there remained two contexts where Howie induction failed to achieve the same results as Magnus induction, and the purpose of this paper is to remedy this situation. We shall use Howie induction and some of the arguments given by Edjvet and Howie [**6**], but neither Magnus induction nor the original Cohen–Lyndon theorem, to prove the following result. The examples immediately following the statement will show that this result is a common generalisation of the original Cohen–Lyndon theorem and the Edjvet–Howie theorem. (Experts will realise that, conversely, this result can be deduced from the original Cohen–Lyndon theorem and the Edjvet–Howie theorem by standard arguments.)

The local-indicability Cohen–Lyndon theorem. *If F is a locally indicable group and T is an F-tree with trivial edge stabilisers, then each element of F that fixes no vertex of T is Cohen–Lyndon aspherical in F.* □

Example 1.2. Let X be a set and let $F = \langle X \mid \rangle$.

Let T be the F-graph with vertex set F and edge set $F \times X$ such that each edge $(f, x) \in F \times X$ has initial vertex f and terminal vertex fx. In a natural way, F acts on T. The stabilisers are trivial. It is well known that T is a tree; see, for example [**5**, Theorem I.7.6].

Thus, the original Cohen–Lyndon theorem is the case of the local-indicability Cohen–Lyndon theorem where F acts freely on T.

Example 1.3. Let A and B be locally indicable groups and let $F = A*B$.

Let T be the F-graph with vertex set $(F/A) \vee (F/B)$ and edge set F such that each edge $f \in F$ has initial vertex fA and terminal vertex fB. In a natural way F acts on T. The edge stabilisers are trivial and the elements of F that fix vertices of T are the F-conjugates of the elements of $A \cup B$. It can be shown that T is a tree; see, for example [**5**, Theorem I.7.6].

Thus, the Edjvet–Howie theorem is the case of the local-indicability Cohen–Lyndon theorem where T has only one edge F-orbit and two vertex F-orbits.

In Section 2 we introduce definitions concerning staggerable subsets, strongly staggerable subsets and other concepts that we shall be using.

In Section 3 we describe the finite descending chain of subgroups of F used in Howie induction. We then see that the staggerable conditions can be moved all the way down the chain.

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In Section 4 we find that at the bottom of any of the chains of Section 3 the conditions of staggerability and local indicability interact to produce Whitehead subsets. This section reproduces parts of Appendix A of [**1**] with some modifications to suit the current applications.

In Section 5 we recall some technical results of Cohen and Lyndon, Karrass and Solitar and Edjvet and Howie. These results then allow us to take information that was spontaneously generated at the bottom of the chain and move it all the way back up the chain; the local-indicability Cohen–Lyndon theorem then follows.

Corollary 2.2 of [**4**] gives a sufficient condition for a subset of a free group F to be Cohen–Lyndon aspherical in F. In Section 6 we find that the preceding machinery implies the more general result that, for any locally indicable group F and any subset R of F, if there exists some F-tree T with trivial edge stabilisers such that R is strongly T-staggerable modulo F, then R is Cohen–Lyndon aspherical in F. Standard arguments then yield consequences concerning the quotient group $G := F/\langle\!\langle R\rangle\!\rangle$; for example, one obtains information about the torsion subgroups of G and the higher homology groups of G.

2. Staggerability. In this section we introduce definitions concerning staggerability and other concepts that we shall be using. Recall that F is a multiplicative group.

NOTATION 2.1. For any set X, we let $|X|$ denote the cardinal of X.

By an *ordering* of a set X, we shall mean a binary relation that *totally* orders X.

We will find it useful to have a notation for intervals in $\mathbb{Z}$ that is different from the notation for intervals in $\mathbb{R}$. Let $i, j \in \mathbb{Z}$. We write $[i\uparrow j] := \{k \in \mathbb{Z} \mid k \geqslant i \text{ and } k \leqslant j\}$, and $]-\infty\uparrow j] := \{k \in \mathbb{Z} \mid k \leqslant j\}$, and $[i\uparrow\infty[\ := \{k \in \mathbb{Z} \mid k \geqslant i\}$.

We shall define families of subscripted symbols by using the following convention. Let v be a symbol. For each $k \in \mathbb{Z}$, we let v_k denote the ordered pair (v, k), and, for each subset I of $\mathbb{Z}$, we let $v_I := (v_k \mid k \in I)$.

For two subsets Y and Z of a set X, the complement of $Y \cap Z$ in Y will be denoted by $Y - Z$ (and not by $Y \setminus Z$ since we let $F\backslash Z$ denote the set of F-orbits of a left F-set Z).

For any subset Y of a left F-set X, we write $\mathrm{glue}(F, Y) := \{f \in F \mid fY \cap Y \neq \emptyset\}$.

We write $F' := \langle\, \{[x, y] \mid x, y \in F\}\,\rangle \trianglelefteq F$ and $F^{\mathrm{ab}} := F/F'$, the abelianization of F.

DEFINITION 2.2. Let r be an element of F.

We say that r *has a unique root in* F if $r = 1$ or r lies in a unique maximal infinite cyclic subgroup of F.

If $r \neq 1$ and r lies in a unique maximal infinite cyclic subgroup C of F, we define *the unique root of* r *in* F, denoted by $\sqrt[F]{r}$, to be the unique generator of C of which r is a positive power. We say that 1 *is the unique root of* 1, and we define $\sqrt[F]{1} := 1$.

If $r \neq 1$ and $\mathbf{C}_F(r)$ is infinite and cyclic, then r has a unique root in F and $\langle\sqrt[F]{r}\,\rangle = \mathbf{C}_F(r)$.

We say that a subset R of F *has unique roots in* F if every element of R has a unique root in F, in which case we let $\sqrt[F]{R}$ denote the set of these unique roots in F.

We now define the staggerability concepts that we need. We shall use [**5**] as our reference for Bass–Serre theory.

DEFINITION 2.3. Let $T = (T, VT, ET, \iota, \tau)$ be an F-tree and suppose ET is F-free.

(i) Let r be an element of F that fixes no vertex of T.

There exists a unique minimal $\langle r \rangle$-subtree of T, which is denoted by axis(r) and has the form of a real line shifted by r; see, for example [**5**, Proposition I.4.11]. In particular, $\langle r \rangle \backslash \operatorname{axis}(r)$ is a finite cyclic graph, and $F \backslash (F(\operatorname{axis}(r)))$ is a quotient thereof. We write $E\operatorname{axis}(r) := E(\operatorname{axis}(r))$, and we shall be particularly interested in the finite set $F \backslash (F(E\operatorname{axis}(r))) := \{Fe \mid e \in E\operatorname{axis}(r)\} \subseteq F \backslash ET$.

For all $f \in F$, $\operatorname{axis}({}^f r) = f \operatorname{axis}(r)$ and $F \backslash (F(E\operatorname{axis}({}^f r))) = F \backslash (F(E\operatorname{axis}(r)))$.

For all $n \in \mathbb{Z} - \{0\}$, $\operatorname{axis}(r^n) = \operatorname{axis}(r)$.

As F acts freely on ET, it can be shown that $\mathbf{C}_F(r)$ acts freely on axis(r), and, by Bass–Serre theory, $\mathbf{C}_F(r)$ is infinite and cyclic. Here r has a unique root in F and $\langle r \rangle \subseteq \langle \sqrt[F]{r} \rangle = \mathbf{C}_F(r) \subseteq \operatorname{glue}(F, E\operatorname{axis}(r))$; notice that if $\operatorname{glue}(F, E\operatorname{axis}(r)) = \langle r \rangle$, then $\sqrt[F]{r} = r$.

(ii) If $<$ is some (total) ordering of ET and R is some subset of F, we say that R is $(T, <)$-*staggered modulo* F if the following three conditions hold.

(S1) Each F-orbit in ET is an interval in $(ET, <)$; there then exists a unique ordering of $F \backslash ET$, again denoted by $<$, with the property that for all $e, e' \in ET$, $Fe < Fe'$ if and only if $e < e'$ and $Fe \neq Fe'$.

(S2) Each element of R fixes no vertex of T.

(S3) For each $(r_1, r_2) \in R \times R$, exactly one of the following three conditions holds.

(a) ${}^F r_1 = {}^F r_2$.

(b) In $(F \backslash ET, <)$,
$\min(F \backslash (F(E\operatorname{axis}(r_1)))) < \min(F \backslash (F(E\operatorname{axis}(r_2))))$ and
$\max(F \backslash (F(E\operatorname{axis}(r_1)))) < \max(F \backslash (F(E\operatorname{axis}(r_2))))$.

(c) In $(F \backslash ET, <)$,
$\min(F \backslash (F(E\operatorname{axis}(r_2)))) < \min(F \backslash (F(E\operatorname{axis}(r_1))))$ and
$\max(F \backslash (F(E\operatorname{axis}(r_2)))) < \max(F \backslash (F(E\operatorname{axis}(r_1))))$.

We say that R is *strongly* $(T, <)$-*staggered modulo* F if, moreover, the following four conditions hold.

(S4) No two distinct elements of R are F-conjugate.

(S5) $F \backslash T$ has a unique maximal subtree.

(S6) $(F \backslash ET, <)$ is order isomorphic to an interval in $\mathbb{Z}$.

(S7) Any two $<$-consecutive edges in $(F \backslash ET, <)$ have a vertex in common in $F \backslash T$.

We have two types of examples in mind satisfying (S5), (S6) and (S7): cases where $F \backslash T$ has only one vertex and cases where $F \backslash T$ has the form of the real line.

(iii) A subset R of F is said to be T-*staggerable modulo* F if there exists some ordering $<$ of ET such that R is $(T, <)$-staggered modulo F.

By observations made in Definitions 2.3(i), we see that if R is T-staggerable modulo F, then R has unique roots in F.

Notice that if $|R| = 1$, then R is T-staggerable modulo F if and only if the unique element of R fixes no vertex of T.

A subset R of F is said to be *strongly* T-*staggerable modulo* F if there exists some ordering $<$ of ET such that R is strongly $(T, <)$-staggered modulo F.

REMARK 2.4. In the case where F is free, the concept of a staggered subset appeared in Howie's paper [**11**, p. 642] and was a generalisation, from the case where $|F \backslash VT| = 1$

and F is free, of a definition of a staggered presentation that is given in Lyndon and Schupp's book [**15**, p. 152]. In the case where $|F\backslash VT| = 1$ and F is free, the above concept of a strongly staggered subset corresponds to a definition that is given in [**15**, p. 104] of what could reasonably be called a strongly staggered presentation but is somewhat confusingly again called a 'staggered presentation'; such strongly staggered presentations arise unnamed in the hypotheses of Lyndon's (non-simple) identity theorem [**14**, Section 7] and Cohen and Lyndon's result [**4**, Corollary 2.2].

In Corollary 6.2, we show that, for any locally indicable group F and any subset R of F, if there exists some F-tree T with trivial edge stabilisers such that R is strongly T-staggerable modulo F, then R is Cohen–Lyndon aspherical in F. The case of Corollary 6.2 where $|F\backslash VT| = 1$ and F is free is precisely the Cohen–Lyndon result for strongly staggered presentations [**4**, Corollary 2.2]; see also [**15**, Proposition III.11.1].

Howie and Pride [**12**, p. 72] consider what they call a *staggered quotient* $F/\langle\!\langle R\rangle\!\rangle$ of a free product F, and this can be related to concepts considered here as follows. They express F as the fundamental group of a graph of groups [**12**, p. 73] and the resulting Bass–Serre tree T has an ordering $<$ on ET such that their R is $(T, <)$-staggered modulo F. However, this is only a small part of their requirements for a staggered quotient.

We shall be using the following well-known observation.

REMARK 2.5. Let X be a generating set of F, let T be an F-tree, and let Y be a subtree of T such that, for some vertex v of T, $Y \supseteq \{v\} \cup Xv$, or, more generally, such that $X \subseteq \mathrm{glue}(F, Y)$.

Consider the F-subforest FY of T, and let T' denote the component of FY that contains Y. The set $\{f \in F \mid fY \subseteq T'\}$ is closed under right multiplication by elements of $X \cup X^{-1}$ and contains 1, hence it is all of F. Thus, $FY = T'$. Hence, FY is an F-subtree of T. Moreover, the map $Y \to F\backslash T'$, $y \mapsto Fy$ is surjective.

3. Moving information down a chain. In this section, without mentioning local indicability, we shall see how to move a staggerable subset S down a finite chain of subgroups to a finitely generated group whose abelianization is virtually generated by S. The finite descending chain of subgroups is essentially the same chain of subgroups that was considered by Howie in his tower arguments in [**9**] and [**10**].

We shall often consider the following situation.

SETTING 3.1. Let S and Φ be finite subsets of F such that $\langle S\cup\Phi\rangle/\langle\!\langle S\rangle\!\rangle$ has no infinite, cyclic quotients.

Let T be an F-tree with trivial edge stabilisers.

Choose an arbitrary vertex v of T. Let Y be the smallest subtree of T that contains $\{v\} \cup Sv \cup \Phi v$. Then Y is finite. Since F acts freely on ET, $\mathrm{glue}(F, EY)$ is finite. We set $S^+ := S \cup \Phi \cup \mathrm{glue}(F, EY)$. Then S^+ is a finite subset of $\mathrm{glue}(F, Y)$. We set $\nu := |S^+| + 1$. For each subgroup H of F, we set $H^\dagger := \langle S^+ \cap H\rangle$. Notice that if H contains $S \cup \Phi$, then $H^\dagger$ also contains $S \cup \Phi$.

We set $F_0 := F$. Suppose that, for some $n \in [0{\uparrow}(\nu-1)]$, we have a subgroup F_n of F containing S. If $F_n^\dagger/\langle\!\langle S\rangle\!\rangle$ has no infinite, cyclic quotients, we choose $F_{n+1} := F_n^\dagger$; otherwise, we choose F_{n+1} to be an arbitrary normal subgroup of $F_n^\dagger$ that contains S such that $F_n^\dagger/F_{n+1}$ is infinite and cyclic. After ν such steps, we would have recursively chosen a finite, monotonically decreasing sequence $F_{[0\uparrow\nu]}$ of subgroups of F that contain S.

PROPOSITION 3.2. *In* Setting 3.1*, the following hold.*

(i) *$F_\nu = \langle S^+ \cap F_\nu \rangle$; $S \cup \Phi \subseteq F_\nu$; and, $F_\nu / \langle\!\langle S \rangle\!\rangle$ has no infinite, cyclic quotients.*

(ii) *If S is T-staggerable modulo F, then S is T-staggerable modulo F_ν.*

Proof. (i) If $n \in [0{\uparrow}(\nu-1)]$ and $S \cup \Phi \subseteq F_n$, then $S \cup \Phi \subseteq F_n^\dagger$ and, since $\langle S \cup \Phi \rangle / \langle\!\langle S \rangle\!\rangle$ has no infinite, cyclic quotients, $S \cup \Phi \subseteq F_{n+1}$. By induction, $S \cup \Phi \subseteq F_\nu$.

Consider any $n \in [0{\uparrow}(\nu-1)]$. If $F_{n+1} < F_n^\dagger \quad (= \langle S^+ \cap F_n \rangle \leqslant F_n)$, then $F_{n+1} \not\supseteq S^+ \cap F_n$, and, hence, $S^+ \cap F_{n+1} \subset S^+ \cap F_n$; notice that the number of those n for which the latter happens is at most $|S^+| = \nu-1$. It follows that there exists some $\mu \in [0{\uparrow}(\nu-1)]$ such that $F_{\mu+1} = F_\mu^\dagger$. Then $F_\mu^\dagger / \langle\!\langle S \rangle\!\rangle$ has no infinite, cyclic quotients, and

$$F_\mu^\dagger = \langle S^+ \cap F_\mu \rangle = \langle S^+ \cap F_\mu^\dagger \rangle = \langle S^+ \cap F_{\mu+1} \rangle = F_{\mu+1}^\dagger.$$

It then follows that the sequence $F^\dagger_{[\mu\uparrow\nu]}$ is constant, and (i) follows.

(ii) By hypothesis, S is T-staggerable modulo $F_0 \quad (= F)$.

Let n be an element of $[0{\uparrow}(\nu-1)]$ such that S is T-staggerable modulo F_n. By induction, it suffices to show that S is T-staggerable modulo F_{n+1}.

There exists some ordering $<$ of ET such that S is $(T, <)$-staggered modulo F_n. By altering the ordering $<$ of ET within each F_n-orbit, we can arrange that every $F_n^\dagger$-orbit is an interval with respect to $<$. We claim that S is now $(T, <)$-staggered modulo $F_n^\dagger$. It suffices to show that if two elements of S are F_n-conjugate, then they are $F_n^\dagger$-conjugate. Suppose we have $f \in F_n$ and $r, {}^f r \in S \subseteq F_n^\dagger$. Now $\operatorname{axis}(r) \subseteq \langle r \rangle(Y) \subseteq F_n^\dagger(Y)$ and $f \operatorname{axis}(r) = \operatorname{axis}({}^f r) \subseteq \langle {}^f r \rangle(Y) \subseteq F_n^\dagger(Y)$. Hence, $f \in \operatorname{glue}(F_n, F_n^\dagger(EY))$. Thus, $F_n^\dagger f F_n^\dagger$ meets

$$\operatorname{glue}(F_n, EY) \subseteq S^+ \cap F_n \subseteq \langle S^+ \cap F_n \rangle = F_n^\dagger.$$

Hence, $f \in F_n^\dagger$. Thus, S is $(T, <)$-staggered modulo $F_n^\dagger$.

If $F_{n+1} = F_n^\dagger$, then S is $(T, <)$-staggered modulo F_{n+1} as desired. Thus, we may assume that $F_n^\dagger / F_{n+1}$ is infinite and cyclic. There exists some $z \in F_n^\dagger$ such that zF_{n+1} generates $F_n^\dagger / F_{n+1}$. Then $F_n^\dagger = \langle z \rangle F_{n+1}$. Since $F_n^\dagger / F_{n+1}$ is infinite, $\langle z \rangle \cap F_{n+1} = \{1\}$. Within each $F_n^\dagger$-orbit in ET, the F_{n+1}-orbits are permuted by z and form a single $\langle z \rangle$-orbit. By altering the ordering $<$ of ET within each $F_n^\dagger$-orbit, we can arrange that every F_{n+1}-orbit is an interval with respect to $<$ such that z moves each F_{n+1}-orbit to the next $<$-largest orbit. It suffices to examine two elements of S that are $F_n^\dagger$-conjugate. Suppose we have $f \in F_{n+1}$ and $i \in \mathbb{Z}$ and $r \in S$ such that ${}^{z^i f} r \in S$. If $i = 0$, then r and ${}^{z^i f} r$ are F_{n+1}-conjugate, which is one of the desired possibilities. We now assume that $i > 0$; the case where $i < 0$ is similar. Now

$$F_{n+1} \backslash (F_{n+1}(E\operatorname{axis}({}^{z^i f} r))) = F_{n+1} \backslash (F_{n+1}(z^i f\, E\operatorname{axis}(r))) = z^i (F_{n+1} \backslash (F_{n+1}(E\operatorname{axis}(r)))).$$

Thus,

$$\begin{aligned}
\min(F_{n+1} \backslash (F_{n+1}(E\operatorname{axis}({}^{z^i f} r)))) &= z^i \min(F_{n+1} \backslash (F_{n+1}(E\operatorname{axis}(r)))) \\
&> \min(F_{n+1} \backslash (F_{n+1}(E\operatorname{axis}(r)))), \\
\max(F_{n+1} \backslash (F_{n+1}(E\operatorname{axis}({}^{z^i f} r)))) &= z^i \max(F_{n+1} \backslash (F_{n+1}(E\operatorname{axis}(r)))) \\
&> \max(F_{n+1} \backslash (F_{n+1}(E\operatorname{axis}(r)))).
\end{aligned}$$

Hence, S is $(T, <)$-staggered modulo F_{n+1}, as desired. □

4. New information generated at the bottom of a chain. In this section we review, with some changes, the main results of [**1**, Appendix A]. The following is a minor extension of [**1**, Lemma A.2.1].

LEMMA 4.1. *Let F be a finitely generated, locally indicable group, let T be an F-tree with trivial edge stabilisers, and let S be a subset of F such that S is T-staggerable modulo F and $\sqrt[F]{S} = S$. Then the following hold.*

(i) *$F/\langle\!\langle S \rangle\!\rangle$ is indicable.*
(ii) *If $F/\langle\!\langle S \rangle\!\rangle$ is trivial, then, for each $r \in S$, $\mathrm{glue}(F, E\mathrm{axis}(r)) = \langle r \rangle$, and F acts freely on T and ${}^F S$ is a Whitehead subset of F.*

Proof. Recall from Definitions 2.3 that S has unique roots in F.

Without loss of generality, we may replace S with ${}^F S$, and we then have ${}^F S = S$.

By hypothesis, there exists some ordering $<$ of ET such that S is $(T, <)$-staggered modulo F.

Let X be a finite generating set of F, let v be a vertex of T, let Y be the smallest subtree of T containing $\{v\} \cup Xv$ and let $T' := FY$. By Remark 2.5, T' is an F-subtree of T and $F\backslash T'$ is finite. Consider any $r \in S$. Then T' is an $\langle r \rangle$-subtree of T. Since $\mathrm{axis}(r)$ is the unique smallest $\langle r \rangle$-subtree of T, we see that T' contains $\mathrm{axis}(r)$. It follows that S is $(T', <)$-staggered modulo F. It now suffices to prove the result with T' in place of T. Thus, we may assume that $F\backslash T$ is finite.

By induction, we may assume that the result holds for all smaller values of $|F\backslash ET|$.

Since F is a finitely generated, locally indicable group, F is indicable, and it can be seen that (i) and (ii) hold when S is empty. Thus, we may assume that S is non-empty. It then follows from Definition 2.3 that $|F\backslash ET| \geqslant 1$. In particular, there exists some $e_{\max} \in \bigcup_{r \in S} E\mathrm{axis}(r)$ such that in $(F\backslash ET, <)$,

$$Fe_{\max} = \max\{Fe \mid e \in \bigcup_{r \in S} E\mathrm{axis}(r)\}.$$

There then exists some $r_{\max} \in S$ such that $e_{\max} \in E\mathrm{axis}(r_{\max})$, and, by the definition of $(T, <)$-staggered modulo F, $Fe_{\max}$ does not meet the axis of any element of $S - {}^F(r_{\max})$. Thus, there exists some pair (r, e), for example $(r, e) = (r_{\max}, e_{\max})$, such that the following hold.

$$r \in S,\ e \in E\mathrm{axis}(r) \text{ and } Fe \text{ does not meet the axis of any element of } S - {}^F r. \quad (1)$$

$$\text{If } \mathrm{glue}(F, E\mathrm{axis}(r)) \neq \langle r \rangle, \text{ then } (r, e) = (r_{\max}, e_{\max}). \quad (2)$$

We consider the F-forest $T - Fe$. Let T_ι denote the component of $T - Fe$ containing ιe, and let T_τ denote the component of $T - Fe$ containing τe. Let F_ι denote the F-stabiliser of $\{T_\iota\}$, and let F_τ denote the F-stabiliser of $\{T_\tau\}$, where we are using set brackets to emphasise that we want to consider each component of $T - Fe$ as a single element. Let $S_\iota := S \cap F_\iota$ and $S_\tau := S \cap F_\tau$. By (1), for each $r' \in S - {}^F r$, $\mathrm{axis}(r')$ lies in $T - Fe$ and hence lies in a component of $T - Fe$. It follows that ${}^F S_\iota \cup {}^F S_\tau \cup {}^F r$ is all of S. Notice that if $F\{T_\iota\} = F\{T_\tau\}$, then ${}^F S_\iota = {}^F S_\tau$.

By applying the Bass–Serre structure theorem to the F-tree whose vertices are the components of $T - Fe$ and whose edge set is Fe, with fe joining fT_ι to fT_τ $(f \in F)$,

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we find that

$$F = \begin{cases} F_\iota * F_\tau & \text{if } F\{T_\iota\} \neq F\{T_\tau\}, \\ F_\iota * \langle f \mid \quad \rangle & \text{if } f \in F \text{ and } fT_\iota = T_\tau. \end{cases} \tag{3}$$

Hence,

$$F/\langle\!\langle S - {}^F r \rangle\!\rangle = \begin{cases} (F_\iota/\langle\!\langle S_\iota \rangle\!\rangle) * (F_\tau/\langle\!\langle S_\tau \rangle\!\rangle) & \text{if } F\{T_\iota\} \neq F\{T_\tau\}, \\ (F_\iota/\langle\!\langle S_\iota \rangle\!\rangle) * \langle f \mid \quad \rangle & \text{if } f \in F \text{ and } fT_\iota = T_\tau. \end{cases} \tag{4}$$

Case 1. Both $F_\iota/\langle\!\langle S_\iota \rangle\!\rangle$ and $F_\tau/\langle\!\langle S_\tau \rangle\!\rangle$ have infinite, cyclic quotients.

By (4), $F/\langle\!\langle S - {}^F r \rangle\!\rangle$ has a rank-two, free-abelian quotient. On incorporating r into the normal subgroup being quotiented out, we see that $F/\langle\!\langle S \rangle\!\rangle$ has an infinite, cyclic quotient, and, hence (i) and (ii) hold in this case.

Case 2. One of $F_\iota/\langle\!\langle S_\iota \rangle\!\rangle$, $F_\tau/\langle\!\langle S_\tau \rangle\!\rangle$ does not have an infinite, cyclic quotient.

By reversing the orientation of every edge of T, if necessary, we may assume that $F_\iota/\langle\!\langle S_\iota \rangle\!\rangle$ does not have an infinite, cyclic quotient.

By the induction hypothesis applied to $(F_\iota, T_\iota, S_\iota)$, we see that $F_\iota/\langle\!\langle S_\iota \rangle\!\rangle$ is trivial, F_ι acts freely on T_ι, and some transversal for the F_ι-action on S_ι by conjugation is a (finite) free-generating set of F_ι, and for each $r_\iota \in S_\iota$, glue$(F_\iota, E\text{axis}(r_\iota)) = \langle r_\iota \rangle$ and, hence, glue$(F, E\text{axis}(r_\iota)) = \langle r_\iota \rangle$.

By inverting every element of S, if necessary, we may assume the following.

$$\text{There exists some segment of axis}(r) \text{ of the form } e, p, re. \tag{5}$$

Consider the case where $F\{T_\iota\} \neq F\{T_\tau\}$. By (5), we have a path $\bar{r}p$ in axis(r) from $\bar{r}\tau e \in \bar{r}T_\tau \neq T_\iota$ to $\iota e \in T_\iota$. Now $\bar{r}p$ necessarily enters T_ι through an oriented edge of the form ge^{-1} for some $g \in F_\iota$, and g then lies in glue$(F_\iota, E\text{axis}(r)) - \langle r \rangle$. This proves the following.

$$\text{If glue}(F_\iota, E\text{axis}(r)) \subseteq \langle r \rangle \text{ then } F\{T_\iota\} = F\{T_\tau\}. \tag{6}$$

Consider the case where S_ι is empty. Here $F_\iota = F_\iota/\langle\!\langle S_\iota \rangle\!\rangle = \{1\}$. By (6), $F\{T_\iota\} = F\{T_\tau\}$, and by (3), there exists some $t \in F$ such that $F = \langle t \mid \rangle$. Now $\{t, \bar{t}\} = \sqrt[F]{F} - \{1\} \supseteq S = \{r\}$. Hence, $F = \langle r \mid \rangle$, and glue$(F, E\text{axis}(r)) = \langle r \rangle$. This proves the following.

$$\text{If glue}(F, E\text{axis}(r)) \neq \langle r \rangle, \text{ then } S_\iota \text{ is non-empty.} \tag{7}$$

Case 2.1. glue$(F, E\text{axis}(r)) = \langle r \rangle$.

Here, by (6), $F\{T_\iota\} = F\{T_\tau\}$. Hence, $F(VT_\iota) = VT$ and $S = {}^F S_\iota \cup {}^F r$. Consider any $v \in VT$. We wish to show that $F_v = 1$, and we may assume that $v \in VT_\iota$. Here $F_v \leqslant F_\iota$, and since F_ι acts freely on T_ι, $F_v = 1$, as desired. Thus, F acts freely on T. In (5), the path p from τe to $r\iota e$ in axis(r) does not meet Fe since glue$(F, E\text{axis}(r)) = \langle r \rangle$, and, hence, p stays within T_τ, and $r\iota e \in T_\tau$. Thus, $rT_\iota = T_\tau$, and, by (3), $F = F_\iota * \langle r \mid \rangle$. Since

$F_\iota/\langle\!\langle S_\iota\rangle\!\rangle$ is trivial, we see that $F/\langle\!\langle S\rangle\!\rangle$ is trivial, and hence $F/\langle\!\langle S\rangle\!\rangle$ is indicable. Here all the required conclusions hold.

Case 2.2. glue$(F, E\text{axis}(r)) \neq \langle r\rangle$.

By (7), S_ι is non-empty, and hence there exists some $e_\iota \in \bigcup_{r_\iota\in S_\iota} E\text{axis}(r_\iota)$ such that

$$Fe_\iota = \min\{Fe \mid e \in \bigcup_{r_\iota\in S_\iota} E\text{axis}(r_\iota)\}.$$

There then exists some $r_\iota \in S_\iota$ such that $e_\iota \in E\text{axis}(r_\iota)$, and we then know that glue$(F, E\text{axis}(r_\iota)) = \langle r_\iota\rangle$. By (2), $(r, e) = (r_{\max}, e_{\max})$. Using the definition of $(T, <)$-staggered modulo F, one can show that Fe_ι does not meet the axis of any element of ${}^F r$, and similarly Fe_ι does not meet the axis of any element of ${}^F S_\iota - {}^F r_\iota$. It is clear that if ${}^F S_\tau \neq {}^F S_\iota$, then Fe_ι does not meet the axis of any element of ${}^F S_\tau$. Hence, Fe_ι does not meet the axis of any element of $S - {}^F r_\iota$. We now replace (r, e) with (r_ι, e_ι) in (1) and (2) and find that the same argument as before now terminates in Case 1 or Case 2.1. Hence, all the desired conclusions hold.

This completes the proof. $\square$

COROLLARY 4.2. *In* Setting 3.1, *suppose that the following hold: F is locally indicable; S is T-staggerable modulo F and $\sqrt[F]{S} = S$. Then the following hold: $S \cup \Phi \subseteq F_\nu$; F_ν acts freely on T; and ${}^{F_\nu}S$ is a Whitehead subset of F_ν.*

Proof. Recall from Definitions 2.3 that S has unique roots in F.

By Proposition 3.2(i), F_ν is finitely generated and $S \cup \Phi \subseteq F_\nu$ and $F_\nu/\langle\!\langle S\rangle\!\rangle$ has no infinite, cyclic quotients. By Proposition 3.2(ii), S is T-staggerable modulo F_ν. By Lemma 4.1(i), $F_\nu/\langle\!\langle S\rangle\!\rangle$ is indicable, and hence trivial. Now the result holds by Lemma 4.1(ii). $\square$

The next result is repeated from Corollary A.2.3 of **[1]**; Section A.3 of **[1]** describes related results of Magnus, Brodskiĭ, Howie, Short and others.

THEOREM 4.3. *Let F be a locally indicable group and let T be an F-tree with trivial edge stabilisers and let R be a subset of F that is T-staggerable modulo F.*

Then the following hold.

(i) (The local-indicability Freiheitssatz) *$\langle {}^F R\rangle$ acts freely on T, and hence $\langle {}^F R\rangle$ is a free group and the F-stabiliser of each vertex of T embeds in $F/\langle\!\langle R\rangle\!\rangle$ under the natural map.*

(ii) *$F/\langle\!\langle \sqrt[F]{R}\rangle\!\rangle$ is locally indicable.*

Proof. Since $\langle {}^F R\rangle \leqslant \langle {}^F(\sqrt[F]{R})\rangle$, we may replace R with $\sqrt[F]{R}$; then we have $\sqrt[F]{R} = R$.

(i) Let S be an arbitrary finite subset of ${}^F R$ and let Φ be the empty set. We may then assume that we are in Setting 3.1. By Corollary 4.2, $\langle S\rangle \leqslant F_\nu$ and F_ν acts freely on T. Hence, $\langle S\rangle$ acts freely on T, and since S is an arbitrary finite subset of ${}^F R$, we see that $\langle {}^F R\rangle$ acts freely on T, that is, the F-stabiliser of each vertex of T embeds in $F/\langle\!\langle R\rangle\!\rangle$ under the natural map. By Reidemeister's theorem, or by Bass–Serre theory, $\langle {}^F R\rangle$ is a free group.

(ii) Let H be an arbitrary finitely generated subgroup of $F/\langle {}^F R\rangle$. It suffices to show that H is indicable. We may assume that H has no infinite, cyclic quotients. Then H^{ab} is finite. Let d denote the exponent of H^{ab}. There exists some finite subset Φ of F such that $H = (\langle\Phi\rangle\langle {}^F R\rangle)/\langle {}^F R\rangle$. Now $H' = (\langle\Phi\rangle'\langle {}^F R\rangle)/\langle {}^F R\rangle$ and we see that $\Phi^d \subseteq \langle\Phi\rangle'\langle {}^F R\rangle$. Hence, there exists some finite subset S of ${}^F R$ such that the finite set

Φ^d lies in $\langle \Phi \rangle' \langle S \rangle$. Then the abelian group $(\langle \Phi \cup S \rangle / \langle\!\langle S \rangle\!\rangle)^{\mathrm{ab}}$ has exponent at most d, and hence $\langle \Phi \cup S \rangle / \langle\!\langle S \rangle\!\rangle$ has no infinite, cyclic quotients. We may then assume that we are in Setting 3.1. By Corollary 4.2, $\Phi \subseteq F_\nu = \langle {}^{F_\nu} S \rangle \leqslant \langle {}^F R \rangle$, and hence H is trivial as desired. □

COROLLARY 4.4 (Brodskiĭ–Howie–Short). *Suppose that A and B are locally indicable groups and that r is an element of $A*B$. If no $A*B$-conjugate of r lies in A, then the natural map embeds A in $(A*B)/\langle\!\langle r \rangle\!\rangle$.*

COROLLARY 4.5 (Magnus' Freiheitssatz). *Suppose that A and B are free groups and that r is an element of $A*B$. If no $A*B$-conjugate of r lies in A, then the natural map embeds A in $(A*B)/\langle\!\langle r \rangle\!\rangle$.*

5. Moving the new information back up the chain. In this section we recover the local-indicability Cohen–Lyndon theorem.

We begin by recording some general results about Whitehead subsets.

LEMMA 5.1. *Let Y be a non-empty subset of F.*

Then Y is a Whitehead subset of F if and only if all of the following hold: each $y \in Y$ freely generates $\mathbf{C}_F(y)$; Y generates F; Y is closed under the F-conjugation action; and Y with the F-conjugation action is the vertex set of an F-tree with trivial edge stabilisers.

Proof. Suppose that Y is a Whitehead subset of F, and let X be a free-generating set of F such that $Y = {}^F X$. Then Y generates F, Y is closed under the F-conjugation action and each $y \in Y$ generates a non-trivial free factor of F and, hence, freely generates $\mathbf{C}_F(y)$. We can express F as the fundamental group of a tree of groups in which the edge groups are trivial and the family of vertex groups is $(\langle x \mid \rangle \mid x \in X)$. Let T denote the corresponding Bass–Serre tree. Thus, $VT = \bigvee\limits_{x \in X} F/\langle x \rangle \simeq \bigvee\limits_{x \in X} {}^F x = {}^F X = Y$, and ET is some free F-set.

Conversely, suppose Y generates F and each $y \in Y$ freely generates $\mathbf{C}_F(y)$, and Y is closed under the F-conjugation action, and Y with the F-conjugation action is the vertex set of an F-tree T with trivial edge stabilisers. By the Bass–Serre structure theorem, there exists some graph of groups $(\mathcal{F}, \overline{T})$ and some maximal subtree $\overline{T}_0$ of $\overline{T}$ such that F is the fundamental group of $(\mathcal{F}, \overline{T}, \overline{T}_0)$. By the centraliser condition, the family of vertex groups can be expressed as $(\langle x \mid \rangle \mid x \in X)$ for some transversal X for the F-conjugation action on Y. Since F is generated by the F-conjugates of vertex groups, it follows that $\overline{T} = \overline{T}_0$. Hence, F is the free product of the family of vertex groups. Thus, X is a free-generating set of F and $Y = {}^F X$, as desired. □

This yields the following result, which we shall use in the proof of Theorem 5.3.

COROLLARY 5.2. *Let R be a non-empty subset of F.*

Then R is Cohen–Lyndon aspherical in F if and only if all of the following hold: no two distinct elements of R are F-conjugate; each $r \in R$ freely generates $\mathbf{C}_{\langle {}^F R \rangle}(r)$; and ${}^F R$ with the $\langle {}^F R \rangle$-conjugation action is the vertex set of an $\langle {}^F R \rangle$-tree with trivial edge stabilisers.

We now give a generalisation of results of Cohen and Lyndon [**4**, Lemma 2.1] and Karrass and Solitar [**13**, Theorem 1] that is similar to results of Chiswell, Collins and Huebschmann [**3**, Theorem 4.6].

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THEOREM 5.3. *Let $(\mathcal{F}, Y)$ be a graph of groups with family of groups $(\mathcal{F}(y) \mid y \in Y)$ and family of edge maps $(\bar{t}_e\colon \mathcal{F}(e) \to \mathcal{F}(\tau_Y e) \mid e \in EY)$, let Y_0 be a maximal subtree of Y and let F be the fundamental group of $(\mathcal{F}, Y, Y_0)$.*

Let $R := (r_v \mid v \in VY)$ be a family of elements of F with the property that, for each $v \in VY$, all of the following hold: $r_v \in \mathcal{F}(v)$; for each $e \in \iota_Y^{-1}(\{v\})$, $\langle {}^{\mathcal{F}(v)} r_v \rangle \cap \mathcal{F}(e) = \{1\}$; and for each $e \in \tau_Y^{-1}(\{v\})$, $\langle {}^{\mathcal{F}(v)} r_v \rangle \cap {}^{\bar{t}_e}\mathcal{F}(e) = \{1\}$.

Then for each $v \in VY$, $\langle {}^F R \rangle \cap \mathcal{F}(v) = \langle {}^{\mathcal{F}(v)} r_v \rangle$.

Let $U := \{v \in VY \mid r_v \neq 1\}$. Then $\{r_u \mid u \in U\}$ is Cohen–Lyndon aspherical in F if and only if, for each $u \in U$, r_u is Cohen–Lyndon aspherical in $\mathcal{F}(u)$.

Proof. For each $v \in VY$, let $\overline{\mathcal{F}}(v) := \mathcal{F}(v)/\langle\!\langle r_v \rangle\!\rangle$, and for each $e \in EY$, let $\overline{\mathcal{F}}(e) := \mathcal{F}(e)$. By the hypotheses on R, the edge maps for $\mathcal{F}$ then induce injective maps that give a graph of groups $(\overline{\mathcal{F}}, Y)$ with family of groups $(\overline{\mathcal{F}}(y) \mid y \in Y)$. Let $\overline{F}$ denote the fundamental group of $(\overline{\mathcal{F}}, Y, Y_0)$. Using the definition of fundamental groups, we then construct a natural group homomorphism $F \to \overline{F}$, which is surjective and has kernel $\langle {}^F R \rangle$. In particular, for each $v \in VY$, $\langle {}^F R \rangle \cap \mathcal{F}(v) = \langle {}^{\mathcal{F}(v)} r_v \rangle$.

Let T denote the Bass–Serre tree for $(\mathcal{F}, Y, Y_0)$, and let $\overline{T}$ denote the Bass–Serre tree for $(\overline{\mathcal{F}}, Y, Y_0)$. Using the definition of Bass–Serre trees, we construct a natural identification $\langle {}^F R \rangle \backslash T = \overline{T}$. Using the Bass–Serre structure theorem for $\langle {}^F R \rangle$ acting on T, we see that $\langle {}^F R \rangle$ is the fundamental group of a tree of groups over $\overline{T}$, with trivial edge groups, and with family of vertex groups of the form $(\langle {}^F R \rangle \cap {}^z\mathcal{F}(v) \mid v \in VY, z \in Z_v)$, where, for each $v \in VY$, Z_v is some transversal for the $\langle {}^F R \rangle \mathcal{F}(v)$-action on F by multiplication on the right. Thus, $\langle {}^F R \rangle = \mathop{\ast}\limits_{v \in VY} \mathop{\ast}\limits_{z \in Z_v} \langle {}^{z\mathcal{F}(v)} r_v \rangle = \mathop{\ast}\limits_{u \in U} \mathop{\ast}\limits_{z \in Z_u} \langle {}^{z\mathcal{F}(u)} r_u \rangle$.

We claim that for each $u \in U$, $\mathbf{C}_F(r_u) = \mathbf{C}_{\mathcal{F}(u)}(r_u)$. If $f \in \mathbf{C}_F(r_u)$, then r_u fixes the path in T from $1\mathcal{F}(u)$ to $f\mathcal{F}(u)$, and, by the hypotheses, r_u fixes no edges of T. Hence, the path under consideration is trivial, and hence $f \in \mathcal{F}(u)$, as claimed.

Suppose that, for each $u \in U$, r_u is Cohen–Lyndon aspherical in $\mathcal{F}(u)$, and let Y_u be a transversal for the $\langle {}^{\mathcal{F}(u)} r_u \rangle \mathbf{C}_{\mathcal{F}(u)}(r_u)$-action on $\mathcal{F}(u)$ by multiplication on the right such that ${}^{Y_u} r_u$ is a free-generating set of $\langle {}^{\mathcal{F}(u)} r_u \rangle$. Then $Z_u Y_u$ is a transversal for the $\langle {}^F R \rangle \mathbf{C}_F(r_u)$-action on F by multiplication on the right and $\bigvee\limits_{u \in U} {}^{Z_u Y_u} r_u$ is a free-generating set of $\langle {}^F R \rangle$. Thus, R is Cohen–Lyndon aspherical in F.

Conversely, suppose R is Cohen–Lyndon aspherical in F and u is an element of U. By Corollary 5.2, each $r \in R$ freely generates $\mathbf{C}_{\langle {}^F R \rangle}(r)$, no two distinct elements of R are F-conjugate and there exists some $\langle {}^F R \rangle$-tree T' with vertex set ${}^F R$ and with trivial edge stabilisers. By Corollary 5.2, it remains to show that ${}^{\mathcal{F}(u)} r_u$ is the vertex set of some $\langle {}^{\mathcal{F}(u)} r_u \rangle$-tree with trivial edge stabilisers. The subgroup $\langle {}^{\mathcal{F}(u)} r_u \rangle$ of $\langle {}^F R \rangle$ acts on T' and for each vertex of T', say ${}^f r_{u'}$ with $f \in F$ and $u' \in U$, the $\langle {}^{\mathcal{F}(u)} r_u \rangle$-stabiliser is $\langle {}^{\mathcal{F}(u)} r_u \rangle \cap {}^f(\mathbf{C}_{\mathcal{F}(u')}(r_{u'}))$. The latter intersection fixes the path in T from $1\mathcal{F}(u)$ to $f\mathcal{F}(u')$, and hence is trivial unless $u' = u$ and $f \in \mathcal{F}(u)$. Thus, the set of vertices of T' with non-trivial $\langle {}^{\mathcal{F}(u)} r_u \rangle$-stabiliser is the subset ${}^{\mathcal{F}(u)} r_u$ of ${}^F R$ $(= VT')$. Since the vertex stabilisers then generate $\langle {}^{\mathcal{F}(u)} r_u \rangle$, we see that $\langle {}^{\mathcal{F}(u)} r_u \rangle \backslash T'$ is a tree. Hence, successively $\langle {}^{\mathcal{F}(u)} r_u \rangle$-equivariantly contracting suitable edges of T' produces a $\langle {}^{\mathcal{F}(u)} r_u \rangle$-tree with vertex set ${}^{\mathcal{F}(u)} r_u$ and trivial edge stabilisers, as desired. $\square$

REMARK 5.4. If A and B are groups and r is an element of A, it follows from Theorem 5.3 that r is Cohen–Lyndon aspherical in A if and only if r is Cohen–Lyndon aspherical in $A \ast B$.

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COROLLARY 5.5. *Suppose A and B are groups and A_1 is a subgroup of A and B_1 is a subgroup of B and r is an element of $A_1{*}B_1$ and the natural maps embed A_1 and B_1 in $(A_1{*}B_1)/\langle\!\langle r\rangle\!\rangle$.*

If r is Cohen–Lyndon aspherical in $A_1{*}B_1$, then r is Cohen–Lyndon aspherical in $A{*}B$.

Proof. By applying Theorem 5.3 to $(A_1{*}B_1){*}_{B_1}B \quad (= A_1{*}B)$, we see that r is Cohen–Lyndon aspherical in $A_1{*}B$ and also the natural map embeds $(A_1{*}B_1)/\langle\!\langle r\rangle\!\rangle$ in $(A_1{*}B)/\langle\!\langle r\rangle\!\rangle$. Hence, the natural map embeds A_1 in $(A_1{*}B)/\langle\!\langle r\rangle\!\rangle$.

By applying Theorem 5.3 to $(A_1{*}B){*}_{A_1}A \quad (= A{*}B)$, we see that r is Cohen–Lyndon aspherical in $A{*}B$. □

COROLLARY 5.6. *Suppose F is locally indicable and T is an F-tree with trivial edge stabilisers and r is an element of F that fixes no vertex of T.*

Let H be a subgroup of F such that $H \supseteq \mathrm{glue}(F, E\mathrm{axis}(r))$ and $H = \langle\,\mathrm{glue}(H, \mathrm{axis}(r))\,\rangle$. If r is Cohen–Lyndon aspherical in H, then r is Cohen–Lyndon aspherical in F.

Proof. Let e be an edge in $\mathrm{axis}(r)$. Let $\overline{T}$ denote the F-tree obtained from T by collapsing all edges in $ET - Fe$. Then $E\overline{T} = Fe$ and r shifts e and $E\mathrm{axis}_{\overline{T}}(r) = Fe \cap E\mathrm{axis}_T(r)$ and $\mathrm{glue}(H, \mathrm{axis}_T(r)) \subseteq \mathrm{glue}(H, \mathrm{axis}_{\overline{T}}(r))$. It follows that we can replace T with $\overline{T}$ and assume that $ET = Fe$.

Let $T' = H(\mathrm{axis}(r))$. By Remark 2.5, T' is an H-subtree of T.

For each $e' \in E\mathrm{axis}(r)$, there exists some $f \in F$ such that $fe = e'$, whence $f \in \mathrm{glue}(F, E\mathrm{axis}(r)) \subseteq H$ and $e' = fe \in He$. Hence, $E\mathrm{axis}(r) \subseteq He$. Hence, $ET' = H(E\mathrm{axis}(r)) \subseteq He$. In summary, $ET = Fe$ and $ET' = He$.

Case 1. $|F\backslash(VT)| = 2$.

Here, we can write $F = A{*}B$ and no F-conjugate of r lies in $A \cup B$ and $H = A_1{*}B_1$ for some $A_1 \leqslant A$ and some $B_1 \leqslant B$. By Corollaries 4.4 and 5.5, r is Cohen–Lyndon aspherical in $A{*}B \ (= F)$.

Case 2. $|F\backslash(VT)| = 1$.

Here we can write $F = A * (b\colon \{1\} \to \{1\})$ and no F-conjugate of r lies in A.

Case 2.1. $|H\backslash(VT')| = 2$.

Here $H = A_1{*}{}^bA_2$ for some $A_1 \leqslant A$ and some $A_2 \leqslant A$.

Then r is Cohen–Lyndon aspherical in $A{*}{}^bA$ by Corollaries 4.4 and 5.5.

Now r is Cohen–Lyndon aspherical in $(A{*}{}^bA) * (b\colon A \to {}^bA) \quad (= A{*}\langle\, b \mid \ \rangle = F)$, by Corollary 4.4 and Theorem 5.3.

Case 2.2. $|H\backslash(VT')| = 1$.

Here $H = A_1{*}(ba\colon \{1\} \to \{1\})$ for some $A_1 \leqslant A$ and some $a \in A$.

If no F-conjugate of r lies in $\langle\, ba \mid \ \rangle$, then r is Cohen–Lyndon aspherical in $A{*}\langle\, ba \mid \ \rangle \quad (= A{*}\langle\, b \mid \ \rangle = F)$ by Corollaries 4.4 and 5.5.

Thus, we may assume that r itself lies in $\langle\, ba \mid \ \rangle$ and is then clearly Cohen–Lyndon aspherical in $\langle\, ba \mid \ \rangle$. Hence, by Remark 5.4, r is Cohen–Lyndon aspherical in $A{*}\langle\, ba \mid \ \rangle \quad (= A{*}\langle\, b \mid \ \rangle = F)$.

Hence, in all cases, r is Cohen–Lyndon aspherical in F. □

NOTATION 5.7. If a and b are elements of F, we shall let ${}^{\langle a|\,\rangle}b := ({}^{a^i}b \mid i \in \mathbb{Z})$. For any subset J of $\mathbb{Z}$, we let ${}^{a^J}b := ({}^{a^j}b \mid j \in J) \subseteq {}^{\langle a|\,\rangle}b$.

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The following will be useful for simplifying calculations.

LEMMA 5.8. *Suppose that f is an element of F and that N is a normal subgroup of F such that $F/N = \langle fN \mid \rangle$. Let z be a symbol, let $\tilde{F} := F{*}\langle z \mid \rangle$ and let $\tilde{N}$ denote the smallest normal subgroup of $\tilde{F}$ containing $N \cup \{f\bar{z}\}$. Then $\tilde{F}/\tilde{N} = \langle f\tilde{N} \mid \rangle = \langle z\tilde{N} \mid \rangle$ and N is a free factor of $\tilde{N}$.*

Proof. It is easy to see that $F = \langle f \mid \rangle \ltimes N$ and $\tilde{F}/\tilde{N} = \langle f\tilde{N} \mid \rangle = \langle z\tilde{N} \mid \rangle$. It is not difficult to use generators and relations to check that $\tilde{F} = F{*}\langle z \mid \rangle = \langle f \mid \rangle \ltimes (N{*}\langle {}^{\langle f \mid \rangle}(f\bar{z}) \mid \rangle)$, and hence $\tilde{N} = N{*}\langle {}^{\langle f \mid \rangle}(f\bar{z}) \mid \rangle$. □

The following is proved but not stated in **[6]**; here, we give an argument with a different rewriting procedure to provide some variety.

LEMMA 5.9. (Edjvet–Howie) *Suppose that A and B are locally indicable groups, and that r is an element of $A{*}B$ such that no $A{*}B$-conjugate of r lies in $A\cup B$.*

Let N be a normal subgroup of $A{*}B$ such that $r \in N$ and $(A{*}B)/N$ is infinite and cyclic. If r is Cohen–Lyndon aspherical in N, then r is Cohen–Lyndon aspherical in $A{*}B$.

Proof. Let $F = A{*}B$.

It follows from Lemma 5.8 and Remark 5.4 that by adjoining an infinite, cyclic free factor to A, if necessary, we may assume that there exists some $a \in A$ such that $F/N = \langle aN \mid \rangle$.

Similarly, by adjoining an infinite, cyclic free factor to B, if necessary, we may further assume that there exists some $b \in B$ such that $F/N = \langle aN \mid \rangle = \langle bN \mid \rangle$.

Let $N_A := A \cap N$ and let $N_B := B \cap N$. Notice that ${}^{a}N_A = N_A$ and ${}^{a}N_B = {}^{a\bar{b}}N_B$. It is not difficult to use generators and relations to check that $F = \langle a \mid \rangle \ltimes (N_A{*}N_B{*}\langle {}^{\langle a \mid \rangle}(a\bar{b}) \mid \rangle)$, and hence $N = N_A{*}N_B{*}\langle {}^{\langle a \mid \rangle}(a\bar{b}) \mid \rangle$.

We may replace r with any F-conjugate of r, since any automorphism of N respects Cohen–Lyndon asphericity.

Let $j \in \mathbb{Z}$. Conjugation by a^j induces an automorphism of N and we obtain the free-product decomposition $N = N_A * {}^{a^j}N_B * \langle {}^{\langle a \mid \rangle}(a\bar{b}) \mid \rangle$; here we consider the resulting cyclically reduced expression for r, and we see that there exists some finite subset J_j in $\mathbb{Z}$ such that this cyclically reduced expression lies in $N_A{*}\, {}^{a^j}N_B{*}\, \langle {}^{a^{J_j}}(a\bar{b}) \mid \rangle$. We may assume that J_j is minimal and r itself lies in this free factor of N. If J_j is empty, then by replacing r with ${}^{\bar{a}^j}r$, we may assume that $r \in N_A{*}N_B$, and then, by Remark 5.4, r is Cohen–Lyndon aspherical in $N_A{*}N_B$. By Corollaries 4.4 and 5.5, r is Cohen–Lyndon aspherical in $A{*}B$ $(= F)$.

Thus, we may assume that, for each $j \in \mathbb{Z}$, J_j is non-empty.

Let $j \in \mathbb{Z}$. Then ${}^{a^{j+1}}N_B = {}^{({}^{a^j}(a\bar{b}))}({}^{a^j}N_B)$, and we find that $J_j \subseteq J_{j+1} \cup \{j\}$. Similarly, $J_{j+1} \subseteq J_j \cup \{j\}$, and hence $J_j \cup \{j\} = J_{j+1} \cup \{j\}$.

It follows that $\{j \in \mathbb{Z} \mid \min J_j < j\} = \{j \in \mathbb{Z} \mid \min J_{j+1} = \min J_j < j\} = \{j \in \mathbb{Z} \mid \min J_{j+1} < j\}$. It is not difficult to see that this set cannot be all of $\mathbb{Z}$. Let $K := \{j \in \mathbb{Z} \mid \min J_j \geqslant j\}$. Then $K = \{j \in \mathbb{Z} \mid \min J_{j+1} \geqslant j\}$ and $K \neq \emptyset$. Let us choose $k \in K$ to minimise the pair $(|J_k|, (\min J_k) - k)$. If $k{+}1 \in K$, then $\min J_{k+1} \geqslant k{+}1$, $k \notin J_{k+1}$, $J_{k+1} \subseteq J_k$ and by the minimality property of k, $J_{k+1} = J_k$ and $(\min J_{k+1}) - (k{+}1) \geqslant (\min J_k) - k$, which is a contradiction. Thus, $k{+}1 \in \mathbb{Z} - K$. It follows that $\min J_{k+1} = k$.

By replacing r with $^{\overline{a}^k}r$, we may assume that $k = 0$, and hence $\min J_1 = 0$. Let $\nu := \max J_1$. Then $\nu \geqslant 0$ and

$$r \in N_A * {}^a N_B * \langle {}^{a^{[0\uparrow\nu]}}(a\overline{b}) \mid \rangle = N_A * {}^{a\overline{b}} N_B * \langle {}^{a^{[0\uparrow\nu]}}(a\overline{b}) \mid \rangle = N_A * N_B * \langle {}^{a^{[0\uparrow\nu]}}(a\overline{b}) \mid \rangle.$$

Since $\min J_1 = 0$, no N-conjugate of r lies in $N_A * {}^a N_B * \langle {}^{a^{[1\uparrow\nu]}}(a\overline{b}) \mid \rangle = N_A * {}^{a\overline{b}} N_B * \langle {}^{a^{[1\uparrow\nu]}}(a\overline{b}) \mid \rangle$.

We claim that no N-conjugate of r lies in $N_A * N_B * \langle {}^{a^{[0\uparrow(\nu-1)]}}(a\overline{b}) \mid \rangle$. The case of the claim where $\nu = 0$ holds because J_0 is non-empty. The case of the claim where $\nu \geqslant 1$ holds because $\max J_1 = \nu$ and hence no N-conjugate of r lies in

$$N_A * {}^a N_B * \langle {}^{a^{[0\uparrow(\nu-1)]}}(a\overline{b}) \mid \rangle = N_A * {}^{a\overline{b}} N_B * \langle {}^{a^{[0\uparrow(\nu-1)]}}(a\overline{b}) \mid \rangle = N_A * N_B * \langle {}^{a^{[0\uparrow(\nu-1)]}}(a\overline{b}) \mid \rangle.$$

This proves the claim.

By Remark 5.4, r is Cohen–Lyndon aspherical in $N_A * N_B * \langle {}^{a^{[0\uparrow\nu]}}(a\overline{b}) \mid \rangle$. By Corollary 4.4 and Theorem 5.3, r is Cohen–Lyndon aspherical in the HNN extension

$$(N_A * N_B * \langle {}^{a^{[0\uparrow\nu]}}(a\overline{b}) \mid \rangle) * (a\colon (N_A * N_B * \langle {}^{a^{[0\uparrow(\nu-1)]}}(a\overline{b}) \mid \rangle) \to (N_A * {}^{a\overline{b}} N_B * \langle {}^{a^{[1\uparrow\nu]}}(a\overline{b}) \mid \rangle)),$$

which is $\langle a \mid \rangle \ltimes (N_A * N_B * \langle {}^{\langle a \mid \rangle}(a\overline{b}) \mid \rangle) \quad (= F)$. $\square$

Corollary 5.10. *Suppose F is locally indicable and T is an F-tree with trivial edge stabilisers and r is an element of F that fixes no vertex of T.*

Let N be a normal subgroup of F such that $r \in N$ and F/N is infinite and cyclic. If r is Cohen–Lyndon aspherical in N, then r is Cohen–Lyndon aspherical in F.

Proof. The argument is similar to the proof of Corollary 5.6. Let e be an edge in axis(r). Let $\overline{T}$ denote the F-tree obtained from T by contracting all edges in $ET - Fe$. Then $E\overline{T} = Fe$ and r shifts e. Hence, r fixes no vertex of $\overline{T}$. Thus, we may replace T with $\overline{T}$ and assume that $ET = Fe$.

Case 1. $|F \backslash (VT)| = 2$.

Here we can write $F = A * B$ and no F-conjugate of r lies in $A \cup B$. Then r is Cohen–Lyndon aspherical in F by Lemma 5.9.

Case 2. $|F \backslash (VT)| = 1$.

Here we can write $F = A * (b\colon \{1\} \to \{1\})$ and no F-conjugate of r lies in A.

If no F-conjugate of r lies in $\langle b \mid \rangle$, then r is Cohen–Lyndon aspherical in $A * \langle b \mid \rangle$ $(= F)$ by Lemma 5.9.

Thus, we may assume that r itself lies in $\langle b \mid \rangle$ and is clearly Cohen–Lyndon aspherical in $\langle b \mid \rangle$. Hence, r is Cohen–Lyndon aspherical in $A * \langle b \mid \rangle$ $(= F)$ by Remark 5.4.

Thus, in all cases, r is Cohen–Lyndon aspherical in F. $\square$

5.11. The local-indicability Cohen–Lyndon theorem. *Suppose that F is a locally indicable group and T is an F-tree with trivial edge stabilisers. If r is an element of F that fixes no vertex of T, then $\mathbf{C}_F(r)$ is infinite and cyclic, and r is Cohen–Lyndon aspherical in F.*

Proof. Recall from Definitions 2.3 that r has a unique root in F and $\mathbf{C}_F(r) = \langle \sqrt[F]{r} \rangle$.

Let $S := \{\sqrt[F]{r}\}$ and $\Phi := \emptyset$. We may then assume that we are in Setting 3.1, and that v is a vertex in axis($\sqrt[F]{r}$). Then $\langle S \rangle Y = \text{axis}(\sqrt[F]{r}) = \text{axis}(r)$. We shall use decreasing induction to show that, for each $n \in [0\uparrow\nu]$, r is Cohen–Lyndon aspherical in F_n.

By Corollary 4.2, ${}^{F_\nu}S$ is a Whitehead subset of F_ν. Since $|S| = 1$, F_ν is cyclic, and hence ${}^{F_\nu}S = S$. Thus, $F_\nu = \langle \sqrt[F]{r} \mid \rangle$. It is then clear that r is Cohen–Lyndon aspherical in F_ν.

Now suppose that $n \in [0{\uparrow}(\nu-1)]$ and r is Cohen–Lyndon aspherical in F_{n+1}.

First, we wish to show that r is Cohen–Lyndon aspherical in $F_n^\dagger$. This is trivial if $F_n^\dagger = F_{n+1}$, and we consider only the case where $F_n^\dagger/F_{n+1}$ is infinite and cyclic. Here r is Cohen–Lyndon aspherical in $F_n^\dagger$ by Corollary 5.10.

We next want to show that r is Cohen–Lyndon aspherical in F_n.

Recall that $F_n^\dagger$ is generated by glue$(F_n^\dagger, Y)$ and that glue$(F_n, EY) \subseteq F_n^\dagger$. Now

$$\mathrm{glue}(F_n^\dagger, Y) \subseteq \mathrm{glue}(F_n^\dagger, \mathrm{axis}(r))$$

and

$$\mathrm{glue}(F_n, E\mathrm{axis}(r)) = \mathrm{glue}(F_n, \langle S \rangle(EY)) \subseteq \langle S \rangle\, \mathrm{glue}(F_n, EY)\langle S \rangle \subseteq F_n^\dagger.$$

Here r is Cohen–Lyndon aspherical in F_n by Corollary 5.6.

By descending induction, r is Cohen–Lyndon aspherical in F_0 $(= F)$. □

6. Applications. We now obtain a sufficient condition for a subset of F to be Cohen–Lyndon aspherical in F.

THEOREM 6.1. *Let $(\mathcal{F}, Y)$ be a graph of groups with family of groups $(\mathcal{F}(y) \mid y \in Y)$ and family of edge maps $(\bar{t}_e\colon \mathcal{F}(e) \to \mathcal{F}(\tau_Y e) \mid e \in EY)$, let Y_0 be a maximal subtree of Y and let F be the fundamental group of $(\mathcal{F}, Y, Y_0)$.*

Let U be a subset of VY and let $(r_v \mid v \in U)$ be a family of elements of F with the property that for each $v \in U$ the following hold.

(a) *$r_v \in \mathcal{F}(v)$ and $\mathcal{F}(v)$ is locally indicable and $\mathcal{F}(v)$ acts on some tree with trivial edge stabilisers and no vertex fixed by r_v.*

(b) *For each $e \in \iota_Y^{-1}(\{v\})$, $\mathcal{F}(e)$ is a free factor of $\mathcal{F}(v)$ that contains no $\mathcal{F}(v)$-conjugate of r_v.*

(c) *For each $e \in \tau_Y^{-1}(\{v\})$, ${}^{\bar{t}_e}\mathcal{F}(e)$ is a free factor of $\mathcal{F}(v)$ that contains no $\mathcal{F}(v)$-conjugate of r_v.*

Then $(r_v \mid v \in U)$ is Cohen–Lyndon aspherical in F.

Proof. This follows from Theorems 5.3 and 5.11. Notice that by Corollary 4.4 the edge groups are not affected by passing to the quotient. □

We shall consider only the case that corresponds to the strongly staggered conditions.

COROLLARY 6.2. *Suppose that F is a locally indicable group and R is a subset of F. If there exists some F-tree T with trivial edge stabilisers such that R is strongly T-staggerable modulo F, then R is Cohen–Lyndon aspherical in F.*

Cohen and Lyndon [**4**, Corollary 2.2] proved the case of this result where F is free and $|F\backslash VT| = 1$.

Proof of Corollary 6.2. Applied to F acting on T, the Bass–Serre structure theorem presents F as the fundamental group of a certain graph of groups $(\mathcal{F}, Y, Y_0)$. Let $<$ be an ordering of ET such that R is strongly $(T, <)$-staggered modulo F. Here $EY = F\backslash ET$ and, by the staggered conditions, the ordering $<$ of ET induces an ordering, again denoted by $<$, of EY. By the strongly staggered conditions, $(EY, <)$ is order isomorphic to an interval in $\mathbb{Z}$. Let us consider only the case where $(EY, <)$ is order isomorphic to $\mathbb{Z}$; the case where $(EY, <)$ is order isomorphic to a proper interval in $\mathbb{Z}$ is handled in a similar way. By choosing an order isomorphism we get an indexing $e_{\mathbb{Z}}$ of the elements of EY.

For any interval I in $\mathbb{Z}$, e_I is an interval in $e_{\mathbb{Z}}$, and we let Y_I denote the subgraph of Y with edge set e_I together with the vertices of Y that are incident to these edges. By the strongly staggered conditions, Y_I is connected and $Y_0 \cap Y_I$ is the unique maximal subtree of Y_I. Then the graph of groups obtained by restricting $\mathcal{F}$ to Y_I has a well-defined fundamental group, which we denote by F_I. It can be seen that F_I is a free factor of F.

By the staggered conditions, the ordering $<$ of EY induces an ordering, again denoted by $<$, of R. Let us consider only the case where R is order isomorphic to $\mathbb{Z}$; the case where R is order isomorphic to a proper interval in $\mathbb{Z}$ is handled in a similar way. By choosing an order isomorphism we get an indexing $r_{\mathbb{Z}}$ of the elements of R. For each $n \in \mathbb{Z}$, we set $\mu_n := \min(F\backslash(F(E\mathrm{axis}(r_n))))$ and $\nu_n := \max(F\backslash(F(E\mathrm{axis}(r_n))))$. Clearly $\mu_n \leqslant \nu_n$. The staggered conditions imply that $\mu_n < \mu_{n+1}$ and $\nu_n < \nu_{n+1}$.

We can then express F as the fundamental group of a second graph of groups whose family of vertex groups is $(F_{[\mu_n \uparrow \nu_n]} \mid n \in \mathbb{Z})$ and whose family of edge groups is $(F_{[\mu_n \uparrow \nu_n]} \cap F_{[\mu_{n+1} \uparrow \nu_{n+1}]} \mid n \in \mathbb{Z})$, with the edge maps being those suggested by the intersection notation. Here the underlying graph has the form of the real line.

For each $n \in \mathbb{Z}$, some F-conjugate of r_n lies in $F_{[\mu_n \uparrow \nu_n]}$ and we may assume that r_n itself lies in the vertex group $F_{[\mu_n \uparrow \nu_n]}$. By the definition of μ_n and ν_n and the staggered conditions, no F-conjugate of r_n lies in either of the two incident-edge groups.

By Theorem 6.1, R is Cohen–Lyndon aspherical in F. $\square$

Cohen–Lyndon asphericity has many applications, and we conclude by mentioning two of them.

SETTING 6.3. Suppose that F is locally indicable and T is an F-tree with trivial edge stabilisers and R_0 is a subset of F that is strongly T-staggerable modulo F.

By Definition 2.3, R_0 has unique roots in F and $\sqrt[F]{R_0}$ is strongly T-staggerable modulo F. For each $r \in R_0$, we have $\mathbf{C}_F(r) = \langle \sqrt[F]{r} \rangle$.

By Corollary 6.2, R_0 and $\sqrt[F]{R_0}$ are Cohen–Lyndon aspherical in F, and we can write the following.

Let $G := F/\langle\!\langle R_0 \rangle\!\rangle$. For each $r \in R_0$, let $G_r := \mathbf{C}_F(r)/\langle r \rangle = \langle \sqrt[F]{r} \mid r \rangle$, a finite, cyclic subgroup of G.

Let $(Y_r \mid r \in R_0)$ be a family of subsets of F with the properties that for each $r \in R_0$, Y_r is a transversal for the $\langle {}^F(\sqrt[F]{R_0}) \rangle$-action on F by multiplication on the left (or right) and $\bigcup\limits_{r \in R_0} ({}^{Y_r}(\sqrt[F]{r}))$ is a free-generating set of $\langle {}^F(\sqrt[F]{R_0}) \rangle$.

Let $(X_r \mid r \in R_0)$ be a family of subsets of F with the properties that for each $r \in R_0$, X_r is a transversal for the $(\langle {}^F R \rangle \langle \sqrt[F]{r} \rangle)$-action on F by multiplication on the right $\bigcup\limits_{r \in R_0} ({}^{X_r} r)$ is a free-generating set of $\langle {}^F R_0 \rangle$. For each element v in the G-graph $\langle {}^F R_0 \rangle \backslash T$, let G_v denote the G-stabiliser of v. Also, let E_0, resp. V_0, be a transversal for the natural G-action on $\langle {}^F R_0 \rangle \backslash ET$, resp. $\langle {}^F R_0 \rangle \backslash VT$. Here F is a free product of a free group and $\mathop{\ast}\limits_{v \in V_0} G_v$.

We first discuss torsion.

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Corollary 6.4. *In* Setting 6.3*, let N denote the* (*normal*) *subgroup of G generated by the elements of G of finite order. Then the following hold.*

(i) *For each $r \in R_0$, Y_r may be viewed as a transversal for the N-action on G by multiplication on the left, and $N = \mathop{*}\limits_{r\in R_0} \mathop{*}\limits_{y \in Y_r} {}^{y}G_r$, and hence N is a free product of finite, cyclic groups.*

(ii) *G/N is locally indicable, and every torsion-free subgroup of G is locally indicable.*

(iii) *Each non-trivial, torsion subgroup of G lies in exactly one of the* (*finite, cyclic*) *subgroups of G of the form ${}^{g}G_r$, $r \in R_0$, $g \in G$, and then r is unique and the coset $gG_r \in G/G_r$ is unique.*

In the case where F acts freely on T, and $|R| = 1$, (iii) can be attributed to Magnus and Lyndon [**14**], (i) is due to Fischer, Karrass and Solitar [**7**] and (ii) can be attributed to Brodskiĭ [**2**].

In the case where $F\backslash T$ has one edge and two vertices, and $|R| = 1$, these results can be attributed to Edjvet and Howie [**6**].

Proof of Corollary 6.4. Let $\overline{G} := F/\langle\!\langle \sqrt[F]{R_0} \rangle\!\rangle$.

By Theorem 4.3(ii), $\overline{G}$ is locally indicable and, in particular, $\overline{G}$ is torsion free.

Let $N_1 := \langle {}^{F}(\sqrt[F]{R_0}) \rangle / \langle {}^{F}R_0 \rangle$. Then $N_1 \trianglelefteq G$ and $G/N_1 = \overline{G}$. Since $\overline{G}$ is torsion free, every element of G of finite order lies in N_1, that is, $N \leqslant N_1$. Also, for each $r \in R_0$, the (faithful) image of Y_r in G is a transversal for the N_1-action on G by multiplication on the left.

Since $\bigcup\limits_{r \in R_0} ({}^{(\langle {}^{F}(\sqrt[F]{R_0}) \rangle Y_r)}r) = \bigcup\limits_{r\in R_0} ({}^{F}r) = {}^{F}R_0$, it follows that

$$N_1 = \langle {}^{F}(\sqrt[F]{R_0}\,) \rangle / \langle {}^{F}R_0 \rangle = \langle \bigvee_{r\in R_0} ({}^{Y_r}(\sqrt[F]{r})) \mid \bigvee_{r \in R_0} ({}^{Y_r}r) \rangle = \mathop{*}_{r\in R_0} \mathop{*}_{y\in Y_r} {}^{y}\langle \sqrt[F]{r} \mid r \rangle = \mathop{*}_{r\in R_0} \mathop{*}_{y \in Y_r} {}^{y}G_r.$$

Thus, N_1 is a free product of finite, cyclic groups. In particular, N_1 is generated by some set of elements of finite order in G, that is, $N_1 \leqslant N$. Hence, $N_1 = N$ and (i) holds. Also, every torsion-free subgroup of N is free. Hence, (ii) holds.

Any non-trivial, torsion subgroup H of G lies in N, and by well-known properties of free products there exists some $n \in N$ such that $H \leqslant {}^{ny}G_r$ for a unique $r \in R_0$ and a unique $y \in Y_r$, and here the coset $n{}^{y}G_r \in N/{}^{y}G_r$ is unique. Since the (faithful) image of Y_r in G is a transversal for the N-action on G by multiplication on the left, we see that (iii) holds. □

We now consider exact sequences and homology groups. For any set X, we let $\mathbb{Z}X$ and $\mathbb{Z}[X]$ denote the $\mathbb{Z}$-module that is free on X; if X is a G-set, this is a $\mathbb{Z}G$-module in a natural way.

Corollary 6.5. *In* Setting 6.3*, the following hold.*

(i) *The left F-action on $\langle {}^{F}R_0 \rangle$ by conjugation induces a left G-action on $\langle {}^{F}R_0 \rangle^{ab}$, and $\langle {}^{F}R_0 \rangle^{ab}$ is then a left $\mathbb{Z}G$-module that is naturally isomorphic to $\bigoplus\limits_{r\in R_0} \mathbb{Z}[G/G_r]$.*

(ii) *There exist natural exact sequences of left $\mathbb{Z}G$-modules that have the form*

$$0 \to \bigoplus_{r\in R_0} \mathbb{Z}[G/G_r] \to \bigoplus_{e \in E_0} \mathbb{Z}[G] \to \bigoplus_{v\in V_0} \mathbb{Z}[G/G_v] \to \mathbb{Z} \to 0, \tag{8}$$

$$0 \to \bigoplus_{r\in R_0} \mathbb{Z}[G] \to \bigoplus_{e\in R_0 \vee E_0} \mathbb{Z}[G] \to \bigoplus_{v \in R_0 \vee V_0} \mathbb{Z}[G/G_v] \to \mathbb{Z} \to 0. \tag{9}$$

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(iii) *For each* $n \in [3\uparrow\infty[$, *the change-of-groups natural transformation* $\bigoplus_{v\in R_0\vee V_0} \mathbf{H}_n(G_v, -) \to \mathbf{H}_n(G, -)$, *between functors from the category of right* $\mathbb{Z}G$*-modules to the category of abelian groups, is an isomorphism of functors, and the change-of-groups natural transformation* $\mathbf{H}^n(G, -) \to \prod_{v\in R_0\vee V_0} \mathbf{H}^n(G_v, -)$, *between functors from the category of left* $\mathbb{Z}G$*-modules to the category of abelian groups, is an isomorphism of functors.*

In the case where F is free and $|F\backslash VT| = 1$, (i) is Lyndon's identity theorem [**14**, Section 7]; in the case where F acts freely and $|R_0| = 1$, (i) is Lyndon's simple identity theorem [**14**, Section 7]. In the case where $|R_0| = 1$, and $F\backslash T$ has one edge and two vertices, (i) is Howie's simple identity theorem [**10**, Theorem 11]. The results (ii) and (iii) are straightforward consequences of (i); see [**14**, Theorem 11.1] and [**10**, Theorem 3].

Proof of Corollary 6.5. For each $r \in R_0$, we have bijective correspondences

$$X_r \simeq F/(\langle {}^F R_0 \rangle \langle \sqrt[F]{r} \rangle) \simeq G/\langle \sqrt[F]{r} \langle {}^F R_0 \rangle \rangle \simeq G/G_r.$$

Since $\langle {}^F R_0 \rangle = \langle \bigvee_{r\in R_0} ({}^{X_r} r) \mid \rangle$, we then have isomorphisms of abelian groups

$$\langle {}^F R_0 \rangle^{\mathrm{ab}} \simeq \mathbb{Z}[\bigvee_{r\in R_0} ({}^{X_r} r)] \simeq \mathbb{Z}[\bigvee_{r\in R_0} (X_r)] \simeq \mathbb{Z}[\bigvee_{r\in R_0} (G/G_r)] \simeq \bigoplus_{r\in R_0} \mathbb{Z}[G/G_r].$$

The composite isomorphism of abelian groups $\langle {}^F R_0 \rangle^{\mathrm{ab}} \simeq \bigoplus_{r\in R_0} \mathbb{Z}[G/G_r]$ is compatible with the G-actions, and we find that (i) holds.

By Theorem 4.3(i), $\langle {}^F R_0 \rangle$ acts freely on T. By [**5**, Definitions I.8.1], the fundamental group of $\langle {}^F R_0 \rangle\backslash T$ can be identified with $\langle {}^F R_0 \rangle$. Then (i) and [**5**, Theorem I.9.2] give a sequence as given in (8). We leave it as an exercise to construct a sequence as in (9), which maps onto the above-constructed sequence in (8) with kernel a short exact sequence. Alternatively, one can arrange for the sequences in (8) and (9) to be the augmented cellular chain complexes of acyclic, simply connected, hence contractible, CW-complexes on which G acts by permuting the cells. Here the second CW-complex is obtained from the first CW-complex by G-equivariantly drawing on each two-cell a point and a finite set of edges joining the new point to old points. This second CW-complex has the property that every non-trivial finite subgroup of G fixes exactly one point of the space and the fixed point is a zero-cell.

For the short exact sequence

$$0 \to \ker\epsilon \to \bigoplus_{v\in R_0\vee V_0} \mathbb{Z}[G/G_v] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

of left $\mathbb{Z}G$-modules, where ϵ is the corresponding augmentation map, we see from (9) that $\ker\epsilon$ has a free $\mathbb{Z}G$-resolution of length at most two. The resulting long exact sequences in homology then show that (iii) holds. □

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