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A PROOF OF THE CALDERON EXTENSION THEOREM

BY

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In this note we outline a proof of the Calderon extension theorem by a technique similar to that for the Whitney extension theorem. For classical proofs, see Calderon [2] and Morrey [4]. See also Palais [6, p. 170]. Our purpose is thus to give a more unified proof of the theorem in the various cases. In addition, the proof applies to the Holder spaces $C^{k+\alpha}$, which was used in [3], and applies to regions satisfying the "cone condition" of Calderon.

Let M be a compact C^∞ manifold with C^∞ boundary embedded as an open submanifold of a compact manifold $\tilde{M}$. Let $\pi:E\to\tilde{M}$ be a vector bundle and let $L^p_k(\pi)$, $L^p_k(\pi \restriction M)$ be the usual Sobolev spaces and $H^k=L^2_k$. See [2], [5],or [6] for the definitions. Here, $\restriction$ denotes restriction. We prove the following for H^s ($s\geq 0$ an integer), but a similar proof also holds for L^p_k, and $C^{k+\alpha}$, $0\leq\alpha\leq 1$.

THEOREM. *There exists a continuous linear map*

$$T:H^s(\pi \restriction M)\to H^s(\pi$$

such that $T(f)\restriction M=f$ *for* $f\in H^s(\pi\restriction M)$.

In particular, this implies

$$H^s(\pi\restriction M) = H^s(\pi)\restriction M.$$

We begin by reducing to the local case:

LEMMA 1. *It suffices, for the theorem to define a linear map* $T:C^\infty_{H,1}\to C^s_1$, *where* $C^\infty_H=\{$*the smooth real functions defined on* $\{x=(x^1,\ldots,x^n)\in\mathbb{R}^n:x^n\leq 0\}$ *and with support in the ball of radius* $1\}$ *and* C^s_1 *is the class* C^s *functions* $f:\mathbb{R}^n\to\mathbb{R}$ *with support in the ball of radius* 1*; such that there is a constant* M *with* $\|Tf\|_s\leq M\|f\|_s$ *and such that* Tf *is an extension of* f.

Proof. Let $(U_1,\phi_1),\ldots,(U_N,\phi_N)$ be a covering of $M\cup\partial M$ by charts in $\tilde{M}$ such that $\phi_i:U_i\to\mathbb{R}^n$; if $U_i\cap\partial M\neq\varnothing$ then

$$\phi_i(U_i\cap\partial M)\subset\{x:x^n=0\}$$
$$\phi_i(U_i\cap M)\subset\{x:x^n\leq 0\}$$

and

$$\phi_i(U_i)\subset\{x:\|x\|\leq 1\}$$

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and assume that the U_i are also bases for vector bundle charts of π. Let $g_1, \ldots, g_N$ be a subordinate partition of unity for this cover. Define T_i on C^∞ real functions on $U_i \cap M$ with support in $U_i \cap M$ by

$$T_i(h) = T(h \circ \phi_i^{-1}) \circ \phi_i$$

where T is the map given by the lemma. Extend this to sections of E with support in $U_i \cap M$ by

$$T_i(h_1, \ldots, h_m) = (T_i(h_1), \ldots, T_i(h_m))$$

where $h_1, \ldots, h_m$ are coordinates for h. Define T by

$$Tf = \sum_{i=1}^{N} g_i T_i(f)$$

for f a C^∞ section. Using these charts to compute the H^s norm and the fact that the derivatives of g_i are bounded, we see that

$$\|Tf\|_s \leq C \|f\|_s$$

for a constant C. It is also clear that Tf is an extension of f. Since the C^∞ sections are dense in $H^s(\pi)$, T has a unique continuous linear extension to $H^s(\pi \restriction M)$. Since $s \geq 0$, Tf is an extension of f for all $f \in H^s(\pi \restriction M)$, for $f_n \to f$ in H^s implies L_2 convergence. Q.E.D.

To construct a T with the properties in Lemma 1, we proceed as follows. First, construct closed cubes K_j as in the Whitney extension theorem (Abraham-Robbin [1, Appendix A]) and a corresponding partition of unity ϕ_j. Let $\tilde{K}_j$ be the reflection of K_j in the hyperplane $x^n = 0$.

Let $x_j \in \tilde{K}_j$ and for f C^∞ with support in the unit ball and defined for $x^n \leq 0$, let f_i denote the i^{th} derivative and let

$$f_{i,j} = f_{i,j}^+ - f_{i,j}^-$$

where

$$f_{i,j}^+ = \frac{1}{\mu(K_j)} \left\{ \int_{\tilde{K}_j} f_i^+(x)^2 \, dx \right\}^{1/2}$$

and

$$f_{i,j}^- = \frac{1}{\mu(K_j)} \left\{ \int_{\tilde{K}_j} f_i^-(x)^2 \, dx \right\}^{1/2}$$

where f_i^+, f_i^- are the positive and negative parts of f_i (as a matrix, or partial derivatives), and $\mu(K_j)$ is the measure of K_j.

Then let (Cf. [Abraham-Robbin, Formula 42])

$$P(x_j, y) = \sum_{i=0} \frac{f_{ij}}{i!} (y - x_j)^i.$$

Now $f_{ij} = f_i(x'_{j,i})$ for some $x'_{j,i} \in \tilde{K}_j$ by the mean value property of integrals. (More precisely, $x'_{j,i}$ may depend on the partial derivatives and not just the total derivative.)

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LEMMA 2. *The function*

$$F(y) = \begin{cases} f(y) & \text{if } y^n \leq 0 \\ \sum_j \phi_j(y) P(x_j, y) \end{cases}$$

is of class C^s and is an extension of f (the sum having N terms for N fixed).

The proof of this follows in exactly the same way as the corresponding result in the Whitney extension theorem. (Note that we still retain the basic property 3.5 (uniformly in the $x'_{i,j}$) of Abraham-Robbin, and this is all that is required for the lemma).

Clearly the association

$$Tf = F$$

is also linear. It remains only to prove this:

LEMMA 3. *There is a constant M so that*

$$\|Tf\|_s \leq M \|f\|_s.$$

Proof. Since Tf has support in the unit sphere, there is a constant M_0 so that if $y \in K_j$,

$$|F(y)| \leq M_0 \|f\|_s^j / \mu(K_j)$$

where $\|f\|_s^j$ is the H^s norm of f restricted to $\tilde{K}_j$. Here we use these facts: (i) the sum over j contains at most N terms for N fixed; (ii) there is a constant P so that if K_k intersects K_j, $\mu(K_j) \leq P\mu(K_k)$ and $\mu(K_j) \geq P^{-1}\mu(K_k)$, and, (iii)

$$|f_{ij}| \leq \|f\|_s^j / \mu(K_j), \qquad i = 0, \ldots, s$$

and all the polynomial terms P are uniformly bounded for x, y in the unit ball. In a similar way the derivatives $F_i(y)$ satisfy

$$|F_i(y)| \leq M_1 \|f\|_s^j / \mu(K_j), \qquad i = 0, \ldots, s$$

for $y \in K_j$. Thus if $M_2 = \max(M_0, M_1)$,

$$\|F\|_s \leq \|f\|_s + \sum_{j=1}^{\infty} [M_2 \|f\|_s^j / \mu(K_j)] \mu(K_j)$$

the sum being over those K_j meeting the unit ball. But

$$\sum_{j=1}^{\infty} \|f\|_s^j = \|f\|_s$$

so that

$$\|F\|_s \leq M \|f\|_s$$

where $M = 1 + M_2$. Q.E.D.

REMARK. The Whitney extension map (constructed in [1]) is continuous in the C^s norm, but not the H^s norm.

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