AN ENUMERATION PROBLEM RELATED TO THE NUMBER OF LABELLED BI-COLOURED GRAPHS

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We will consider the following enumeration problem. Let A and B be finite sets with α and β elements in each set respectively. Let n be some positive integer such that $n \leqq \alpha\beta$. A subset S of the product set $A \times B$ of exactly n distinct ordered pairs (a_i, b_j) is said to be admissible if given any $a \in A$ and $b \in B$, there exist elements (a_i, b_j) and (a_k, b_l) (they may be the same) in S such that $a_i = a$ and $b_l = b$. We shall find here a generating function for the number $N(\alpha, \beta; n)$ of distinct admissible subsets of $A \times B$ and from this generating function, an explicit expression for $N(\alpha, \beta; n)$. In obtaining this result, the idea of a cut probability is used. This approach in a problem of enumeration may be of interest.

One may consider A and B as (say) two chess teams competing with each other. $N(\alpha, \beta; n)$ is then the number of ways of having n simultaneous matches between the two teams such that a player may be involved in several matches but there is at most one match between a pair of players and such that no player is left idle.

In terms of graph theory, $N(\alpha, \beta; n)$ is interpreted as follows. Consider a set of $\alpha + \beta$ labelled nodes of which α are in one colour and β are in another. $N(\alpha, \beta; n)$ is then the number of distinct 2-coloured graphs having exactly n branches on this set of nodes such that no node is allowed to be isolated.

In reference **(2)**, using Polya's theorem **(1)**, Harary obtained expressions for the number of bi-coloured graphs. His results differ from ours first in the respective methods of approach and second in the fact that his enumeration was for graphs with unlabelled nodes.

THEOREM. *Let F be a generating function for $N(\alpha, \beta; n)$:*

$$F(x; \alpha, \beta) = \sum_{n=1}^{\alpha\beta} N(\alpha, \beta; n)\, x^n.$$

Then,

$$F(x; \alpha, \beta) = \sum_{k=0}^{\alpha} (-1)^{\alpha+\beta-k} \binom{\alpha}{k} (1 - (1 + x)^k)^\beta.$$

The idea of the proof goes as follows. We consider a certain bi-rooted graph G and define for G a cut probability P. This probability P can be computed

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in two ways in one of which $N(\alpha, \beta; n)$, except for sign, enters as certain coefficients. The theorem is then proved by extracting certain coefficients of P and relating them to our enumeration problem.

Let G be the bi-rooted graph shown in Fig. 1,

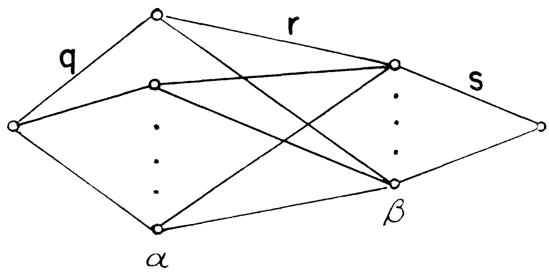

FIG. 1.

in which G has α nodes next to the left root-node and β nodes next to the right root-node. There is one branch connecting each of the α nodes to the left root-node, one branch connecting each of the β nodes to the right root-node, and one branch between every α-node and every β-node.

Consider now each branch as a piece of string and let $1 - q$, $1 - r$, and $1 - s$ be respectively the probability that a left, middle, and right branch be cut in two (disconnected), and let us assume that the random variables (one for each branch) are independent. The cut probability P for the graph G is then the probability that G is cut in the sense that the left root-node and the right root-node are disconnected.

LEMMA 1. *The cut probability P for the graph G is given by*

$$P(q, r, s) = \sum_{k=0}^{\alpha} \binom{\alpha}{k} (1 - q)^{\alpha-k} q^k (s(1 - r)^k + (1 - s))^{\beta}.$$

Proof. Break the α left branches into two sets L_1, L_2 of k and $\alpha - k$ elements in each and break the β right branches into two sets R_1, R_2 of j and $\beta - j$ elements in each. Let E_{kj} be the event that every branch in L_2 and in R_2 is cut and every branch in L_1 and R_1 is left uncut, and that the graph G is itself cut. Then

$$Pr\{E_{kj}\} = q^k(1 - q)^{\alpha-k}s^j(1 - s)^{\beta-j}(1 - r)^{jk}.$$

It then follows that

$$P(q, r, s) = \sum_{k=0}^{\alpha} \sum_{j=0}^{\beta} \binom{\alpha}{k}\binom{\beta}{j} Pr\{E_{kj}\}$$
$$= \sum_{k=0}^{\alpha} \binom{\alpha}{k} q^k(1 - q)^{\alpha-k}(s(1 - r)^k + (1 - s))^{\beta}.$$

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LEMMA 2. *Let $f(r)$ be the coefficient of the term $q^\alpha s^\beta$ in $P(q, r, s)$. Then*

(i) $$f(r) = \sum_{k=0}^{\alpha} (-1)^{\alpha+\beta-k} \binom{\alpha}{k} (1 - (1-r)^k)^\beta;$$

and

(ii) *Writing $f(r)$ as*

$$f(r) = C_0 + C_1 r + \ldots + C_\alpha r^\alpha,$$

the coefficient C_n of r^n is $(-1)^n N(\alpha, \beta; n)$.

Proof. By expanding the expression $P(q, r, s)$, (i) follows. To prove (ii), we note first that each middle branch defines a unique path from the left root-node to the right root-node. Therefore there are $\alpha\beta$ distinct (not independent) such paths. Let D_i be the event that the ith path is uncut. Then

$$\begin{aligned} P(q, r, s) &= 1 - Pr\{D_1 \cup D_2 \cup \ldots \cup D_{\alpha\beta}\} \\ &= 1 - \sum_i Pr\{D_i\} + \sum_{i,j} Pr\{D_i \cap D_j\} \\ &\qquad - \ldots + (-1)^{\alpha\beta} Pr\{D_1 \cap \ldots \cap D_{\alpha\beta}\}. \end{aligned}$$

Now each sum can be expressed as

$$\sum_{i_1, \ldots, i_n} Pr\{D_{i_1} \cap \ldots \cap D_{i_n}\} = r^n g(q, s)$$

where $g(q, s)$ is a polynomial in q and s, and r^n can appear nowhere else in the above expression for $P(q, r, s)$. Let $d(\alpha, \beta)$ be the coefficient of $q^\alpha s^\beta$ in $g(q, s)$. Then the number $d(\alpha, \beta)$, except for sign, is the number of distinct graphs involving $\alpha + \beta$ nodes and n branches such that each graph satisfies the conditions given in the beginning. A check of sign yields $d(\alpha, \beta) = (-1)^n N(\alpha, \beta; n)$. Since $d(\alpha, \beta)$ is just the coefficient of r^n in $f(r)$, the lemma follows.

Proof of theorem. It follows from Lemma 2 that

$$\sum_{n=0}^{\alpha\beta} (-1)^n N(\alpha, \beta; n) r^n = f(r).$$

Hence, the generating function F is

$$\begin{aligned} F(x; \alpha, \beta) &= \sum_n N(\alpha, \beta; n) x^n = f(-x) \\ &= \sum_{k=0}^{\alpha} (-1)^{\alpha+\beta-k} \binom{\alpha}{k} (1 - (1+x)^k)^\beta, \end{aligned}$$

and the theorem follows.

As an example, we find for $\alpha = 3$, $\beta = 2$ that

$$F(x; 3, 2) = 6x^3 + 12x^4 + 6x^5 + x^6$$

so that $N(3, 2; 3) = 6$, $N(3, 2; 4) = 12$, $N(3, 2; 5) = 6$, and $N(3, 2; 6) = 1$.

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From this theorem, the expression for $N(\alpha, \beta; n)$ can be derived explicitly. Let $\langle x\rangle$ denote the least integer greater than or equal to x, the following iteration of summations obtains.

LEMMA 3.

$$\sum_{j=0}^{\beta} \sum_{i=0}^{kj} f(i,j) = \sum_{i=0}^{k\beta} \sum_{j=\langle i/k\rangle}^{\beta} f(i,j).$$

Using this lemma and expanding the generating function for $N(\alpha, \beta; n)$, one gets

COROLLARY.

$$N(\alpha, \beta; n) = \sum_{k=\langle n/\beta\rangle}^{\alpha} \sum_{j=\langle n/k\rangle}^{\beta} (-1)^{\alpha+\beta+k+j} \binom{\alpha}{k}\binom{\beta}{j}\binom{kj}{n}.$$

The writer is indebted to J. Riordan for pointing out the following identity, which appears novel, and for other enlightening comments. In the above corollary, if we set $\alpha = \beta = n$, then $N(n, n, n)$ is the number of permutations on n objects. The following identity therefore obtains:

$$\sum_{k=1}^{n} \sum_{j=\langle n/k\rangle}^{n} (-1)^{k+j} \binom{n}{k}\binom{n}{j}\binom{kj}{n} = n!\,.$$

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