LONG-PERIOD PERTURBATIONS IN TERRESTRIAL REFERENCE FRAMES

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ABSTRACT. Oceanic and fluid core effects inherent in polar motion and l.o.d.-data were analyzed and related results are discussed in detail.

A new exact analytic solution to the hydrodynamic equations is obtained, which describes tidal motions at low frequencies in a homogeneous, incompressible, inviscid liquid core with arbitrary core-mantle topography. Some geophysical and astrometrical consequences of this solution are considered.

The numerical estimation of the deviations of the pole tide from the static one is obtained. A new hypothesis is proposed that the known interrelation between long-term amplitude and frequency variations in Chandler wobble may be attributed to the influence of turbulent friction of non-equilibrium pole tide.

1. INTRODUCTION

Astrometric space missions such as Hipparcos may substantially contribute to relative accuracy of stellar and other (fundamental) catalogues. On the other hand, the associated absolute systems of reference are still basically referred to the Earth. Consequently, any improvement in modeling the irregular rotation of the Earth with respect to the celestial frame or system or reference contributes also to the implementation of celestial reference systems. This holds for optical as well as for radio catalogues (de Vegt et al., 1988, Argue, 1989).

In this paper we focus on the long-period variations of Earth rotation which are related to the fluid parts of the Earth : (a) oceanic effects and (b) outer core perturbations. By "long-period" effects of periods longer than a few months up to about 14 months (Chandlerian period) is meant. The emphasis is here put, as far as the oceanic effects are concerned, on pole tide which is basically the variation of sea surface caused by the varying centrifugal force of polar motion. The latter reflects the changes between celestial and terrestrial (in the sense of CTS - Conventional Terrestrial System) systems and therefore represents the transformation parameters for the transition from celestial to terrestrial systems and vice versa. As pole tide is excited in a thin layer of liquid (in comparison with the Earth's radius) two-dimensional integrations of related differential equations along the Earth's surface are sufficient in most cases. The fundamental phenomena were investigated by J. Wahr, S. Dickman and others which led to the conclusion that basically pole tides are in equilibrium. Therefore, we focus here on the finer structure of such effects; this is necessary in view of the increased accuracy of modern observations and analysis methods. It affects mainly effects such as the dependence of polar motion frequency variations on its amplitude variations.

The influence of inner and outer core effects on polar motion and LOD-data is of particular importance because there is limited information on the Earth's core; there are relatively few phenomena such as Earth tides, free vibrations related to very big earthquakes, geomagnetism, seismology, to some extent, and a few others which can really give reliable information on the detailed structure of the Earth's core. Therefore, the analysis of Earth rotation data in terms of LOD- and polar motion data is of utmost importance. Mainly the detailed geometry and topography of the core-mantle-boundary (CMB) is important in that respect as well as the physics around it such as questions of hydrostatic equilibrium etc. which are closely related to its topography. In spite of impressive recent results and of rather general agreement, within certain limits, there is still a wide disagreement on details. This paper aims at contributing to a clearer understanding of related phenomena.

In the first part of this report the influence of the CMB, both on the Chandler wobble and on the length of day variation, is considered.

A new analytical solution to the hydrodynamic equations is obtained, which describes tidal motions at low frequencies in a homogeneous, incompressible, inviscid liquid core with arbitrary core-mantle topography. The result is applied to the estimation of the influence of the core-mantle boundary topography on the Chandler wobble and on the long-periodic tidal variations of the length of day.

It is found, that the influence of CMB topography is manifested not only in the changes of the parameters, describing these events (period and ellipticity of the Chandler wobble, amplitudes of the tidal variations of the length of day), but also leads to a new "cross-coupling" effect of (1) the excitation of the length of day variation with the Chandlerian period and (2) the excitation of polar motion with the periods of zonal tidal waves.

The orders of values of all these effects are strongly dependent on the values of gradients of the CMB topography. The numerical estimation for the reasonable models of CMB-topography shows, that the influence of CMB-topography on the period and ellipticity of the Chandler wobble as well as the excitation of the polar motion by the long-period tidal waves are, however, less than the errors of the modern VLBI-measurements. But at the same time, the excitation of the length of day variations by the Chandler wobble is significant. As a result, the analysis of the observed values of the length of day amplitudes at Chandler frequency makes it possible to obtain new information concer-

ning the CMB topography of the actual Earth. Some numerical estimations of this type are
obtained which are based on the analysis of modern VLBI data.

In the second part of this paper the influence of dynamical pole tide on the Chandler wobble is considered. The results of new numerical calculations of the planetary vorticity maps for the actual ocean are presented. Some qualitative and quantitative estimations of the possible dynamical effects are obtained.

An interesting consequence of the dynamical theory of the pole tide is the conclusion, that the system (Earth + ocean) is not linear. It is known, that the frequencies of free oscillations of such systems are dependent on the amplitudes of the oscillation. As a result, the frequency of the Chandler wobble must be dependent on its amplitude.

The qualitative theoretical analysis of this effect makes it possible to conclude, that the Chandler period is an increasing function of the amplitude. It is known, that a similar conclusion was made by Melchior (1957) (see also Munk & McDonald, 1960) based on the analysis of polar motion data since 1900,0. Thus we may conclude that it is possible to attribute this event to the influence of non-equilibrium pole tide.

2. THE SMALL LONG-PERIODIC OSCILLATIONS OF THE HOMOGENEOUS INCOMPRESSIBLE INVISCID LIQUID, CLOSED IN THE RIGID NONUNIFORMLY ROTATING CONTAINER WITH ARBITRARY GEOMETRY.

The small oscillations of the homogeneous incompressible inviscid liquid are described in the uniformly rotating system of Cartesian coordinates by the known system of governing equations and boundary conditions (Lamb, 1932):

$$\dot{\vec{v}} + 2\vec{\omega}\times\vec{v} + \vec{\omega}\times\vec{\omega}\times\vec{r} + \dot{\vec{\omega}}\times\vec{r} = - \nabla \left(\frac{p}{\rho} + V\right) , \tag{1a}$$

$$\nabla \cdot \vec{v} = 0 , \tag{1b}$$

$$(\vec{v},\vec{N})\big|_s = 0 , \tag{2}$$

where $\vec{\omega}$ is the vector of the angular velocity of the system of coordinates, $\vec{r}$ is the radius-vector,

$\vec{v} = \dot{\vec{r}}$ is the velocity of the element of fluid with respect to this system, p = pressure, ρ = density, V is the gravitational potential, the dot above a symbol denotes the time derivative with respect to the non-uniformly rotating system of coordinates, $\vec{N}$ is the outer normal to the boundary surface s.

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We adopt the system of Cartesian coordinates (x,y,z) which is rigidly connected with the container in such way, that $|\omega_x| \ll \omega_z$, $|\omega_y| \ll \omega_z$ (z is the direction of the uniform rotation). In this case, by taking into account only the linear term with respect to ω_x/ω_z, ω_y/ω_z, we write eq. (1a) in the form

$$\dot{\vec{v}} + 2\omega_z \vec{e}_z \times \vec{v} = - \nabla\psi + \vec{\chi} \quad , \tag{2}$$

$$\psi = \frac{p}{\rho} + V - \frac{\omega_z^2}{2}(x^2+y^2) + (\omega_x\omega_z + \dot{\omega}_y)\, xz + (\omega_y\omega_z - \dot{\omega}_x)\, yz \quad , \tag{3}$$

$$\vec{\chi} = \dot{\omega}_z\, (y\vec{e}_x - x\vec{e}_y) + 2\vec{e}_z\, (\dot{\omega}_y x - \dot{\omega}_x y), \tag{4}$$

$\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ are the unit vectors, which are oriented along the direction of the axes x,y and z, correspondingly.

To present eq. (1a) in a form which is suitable for the application of the method of perturbation, we calculate the curl of left and right sides of eq. (2). Taking into account conditions (1b) and (4), we get:

$$\mathrm{curl}(\vec{e}_z \times \vec{v}) = \vec{e}_z (\nabla \cdot \vec{v}) - (\vec{e}_z, \nabla)\, \vec{v} = - \frac{\partial \vec{v}}{\partial z} \, ,$$

$$\mathrm{curl}\ \vec{\chi} = - 2\dot{\vec{\omega}} \, ,$$

and

$$\frac{\partial \vec{v}}{\partial z} = \frac{\dot{\vec{\omega}}}{\omega_z} + \frac{1}{2\omega_z}\,\mathrm{curl}\ \dot{\vec{v}} = \frac{i\sigma}{\omega_z}\left(\vec{\omega}_1 + \frac{1}{2}\,\mathrm{curl}\ \vec{v}\right) \quad , \tag{5}$$

where σ is the frequency of oscillations and $\vec{\omega}_1 = \vec{\omega} - \omega_z\, \vec{e}_z$ is the variable part of $\vec{\omega}$.

In the limiting case $\sigma \to 0$, the right side of (5) tends to zero, too, and eq. (5) is reduced to the well known Proudman-Taylor theorem, in accordance with which the stationary flows in rotating fluids ("geostrophic flows") satisfy the equation

$$\partial \vec{v}^{(0)}/\partial z = 0 \; ; \tag{6}$$

and as a result, the components $v_x^{(0)}$, $v_y^{(0)}$, $v_z^{(0)}$ are functions of x,y only.

It is known (Greenspan, 1969), that the geostrophic flows in the bounded volume are described by the conditions:

1) the lines of flow coincide with the isolines

$$\tilde{z} = z_2(x,y) - z_1(x,y) = \text{const} \quad , \tag{7}$$

where $z_2(x,y)$ and $z_1(x,y)$ are consequently the equations of the upper and lower boundary surfaces.

2) the velocities of geostrophic flows are described by the conditions:

$$v_x^{(0)} = v_x^{(0)}(x,y) = -\phi(\tilde{z}) \frac{\partial \tilde{z}}{\partial x} ,$$

$$v_y^{(0)} = v_y^{(0)}(x,y) = \phi(\tilde{z}) \frac{\partial \tilde{z}}{\partial y} , \tag{8}$$

$$v_z^{(0)} = v_z^{(0)}(x,y) = \gamma(x,y)\ \phi(\tilde{z}) \quad ,$$

where

$$\gamma(x,y) = \frac{\partial \tilde{z}}{\partial x}\frac{\partial \bar{z}}{\partial y} - \frac{\partial \tilde{z}}{\partial y}\frac{\partial \bar{z}}{\partial x} ,$$

$$\bar{z} = \bar{z}(x,y) = \frac{z_1(x,y)+z_2(x,y)}{2} ,$$

and $\phi(\tilde{z})$ is an arbitrary function of $\tilde{z}$ which is determined by the initial conditions only. It is easy to see, that the components of $\vec{v}$ described by (8) satisfy the boundary condition (2), the condition of incompressibility (1b) and the dynamical equation (6).

Subsequently, we can use this solution as a zero approximation. Obviously, when we consider the case of forced oscillations instead of the case of stationary flows, then the function $\phi(\tilde{z})$ must be determined uniquely. Let us consider this condition:

To use the method of perturbations, we present the vector of velocity $\vec{v}$ as a sum of the zeroth-order term in the form (8) and as a first-order term:

$$\vec{v} = \vec{v}^{(0)} + \vec{v}^{(1)} . \tag{9}$$

Substituting (9) into (5) and taking into account the first-order terms only, we get:

$$\frac{\partial \vec{v}_1}{\partial z} = \vec{K}(x,y) \quad , \tag{10}$$

where

$$\vec{K}(x,y) = \frac{i\sigma}{\omega_z}\left[\vec{\omega}_1 + \frac{1}{2}\,\mathrm{curl}\,\vec{v}^{(0)}(x,y)\right] . \tag{11}$$

After the integration of (8) with respect to z in the limits from $z_1(x,y)$ to $z_2(x,y)$ we get:

$$\vec{v}^{(1)}(x,y,z) = v_1(x,y)\,\Big|_{z=0} + \vec{K}z \ . \tag{12}$$

Let us consider now the condition of incompressibility and the boundary conditions for the vector $\vec{v}^{(1)}$. Substituting (9) into (1b) and (2) and taking into account, that, in accordance with (8), $\vec{v}^{(0)}$ satisfies the conditions (1b) and (2) automatically, we get:

$$\nabla \cdot \vec{v}^{(1)} = 0 \ , \tag{13a}$$

$$(\vec{v}^{(1)},\vec{N})\big|_s = 0 \ . \tag{13b}$$

Substituting (12) into (13a) and taking into account, that, in accordance with (11),

$$\nabla \cdot \vec{K} = 0 \ ,$$

we obtain:

$$\nabla \cdot \vec{v}_1 = \nabla \cdot \vec{v}_1(x,y)\big|_{z=0} + z\nabla \cdot \vec{K} + (\vec{K},\nabla z) = 0 \quad ,$$

$$\frac{\partial v_x^{(1)}(x,y)\big|_{z=0}}{\partial x} + \frac{\partial v_y^{(1)}(x,y)\big|_{z=0}}{\partial y} + K_z(x,y) = 0 \ . \tag{14}$$

Taking into account, that $\vec{N} = \vec{N}_1 = \left[\frac{\partial z_1}{\partial x}, \frac{\partial z_1}{\partial y}, -1\right]$

on the surface $z_1(x,y)$ and

$\vec{N} = \vec{N}_2 = \left[-\frac{\partial z_2}{\partial x}, -\frac{\partial z_2}{\partial y}, 1\right]$ on the surface $z_2(x,y)$,

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we can present the condition (13b) in the form:

$$(\vec{v}_1(x,y)|_{z=0}, \vec{N}_1) + z_1(\vec{K}, \vec{N}_1) = 0, \tag{15a}$$

$$(\vec{v}_1(x,y)|_{z=0}, \vec{N}_2) + z_2(\vec{K}, \vec{N}_2) = 0 . \tag{15b}$$

To exclude from the equations (15) the component $v_z^{(1)}(x,y)|_{z=0}$, we sum up (15a) and (15b). We then get:

$$v_x^{(1)}(x,y)|_{z=0} \frac{\partial \tilde{z}}{\partial x} + v_y^{(1)}(x,y)|_{z=0} \frac{\partial \tilde{z}}{\partial y} + K_x \frac{\partial}{\partial x} (\tilde{z}\bar{z}) + K_y \frac{\partial}{\partial y} (\tilde{z}\bar{z}) - K_z \tilde{z} = 0. \tag{16}$$

The conditions (14) and (16) determine the unknown function $\phi(\tilde{z})$ uniquely. To prove this, let us reduce the system (14), (16) to a single integro-differential equation. It is easy to see, that, if eq. (14) is valid, then the components of $\vec{v}$ on the surface z=0 can be presented in the form:

$$v_x^{(1)}(x,y)|_{z=0} = \frac{\partial \xi(x,y)}{\partial y} - \int_0^x K_z(x',y)dx' , \tag{17}$$

$$v_y^{(1)}(x,y)|_{z=0} = - \frac{\partial \xi(x,y)}{\partial x} ,$$

where ξ is an arbitrary single-valued twice differentiable function of x,y. Substitution of (17) into (16) reduces the system (14), (16) to a single integro-differential equation:

$$\frac{\partial \tilde{z}}{\partial x}\frac{\partial \xi}{\partial y} - \frac{\partial \tilde{z}}{\partial y}\frac{\partial \xi}{\partial x} + F(x,y) = 0 , \tag{18}$$

where

$$F(x,y) = K_x \frac{\partial}{\partial x} (\tilde{z},\bar{z}) + K_y \frac{\partial}{\partial y} (\tilde{z}\bar{z}) - K_z \tilde{z} - \frac{\partial \tilde{z}}{\partial x} \int_0^x K_z(x',y)dx' . \tag{19}$$

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Now our analysis is very close to the analysis given in (Molodensky, 1989) for the two-dimensional case.

The form of eq. (18) makes it possible to find the increment of the function $\xi(x,y)$ along some contours in terms of F in a manner similar to Cauchy's method for the integration of first-order quasi-linear equations in partial derivatives (Kamke, 1966). Indeed, relation (18) may be regarded as an orthogonality condition for vectors with the Cartesian coordinates

$$\vec{e}_1 = \left(\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, -1 \right) ,$$

and

$$\vec{e}_2 = \left(-\frac{\partial \tilde{z}}{\partial y}, \frac{\partial \tilde{z}}{\partial x}, -F \right) .$$

Since

$$d\xi = \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy,$$

the vector $\vec{e}_1$ is also orthogonal to the vector

$$\vec{e}_3 = (dx, dy, d\xi)$$

which is tangent to the surface $\xi(x,y)$. The vector $\vec{e}_2$ is thus perpendicular to the normal to the surface $\xi(x,y)$, hence it lies in a plane that is tangent to that surface. Consequently, the curves defined by the equations

$$\frac{dx}{\left(-\frac{\partial \tilde{z}}{\partial y} \right)} = \frac{dy}{\left(\frac{\partial \tilde{z}}{\partial x} \right)} = \frac{d\xi}{(-F)} \tag{20}$$

belong to the surface $\xi(x,y)$. The first part of this equation is equal to

$$\frac{\partial \tilde{z}}{\partial x} dx + \frac{\partial \tilde{z}}{\partial y} dy = 0 ,$$

i.e. the curves under consideration coincide with the geostrophic contours $\tilde{z}$ = const. The second part of (20) determines the increment of ξ along these contours:

$$d\xi|_\Gamma = \frac{F\,dx}{\partial\tilde{z}/\partial y} = -\frac{F\,dy}{\partial\tilde{z}/\partial x} = -\frac{F\,d\ell}{\partial\tilde{z}/\partial n} \,. \tag{21}$$

where $d\ell = (dx^2+dy^2)^{\frac{1}{2}}$ is the element of length of this contour and $\partial/\partial n$ is the derivative along the outer normal to it.

Equation (21) is fully equivalent to the original integro-differential equation (18), in the sense that any integral curve of (21) belongs to the surface $\xi(x,y)$ defined by (18) and, conversely, any solution of (18) can be represented as a family of integral curves of (21). For this reason, the condition of existence for solutions of the partial equation is equivalent to that for solutions of the equation (21). It is easy to see, that this latter is reduced to the single requirement that

$$\oint_\Gamma d\xi = -\oint_\Gamma \frac{F\,d\ell}{\partial\tilde{z}/\partial n} = 0 \,. \tag{22}$$

When (22) does not hold, the increment of ξ along a closed geostrophic contour Γ does not vanish, which is incompatible with the assumption of ξ being a single-valued function of the coordinates.

Let us show now that the condition of existence for the first-order terms in (22) determines uniquely the function $\phi(\tilde{z})$ which enters into the zero-order equation. To show this, one expresses the function $\underline{F}$ which enters into (22) in terms of ϕ and the known functions $\tilde{z}$, $\bar{z}$, γ.

Substitution of (8) into (11) yields

$$\vec{K}(x,y) = \frac{i\sigma}{\omega_z}\left[\vec{\omega}_1 + \frac{1}{2}\left[\vec{e}_x\left(\phi'\gamma\frac{\partial\tilde{z}}{\partial y} + \phi\frac{\partial\gamma}{\partial y}\right) - \vec{e}_y\left(\phi'\gamma\frac{\partial\tilde{z}}{\partial x} + \phi\frac{\partial\gamma}{\partial x}\right)\right] + \vec{e}_z\left(\phi'(\nabla\tilde{z})^2 + \phi\,\Delta\tilde{z}\right)\right] ,$$

where

$$\phi' = \frac{d\phi(\tilde{z})}{d\tilde{z}} \,.$$

Taking into account this expression and (19), one can write eq. (22) in the form:

$$\oint_{\Gamma} \frac{d\ell}{\partial\tilde{z}/\partial n} \left[\left(\omega_x + \frac{1}{2} \left(\phi'\gamma \frac{\partial\tilde{z}}{\partial y} + \phi \frac{\partial\gamma}{\partial y} \right) \right) \frac{\partial(\tilde{z}\bar{z})}{\partial x} + \right.$$

$$+ \left(\omega_y - \frac{1}{2} \left(\phi'\gamma \frac{\partial\tilde{z}}{\partial x} + \phi \frac{\partial\gamma}{\partial x} \right) \right) \frac{\partial(\tilde{z}\bar{z})}{\partial y} - (\phi'(\nabla\tilde{z})^2 + \phi\Delta\tilde{z})\, \tilde{z} - \tag{23}$$

$$\left. - \frac{\partial\tilde{z}}{\partial x} \int_0^x (\phi'(\nabla\tilde{z})^2 + \phi\Delta\tilde{z})\, dx' \right] = 0.$$

The last term in the left hand side of (23) can be transformed in the following way: By taking into account that on the contour Γ in accordance with (21)

$$\frac{d\ell}{\partial\tilde{z}/\partial n} \frac{\partial\tilde{z}}{\partial x} = dy \quad ,$$

one can write

$$- \oint_{\Gamma} \frac{d\ell}{\partial\tilde{z}/\partial n} \frac{\partial\tilde{z}}{\partial x} \int_0^x (\phi'(\nabla\tilde{z})^2 + \phi\Delta\tilde{z})\, dx' = - \iint_s (\phi'(\nabla\tilde{z})^2 + \phi\Delta\tilde{z})\, ds \quad ,$$

where $ds = dx\, dy$ and s is the area of the region, which lays in the plane (x,y) and is bounded by the contour Γ. Taking into account now that

$$\phi'(\nabla\tilde{z})^2 + \phi\Delta\tilde{z} = \mathrm{div}\,(\phi\nabla\tilde{z}) \quad ,$$

and using the well known Gaussian formula, one writes this term in the form:

$$- \iint_s (\phi'(\nabla\tilde{z})^2 + \phi\Delta\tilde{z})\, ds = - \oint_{\Gamma} \phi(\nabla\tilde{z},\vec{n})\, d\ell = - \oint_{\Gamma} \phi \frac{\partial\tilde{z}}{\partial n}\, d\ell \ .$$

In accordance with (8), ϕ is a function of the single argument $\tilde{z}$, and the values ϕ, ϕ' are constants on the contour Γ, where $\tilde{z}$ = const. Therefore, the values ϕ,ϕ' can be shifted in front of the symbol of integration, and eq. (23) can be presented in the form of an ordinary differential equation with respect to the single unknown function $\phi(\tilde{z})$:

$$c_1(\tilde{z})\phi'(\tilde{z}) + c_2(\tilde{z})\phi(\tilde{z}) + c_3(\tilde{z}) = 0 \tag{24}$$

where

$$c_1(\tilde{z}) = \frac{1}{2}\oint_\Gamma \frac{d\ell}{\partial\tilde{z}/\partial n}\left[\gamma\left(\frac{\partial\tilde{z}}{\partial y}\frac{\partial(\tilde{z}\bar{z})}{\partial x} - \frac{\partial\tilde{z}}{\partial x}\frac{\partial(\tilde{z}\bar{z})}{\partial y}\right) - \tilde{z}\left(\frac{\partial\tilde{z}}{\partial n}\right)^2\right]$$

$$= -\frac{\tilde{z}}{2}\oint_\Gamma \frac{d\ell}{\partial\tilde{z}/\partial n}\left\{\gamma^2 + \left(\frac{\partial\tilde{z}}{\partial n}\right)^2\right\}; \tag{25}$$

$$c_2(\tilde{z}) = \frac{1}{2}\oint_\Gamma \frac{d\ell}{\partial\tilde{z}/\partial n}\left[\left(\frac{\partial\gamma}{\partial y}\frac{\partial(\tilde{z}\bar{z})}{\partial x} - \frac{\partial\gamma}{\partial x}\frac{\partial(\tilde{z}\bar{z})}{\partial y}\right) - \tilde{z}\Delta\tilde{z} - \left(\frac{\partial\tilde{z}}{\partial n}\right)^2\right];$$

$$c_3(\tilde{z}) = -(\omega_1)_z\, s(\tilde{z}) + \oint_\Gamma \frac{d\ell}{\partial\tilde{z}/\partial n}\left\{\omega_x\frac{\partial(\tilde{z}\bar{z})}{\partial x} + \omega_y\frac{\partial(\tilde{z}\bar{z})}{\partial y} - (\omega_z)_1\tilde{z}\right\}.$$

Taking into account now that the element of the surface ds is equal to

$$ds = \frac{d\ell d\tilde{z}}{\partial\tilde{z}/\partial n}, \tag{26}$$

we present the expression for $c_2(\tilde{z})$ in the form:

$$c_2(\tilde{z}) = \frac{1}{2}\frac{d}{d\tilde{z}}\iint_s \left\{\frac{\partial\gamma}{\partial y}\frac{\partial(\tilde{z}\bar{z})}{\partial x} - \frac{\partial\gamma}{\partial x}\frac{\partial(\tilde{z}\bar{z})}{\partial y} - \tilde{z}\Delta\tilde{z} - \left(\frac{\partial\tilde{z}}{\partial n}\right)^2\right\} ds =$$

$$\frac{1}{2}\frac{d}{d\tilde{z}}\iint_s \operatorname{div}\vec{A}\, ds = \frac{1}{2}\frac{d}{d\tilde{z}}\oint_\Gamma (\vec{A},\vec{n})\, d\ell = \tag{27}$$

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$$\frac{1}{2}\frac{d}{d\tilde{z}}\oint_\Gamma \frac{d\ell}{\partial\tilde{z}/\partial n}\ (\vec{A},\nabla\tilde{z})\ ,$$

where

$$\vec{A} = \vec{e}_x\left[-\gamma\frac{\partial(\tilde{z}\bar{z})}{\partial y} - \tilde{z}\frac{\partial\tilde{z}}{\partial x}\right] + \vec{e}_y\left[\gamma\frac{\partial(\tilde{z}\bar{z})}{\partial x} - \tilde{z}\frac{\partial\tilde{z}}{\partial y}\right] . \qquad (28)$$

Substituting (28) into (27), one realizes that

$$c_2(\tilde{z}) = \frac{dc_1(\tilde{z})}{d\tilde{z}} . \qquad (29)$$

The expression for the coefficient $c_3(\tilde{z})$ is performed analogously. Using (26), we get:

$$c_3(\tilde{z}) = -(\omega_1)_z\left[s(\tilde{z}) + \tilde{z}\frac{ds(\tilde{z})}{d\tilde{z}}\right] + \frac{d}{d\tilde{z}}\left[\iint_s \operatorname{div}\vec{B}\ ds\right] =$$

$$= -(\omega_1)_z\frac{d}{d\tilde{z}}(\tilde{z}s(\tilde{z})) + \frac{d}{d\tilde{z}}\left[\oint_\Gamma\frac{d\ell}{\partial\tilde{z}/\partial n}\left(\frac{\partial\tilde{z}}{\partial x}B_x + \frac{\partial\tilde{z}}{\partial y}B_y\right)\right], \qquad (30)$$

where

$$\vec{B} = \tilde{z}\bar{z}(\omega_x\vec{e}_x + \omega_y\vec{e}_y). \qquad (31)$$

Substituting (31) into (30) and taking into account, that $\tilde{z}|_\Gamma =$ const., we get

$$c_3(\tilde{z}) = \frac{dD(\tilde{z})}{d\tilde{z}} , \qquad (32)$$

where

$$D(\tilde{z}) = -\tilde{z}(\omega_1)_z s(\tilde{z}) + \tilde{z}\oint_\Gamma\frac{d\ell\bar{z}}{\partial\tilde{z}/\partial n}\left(\omega_x\frac{\partial\tilde{z}}{\partial x} + \omega_y\frac{\partial\tilde{z}}{\partial y}\right) . \qquad (33)$$

The conditions (29), (32) make it possible to integrate the differential equation (24) analytically. Indeed, using (29), we get

$$c_1(\tilde{z})\phi'(\tilde{z}) + c_2(\tilde{z})\phi(z) = (c_1(\tilde{z})\phi(\tilde{z}))' ,$$

and equation (24) is reduced to

$$(c_1(\tilde{z})\phi(\tilde{z}) + D(\tilde{z}))' = 0 .$$

After the integration of this equation with respect to $\tilde{z}$ we get:

$$\phi(\tilde{z}) = -\frac{D(\tilde{z})}{c_1(\tilde{z})} + \frac{\text{const.}(\tilde{z})}{c_1(\tilde{z})} .$$

Using the expression for $c_1(\tilde{z})$ (25), one sees that the coefficient $c_1(\tilde{z})$ is equal to zero in the points of extrema of the function $\tilde{z}(x,y)$. Taking into account, that the velocities $\vec{v}^{(0)}$ are bounded in the vicinities of these points, we can counclude, that

$$\text{const.}\ (\tilde{z}) = 0 ,$$

and, as a final result,

$$\phi(\tilde{z}) = -\frac{D(\tilde{z})}{c_1(\tilde{z})} - 2\,\frac{-s(\tilde{z})(\omega_1)_z + \oint_\Gamma \frac{d\ell\bar{z}}{\partial\tilde{z}/\partial n}\left[\frac{\partial\tilde{z}}{\partial x}\,\omega_x + \frac{\partial\tilde{z}}{\partial y}\,\omega_y\right]}{\oint_\Gamma \frac{d\ell}{\partial\tilde{z}/\partial n}\left\{\gamma^2 + \left[\frac{\partial\tilde{z}}{\partial n}\right]^2\right\}} . \tag{34}$$

Equations (34) and (8) give the full solution of the problem under consideration.

2.1 THE QUALITATIVE ANALYSIS OF THE RESULTS

It is interesting to compare (34) with the well known Poincare's solution (Lamb, 1932). It is known that the last one describes the oscillations (generally, with finite amplitude and arbitrary frequency) of the homogeneous, incompressible, inviscid liquid, which is surrounded by a non-uniformly rotating rigid container with an ellipsoidal boundary.

Our solution (34) describes the more particular case in so far as we consider only the small oscilations for the limiting case of the very long periods ($\sigma/\omega \rightarrow 0$). At the same time, in some aspects, it is essentially more general because it describes the motions not only in the ellipsoidal cavity, but in the cavity of arbitrary geometry.

It is easy to show that this solution predicts some new effects, which are absent in the case of the Earth model with an elliptical core-mantle boundary (which is axially symmetrical with respect to the axis of rotation). They are as follows:

1. It is known, that the free Eulerian (Chandler) wobble of the Earth model with an elliptical core-mantle boundary excites the motions in the liquid core with an invariant z-component of angular momentum. Consequently, the Chandler wobble is not accompanied by the l.o.d. (length of day) variations at the same (Chandlerian) period.

2. Inversely, the tidal variations of the l.o.d. excite the currents inside the liquid core without the x- and y- components of angular momentum. As a result, the long periodic tidal waves don't excite the polar motion.

Using our expression (34), one can see, that these both properties don't take place in the general case of an arbitrary core-mantle boundary. Moreover, in the case of Chandler wobble the amplitude of x-component of the angular momentum in the liquid core does not generally coincide with the amplitude of the y-component. As a result, the trajectory of the Chandler wobble is not circular, but elliptical.

Taking these circumstances into account, it is possible to formulate the inverse problem of estimation of the possible core-mantle boundary heterogeneities based on modern astrometrical data. To make this, we shall consider first the dynamics of the liquid core for some very simple models of the core-mantle boundary.

Let us begin the qualitative analysis of equ. (34) from the consideration of some very simple cases.

2.1.1. If the container is symmetrical with respect to the plane $z = 0$, then $z_2(x,y) = -z_1(x,y)$, and $\bar{z} = \gamma = 0$. Substituting these values into (34), we get:

$$\phi(\tilde{z}) = -2(\omega_1)_z \frac{s(\tilde{z})}{\oint_\Gamma \frac{\partial \tilde{z}}{\partial n} d\ell} . \tag{35}$$

It is interesting to note, that the geostrophic flow determined by (35) is not dependent on the components ω_x, ω_y. Probably, the physical sense of this conclusion can be interpreted as follows: it is known (Greenspan, 1969), that the geostrophic currents organize the system of Proudman-Taylor columns, which are similar to the rigid bodies in several aspects. For example, these columns have the tendency to conserve their form and sizes. It is, indeed, easy to see, that the stationary geostrophic flow described in section 1 is possible only in the case, where the sizes of the columns in the direction parallel to $\vec{\omega}$ is not dependent on time (in the opposite case the stationary flow along the geostrophic contours $\tilde{z}$=const. does not satisfy the condition (6)). If the boundary surfaces are mobile, with respect to the vector $\vec{\omega}$, then, in general, the flow is not stationary and the kinetic energy of the fluid is not constant.

The compression of the Proudman-Taylor columns in the z-direction is accompanied by a decreasing z-component of circulation $(\mathrm{curl}\ \vec{v})_z$ and of the total knetic energy; in the case of stretching the signs are opposite.

From the simple geometrical considerations it is easy to see, that for the case $z_2(x,y) = -z_1(x,y)$, the small tilt of the vector $\vec{\omega}$ is not accompanied by any compression or stretching of the Proudman-Taylor's columns, and the geostrophic flows are not excited. This is why $\phi(\tilde{z})$ is not dependent on the components ω_x, ω_y.

Using the general relation (34), one can see that the ratio of the velocities of geostrophic flow to the velocities of the column's compression or stretching are in the general case of the order of ω/σ. In the limiting case $\sigma \to 0$ this ratio tends to infinity. It means, that even very small long-periodic polar motion (such as the Chandler wobble) results in significant geostrophic motions. In the case of Chandler wobble of the real Earth the boundary of the liquid core is close to the sphere, and the geostrophic contours are close to the circles with centers on the axis of the Earth's rotation. Such motion has an angular momentum mainly in the direction of the z axis and must result in variations of the length of day with Chandler period. We shall consider the numerical estimation of this effect in section 2.3.

2.1.2 For the most simple case when the container is symmetrical both with respect to the plane $z = 0$ and to the axis $x = y = 0$, the values $\partial\tilde{z}/\partial n$ are constant on the contour Γ, and relation (35) is reduced to

$$\phi(\tilde{z}) = -\frac{2s(\tilde{z})(\omega_1)_z}{\ell(\tilde{z})\ \partial\tilde{z}/\partial n},$$

where $\ell(\tilde{z})$ is the length of the contour line $\tilde{z}$ = const. Taking into account, that this contour coincides with the circle, we get:

$$s(\tilde{z}) = \pi r^2,\ \ell(\tilde{z}) = 2\pi r,\ \phi(\tilde{z}) = -r/\partial z/\partial n, \text{ and } \vec{v}^{(0)}$$

$$= -(\omega_1)_z\ \vec{e}_z \times \vec{n},$$

(where $r = (x^2+y^2)^{\frac{1}{2}}$ is the radius of circle, and $\vec{r}$, as before, is the radius-vector). This result has a trivial physical meaning: Obviously, the non-uniform rotation of the container with respect to its axis of symmetry does not excite any differential motion in the liquid, and the liquid conserves its uniform rotation around the z axis in space. In the non-uniformly rotating system of coordinates (x,y,z) this motion is described similarly to the non-uniform rotation of a rigid body.

Using the relation (35), one can see, that the dynamic coupling between the liquid core and mantle is determined not by the value of the deviation of the core-mantle boundary with respect to the axially symmetric geometry but only by the ratios of the bounded surfaces $s(\tilde{z})$ inside the contours Γ to the lengths of the contours

$$\ell = \oint_{\Gamma} d\ell \ .$$

The dynamic coupling is significant when the ratios s/ℓ are small enough. This situation takes place, for example, when the contours $\tilde{z} = \text{const.}$ present the system of closed contours with a relatively small scale of lengths.

2.1.3. Now we can consider the more realistic case where the core-mantle boundary is close enough to the sphere. If we propose, in addition, that the partial derivatives of the core-mantle boundary (with respect to x,y) are close enough to the same derivatives of the unperturbed (spherical) boundary, then we can write

$$\tilde{z} \approx 2b\cos\theta, \quad \frac{\partial \tilde{z}}{\partial n} \approx -2 \,\mathrm{tg}\,\theta \quad ,$$

and

$$\gamma^2 << \left(\frac{\partial \tilde{z}}{\partial n}\right)^2 ,$$

where θ is the co-latitude and a is the mean radius of the core-mantle boundary.

The geostrophic contours Γ are close to the circles, $d\ell = a\sin\theta \, d\lambda$ and one can estimate the contribution to the integral in (34) as follows:

$$\oint_{\Gamma} \frac{d\ell}{\partial \tilde{z}/\partial n} \left(\gamma^2 + \left(\frac{\partial z}{\partial n}\right)^2 \right) \approx \oint_{\Gamma} \frac{\partial \tilde{z}}{\partial n} \, d\ell \approx -4\pi a \frac{\sin^2\theta}{\cos\theta} \ .$$

Taking into account that

$$\frac{\partial \tilde{z}/\partial x}{\partial \tilde{z}/\partial n} = \cos\lambda \ , \quad \frac{\partial \tilde{z}/\partial x}{\partial \tilde{z}/\partial n} = \sin\lambda \quad ,$$

where λ is longitude, equ. (34) is presented in the form:

$$\phi(\tilde{z}) = -\frac{1}{4\pi} \,\mathrm{ctg}\,\theta \int_0^{2\pi} \bar{z}(\cos\lambda\delta\omega_x + \sin\lambda\delta\omega_y)\, d\lambda \ . \qquad (36)$$

Using the presentation of function $\bar{z}(x,y)$ in the form of a Fourier series

$$\tilde{z} = \sum_{n=0}^{\infty} \bar{z}_n^c (R) \cos n\lambda + \bar{z}_n^s (R) \sin n\lambda \quad ,$$

$$\left\{ \text{where } R = \sqrt{x^2+y^2} \right\} ,$$

one realizes that the integral (36) does not vanish only for the harmonics of degree n = 1. After integration of (36) with respect to λ, we get:

$$\phi(\tilde{z}) = - \frac{1}{4\pi} \operatorname{ctg} \theta \, (\bar{z}_1^c \, \delta\omega_x + \bar{z}_1^s \, \delta\omega_y) \ . \tag{37}$$

The substitution of this expression into (8) yields all three components $v_x^{(0)}$, $v_y^{(0)}$, $v_z^{(0)}$ and the angular momentum of the liquid core uniquely. Using the definition of the z-component of the angular momentum, we estimate:

$$M_z = \int_\tau (xv_y^{(0)} - yv_x^{(0)}) \, \rho \, d\tau \sim \frac{c_1 \bar{z}_1}{a} \delta\omega \quad , \tag{38}$$

where c_1 is the moment of inertia of the liquid core, $\bar{z}_1 = (\bar{z}_1^{c\,2} + \bar{z}_1^{s\,2})^{\frac{1}{2}}$,

a is the mean radius of the liquid core and $\delta\omega = (\delta\omega_x^2 + \delta\omega_y^2)^{\frac{1}{2}}$.

Using the numerical value $\delta\omega \sim 10^{-6}\ \omega_z$ (which corresponds to an amplitude of Chandler wobble of ~ 0,2 arc sec), and $c_1 \sim 0.1$ c (where c is the moment of inertia of the mantle) we find

$$\frac{\delta M_z}{M} = - \frac{(\delta\omega_z)_{mantle}}{\omega_z} \sim 10^{-7} \frac{\bar{z}_1}{a} \ , \tag{39}$$

where $M = c\omega_z$ is the angular momentum of the mantle and $(\delta\omega_z)_{mantle}$ is the amplitude of the periodic variation of $(\omega_z)_{mantle}$ at Chandler frequency.

To make this relation more obvious let us consider the case where $\bar{z}_1$ is a linear function of R, i.e. $\bar{z}_1 = K\,R$. It is easy to see that the value d = K/e (where e is the geometric flattening of the liquid core) is equal to the tilt of the main axis of the liquid core's ellipsoid of inertia with respect to the axis of rotation $\vec{\omega}$. Taking this relation into account and introducing the values $\delta\lambda = \int(\delta\omega_z)_{mantle}\, dt$, which is equal to the angular displacement of the mantle in the direction of longitude, and $(\delta\lambda_0)$ which is the amplitude of $\delta\lambda$, then we may

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write (39) in the form

$$\delta\lambda_0 \approx 6 \text{ m arc sec.} \cdot \alpha \ . \tag{40}$$

If we assume that the accuracy of modern VLBI-measurements of periodical processes is of the order of 0.15 *τρ* 0.18 m arc sec. (Gwinn, and Shapiro, 1986), then the measurements of $\delta\lambda_0$ make it possible to determine α with the accuracy of the order of 0.025 - 0.030 rad or 1.5 - 2 degrees.

We may thus conclude that the measurements of l.o.d.-variations at the Chandler period can be considered as a very sensitive method for investigating the core-mantle boundary. Some numerical estimations and examples are given in Section 3.

Inversely, using (34), it is easy to show, that the influence of the tidal l.o.d. variations on polar motion is extremely weak. Under no circumstances do they exceed a value of the order of 10^{-3} m arc s, which is two orders of magnitude smaller than the current accuracy inherent in the harmonic analysis of VLBI measurements.

2.1.4. The influence of the core-mantle topography on the Chandler period and on the ellipticity of the Chandler wobble can be estimated as follows. Using Poincare's presentation for the velocities within the ellipsoidal liquid core in the form

$$\tilde{v}_x \sim - \frac{z\sigma}{\omega_z} \delta\omega_y \ ,$$

$$\tilde{v}_y \sim \frac{z\sigma}{\omega_z} \delta\omega_x \ ,$$

$$\tilde{v}_z \sim \frac{\sigma}{\omega_z} (x\delta\omega_y - y\,\delta\omega_x) \ ,$$

and comparing these expressions with our solutions (8) and (37), we realize that

$$v_x^{(0)}/\tilde{v}_x \sim v_y^{(0)}/\tilde{v}_y \sim \frac{\bar{z}_1}{z} \frac{\omega_z}{\sigma} \ ;$$

$$v_z^{(0)}/\tilde{v}_z \sim \frac{\bar{z}_1 \ \partial\bar{z}_1/\partial\lambda}{2R^2} \frac{\omega_z}{\sigma} \ .$$

In the case of the Chandler wobble ($\omega_z/\sigma \sim 400$) the first ratio is equal to unity if $\bar{z}_1/z \sim 1/400$, i.e. whenever the deviations of core-mantle boundary with respect to the ellipsoid are of the order of only 10 km. Consequently, the topography of the core-mantle boundary for the real Earth model can exert an extremely strong influence on the distribution of the currents in the liquid core. Nevertheless, the influence of these currents on the Chandler period T_{Ch} and on the ellipticity of the Chandler wobble is comparatively weak.

To show this it is enough to remember that the hydrodynamic motions under consideration have mainly z-component of the angular momentum and, consequently, they influence mainly the l.o.d. variations.

2.2 THE COMPARISON WITH THE MEASUREMENTS

The results of the Maximum Entropy Spectrum Analysis of the modern V.L.B.I.-l.o.d. data are presented in Fig. 1. It is necessary to note, that the amplitudes obtained by MESA-technique are well known to be problematic in general. One can see, that some peak with very small amplitude in the vicinity of Chandler frequency probably exist, but its ratio to the level of noise is too small to identify it with the necessary reliability.

3. LINEAR AND NONLINEAR MODELS OF THE DYNAMICAL POLE TIDES

The asymtotic behaviour of solutions to Laplace's tidal equations (L.t.e.) at low (for example, Chandlerian) frequencies was considered in recent years in may papers (see, for example, Dickman 1985, 1986; O'Connor, 1986; Carton, Wahr, 1986; Molodensky, 1989; Groten, Lenhardt, Molodensky, 1990). It was shown in the last two papers, that for the limiting case $\sigma/\omega \to 0$ (where σ is the tidal frequency and ω is the angular velocity of the Earth's diurnal rotation)) these solutions are unstable in that the functions involved in the zero-order approximation are not uniquely determined by the zero-order equations, but depend on first-order terms (terms of the order of σ/ω) as well. As a result, the solutions of L.t.e. significantly depend on the very small terms entering the L.t.e. In the most general case the equations describing the pole tides in the thin (in comparison with the Earth's radius r) layer of the liquid are described by the known system of governing equations (see, for example, Kagan, Monin, 1978) read:

$$\dot{\vec{v}} + (\vec{v},\nabla)\vec{v} + 2\omega\cos\theta\ [\vec{e}_r\vec{v}] = -g\nabla(\zeta-\bar{\zeta}) + \vec{F}\ ,$$

$$\dot{\zeta} = -\operatorname{div}_2(\vec{v}h)\quad , \qquad (41)$$

where $\vec{v}=(v_\theta,\ v_\lambda)$ is the two-dimensional vector of tidal velocity, ζ is the level of the ocean, $\bar{\zeta}=\bar{\zeta}(\theta,\lambda)$ is the associated equipotential surface, ω is again the angular velocity of the Earth's diurnal rotation, $\vec{e}_r$ is the radius-vector, g is the acceleration due to gra-

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vity at the Earth's surface, $h = h(\theta,\lambda)$ is the ocean depth, $\mathrm{div}_2(\vec{v}h)$ is the divergence of two-dimensional vector $\vec{v}h = h\cdot(v_\theta, v_\lambda)$, $\vec{F} = (F_\theta, F_\lambda)$ are the components of the force of friction, which act on the element of the liquid. In the most general case the vector $\vec{F}$ is presented in the form

$$\vec{F} = -k_0\vec{v} + k_h\Delta\vec{v},$$

where k_0 is the coefficient of the bottom friction and k_h is the coefficient of turbulent horizontal friction. In case of turbulent motion the coefficient k_0 is proportional to $|\vec{v}| = (v_\theta^2 + v_\lambda^2)^{\frac{1}{2}}$ being a function of the depth distribution $h(\theta,\lambda)$; in case of laminar motion k_0 is independent of $\vec{v}$ and is a function of the distribution $h(\theta,\lambda)$ only.

3.1 LINEAR MODEL

The asymtotic behaviour of solutions to eq. (41) for the linear approximation is described by the conditions:

1) the lines of flow and isolines $\tilde{\zeta} = \zeta - \bar{\zeta}$ coincide with the geostrophic contours Γ as determined above which, for the case of a thin layer of liquid, are described by

$$\alpha = \frac{g}{2\omega a^2}\,\frac{h(\theta,\lambda)}{\cos\theta} = \mathrm{const.}$$

2) the dependence of $\zeta(\alpha)$ is determined by the ordinary differential equation (Molodensky, 1989):

$$[c_1(\alpha)\,\zeta'(\alpha)]' + c_3(\alpha)\zeta(\alpha) = b(\alpha)\ , \tag{42}$$

where the prime denotes differentiation with respect to α and c_1, c_3, b are the known functions of the depth distribution, which are described by the relations:

$$c_1(\alpha) = \alpha \oint_\Gamma \frac{(\kappa + i\sigma)\partial\alpha/\partial n}{\cos\theta}\, d\ell\ ,$$

$$c_3(\alpha) = -\frac{2i\sigma\omega}{a^2}\oint_\Gamma \frac{d\ell}{\partial\alpha/\partial n}\ , \tag{43}$$

$$b(\alpha) = \frac{2i\sigma\omega}{a^2} \oint_{\Gamma} \frac{\bar{\zeta}\, d\ell}{\partial\alpha/\partial n},$$

where a is the mean radius of the earth, $d\ell$ is the element of length of the geostrophic contour Γ (here for simplicity we put k_h=0).

The solutions of this equation depend mainly on the dimensionless value

$$H = \frac{g}{4\omega^2\cos^2\theta} \frac{h}{r_0{}^2} = \frac{73}{\cos^2\theta} \frac{ha}{r_0{}^2},$$

r_0 is the horizontal scale of length of the closed contours α = const. Taking into consideration the dimensionless value γ, which is equal to the ratio of the mean values of ζ to the mean value of $\bar{\zeta}$ in the same region, one obtains a simple estimate of the relation $\gamma(H)$ for the dissipationless case as follows (Molodensky, 1989):

H = 0,1	0,2	0,5	1,0	2,0	5,0
γ = 0,520	0,656	0,812	0,893	0,944	0,976

From this table one realizes that, when depth increases (or in, other words, equivalently: when the horizontal dimensions of the closed geostrophic contours α = const. decrease), the dynamic tide approaches the static one. When $\cos^2\theta$=0.5 and h=4 km, H=0.1 corresponds to r_0=$6\cdot10^3$ km and H=5 to r_0=$8\cdot10^2$ km. One sees from the aforementioned table that, for the first case, the deviation of the dynamic pole tide from the statical one is significant, whereas for the second case it is very small.

The isolines α=const., for the real ocean model, are given in Levitus (1982) and were recalculated by us on the ground of the spherical harmonics expansion of the depths distribution for degrees $\ell\leq180$. The results are shown on Fig. 20, 20b. These pictures reveal that in most regions of the real ocean the characteristic scale of the length r_0 is in almost any case less than $(2\text{-}3)\cdot10^3$ km. Moreover, the regions, where isolines α=const. are closed, cover a comparatively small part of the oceanic surface. Using the simple estimates based on the aforementioned relation $\gamma(H)$, one sees that, in linear approximation, everywhere in the ocean we have

$$\zeta < 0.1\,\bar{\zeta},$$

and the influence of the dynamic pole tide on the Chandler period is less than the errors inherent in modern measurements: $\delta T_{Ch} < 1$ day.

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3.2 NON-LINEAR MODELS

As was mentioned above, the instability of the solutions of the equations (41) results in the strong dependence of the solutions on small perturbing terms. To estimate the non-linear effects in the dynamic theory of pole tide, it is necessary to compare two groups of small perturbations.

1) the linear terms $\dot{\vec{v}}$, $(\vec{F})_{laminar}$ and

2) the non-linear terms $(\vec{F})_{turbulent}$, $(\vec{v},\nabla)\,\vec{v}$.

To estimate the non-linear terms, it is necessary to take into account not only the tidal currents, but also the nontidal stationary currents in the real ocean, i.e. to present the velocity vector $\vec{v}$ as a sum:

$$\vec{v} = \vec{v}_0 + \vec{v}_1$$

where $\vec{v}_0$ is the vector of velocity of the nontidal stationary ocean currents and $\vec{v}_1$ is relatively small vector tidal flow, in comparison with $\vec{v}_0$. By taking into account that, for the real ocean, $\vec{v}_0$ is of the order of a few cm s^{-1} one realizes that the non-linear group of perturbing terms is greater than, or of the same order as, the linear group, and they must consequently be included in our considerations. The influence of non-linear terms is manifested in the following new properties of the governing equations:

1) The geostrophic contours (lines of flow) are determined not only by the depth distribution $h(\theta,\lambda)$, but also by the distribution of the world ocean currents $\vec{v}_0$. Consequently, instead of contours $h/\cos\theta$ = const., it is necessary to consider the isolines of "potential vorticity"

$$P = \frac{2\omega\cos\theta + (\mathrm{rot}\,\vec{v}_0)_r}{h} = \mathrm{const.};$$

in this case, the role of the parameter r_0 plays the role of horizontal scale of length of the closed contours P=const.

2) As the values k_0 are functions of $|\vec{v}|$, entering into (43) the coefficients c_1, c_3 are functions of $|\vec{v}|$ too. This means that the governing equations (42) are significantly non-linear.

The distribution of the surface currents $\vec{v}_0(\theta,\lambda)$ in the actual global ocean (Fahrbach et al., 1985) is presented in Fig. 3. Comparing Fig. 2 and 3, one realizes, that the difference between the isolines α=const. and P=const. is significant and that it is manifested in the bigger scale length of the regions bounded by isolines P=const. in comparison with the regions, bounded by α=const. As a result, the in-

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fluence of the dynamic pole tide on the Chandler period is essentially greater for the non-linear model than for the linear one.

From Fig. 3 is seen that, for the non-linear model, the typical values r_0 are of the order of $(3\div5)\cdot10^3$ km. Taking into account the relation $\gamma(H)$ given in the aforementioned table we conclude, that for the non-linear model without dissipation (i.e. for the model which takes into account the non-linear term $(v,\nabla)\ v$ only) the possible values of γ are of the order of $0.6\div0.8$. Consequently, the influence of the dynamical pole tides on the period of Chandler wobble may be of the order of $6\div12$ days.

The more exact numerical estimation of the dynamical pole tides for the non-linear approximation is complicated mainly by the following circumstances:

1. The absence of the rigorous mathematical models of oceanic bottom and horizontal turbulent friction. As a result, the models of the dependence of the coefficients k_1, k_2 on $|\vec{v}|$, h are based on some inexact empirical and semi-empirical laws (Schwiderski, 1980);

2. The absence of the detailed models of the world-ocean currents distribution with the depth.

As a result, the exact estimation of the bottom friction is impossible even in the case when the law of bottom friction is known.

Nevertheless the sign of the effects of the ocean friction is determined uniquely. As a matter of fact, when $\sigma/k \to 0$, then the deviation of the dynamic pole tide from the statical one is negative and tends to zero (Molodenski, 1989). Conseqeuntly, the period of Chandler wobble is an increasing function of the coefficients of friction k_0 and k_h. Moreover, it is known that, for turbulent motions, the values of these coefficient present an increasing (usually linear) function of the velocities (Kagan, Monin, 1978). As a result, we may claim, that the period of Chandler wobble must present an increasing function of its amplitude, and the range of the variation of the period is of the order of $6\div12$ days. It is interesting to note, that exactly the same conclusion was obtained by Melchior (1957) based on the analysis of empirical data.

It is interesting to compare the Melchior's results with the results of the latest analysis. The maximum entropy spectral analysis of the intervals 1900-1920; 1920-1940; 1940-1960; 1960-1978 and 1967-1984 was performed by Lenhardt, Groten (1985). The results are presented on Fig. 4 (circles). The results of Melchior are presented on the same picture as points. One can realize, that there is a very high probability, that hte correlation between the amplitudes and periods of Chandler wobble exist indeed.

Thus we may conclude that the interrelation between long-tern amplitude and frequency variations in polar motion may be attributed with the high probability to the influence of turbulent friction for the non-equilibrium pole tide.

CONCLUSIONS

The efforts discussed in this paper basically refer to the axis of rotation and the associated modulus of the earth rotation vector which might be expressed in terms of LOD. Insofar the title of this paper might be questioned if we assume that, by definition, the quantities considered here are related to the celestial system of reference instead of the terrestrial frame. As polar motion defines two of the Eulerian angles relating celestial to terrestial system it is more or less a matter of personal judgment whether we discuss those perturbations with respect to terrestrial or celestial frames. The sources of these perturbations are so closely related to the earth itself that our choice of title appears appropriate in order to avoid misunderstanding.

Consequently, we did not refer to a particular type of a CTRS (Boucher, 1990) such as IERS-TRF and rather treated the topic in general terms.

Two aspects have to be stressed : (1) It still appears possible to clear up existing open problems related to the fluid parts of the earth-fluid outer core and ocean - by precise analysis of polar motion and LOD-data. Consequently, astrometry has not yet been fully exploited in giving information on geophysics in a domain of frequencies where little alternative information is available and (2) by still improving the accuracy in measuring polar motion and LOD we may still get substantially better information on the physics of the earth which, together with improved atmospheric (AAM etc.) data, could lead to the possibility to model and predict polar motion and LOD better than it is carried out now. This paper fills a gap in that respect.

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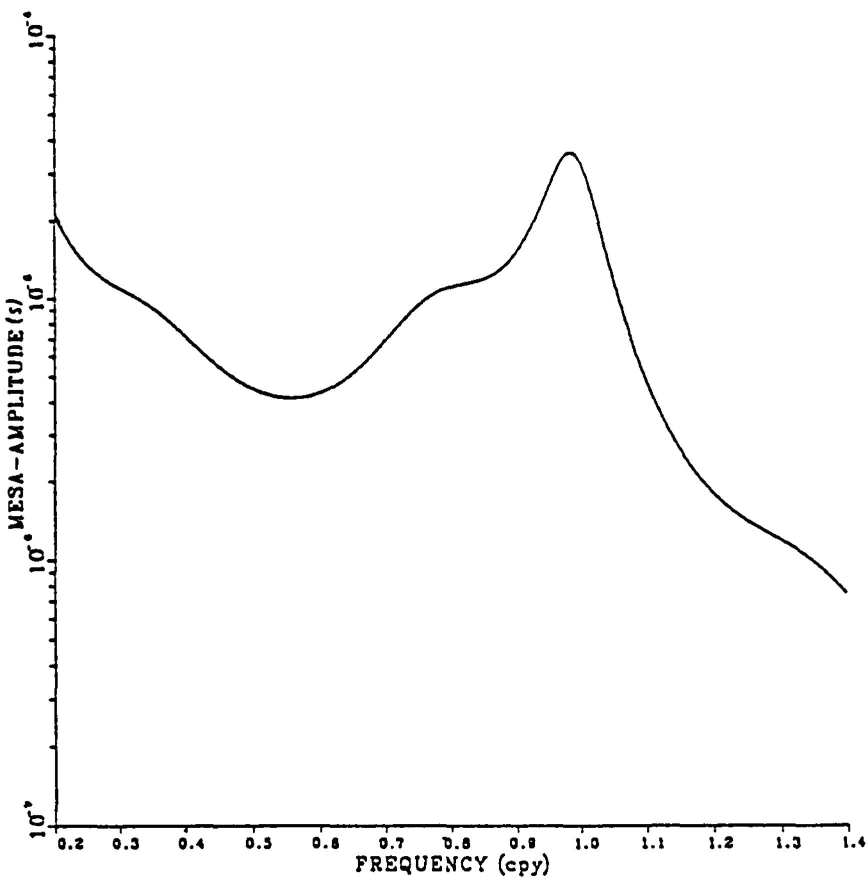

FIG.1 SPECTRUM OF L.O.D.

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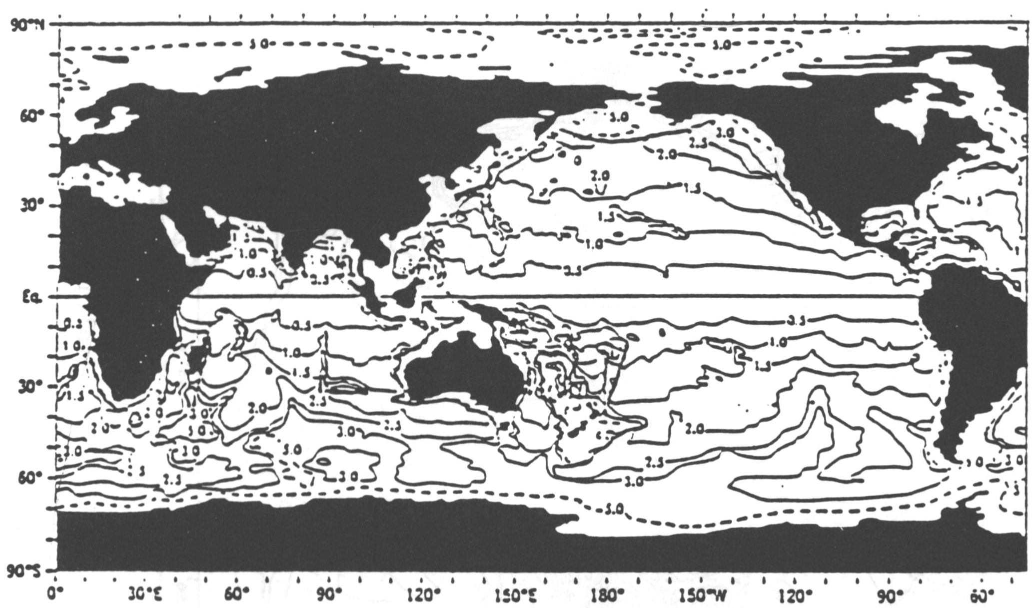

Fig.2a - Planetary vorticity ($10^{-8}m^{-1}sec^{-1}$)
(from Levitus, 1982)

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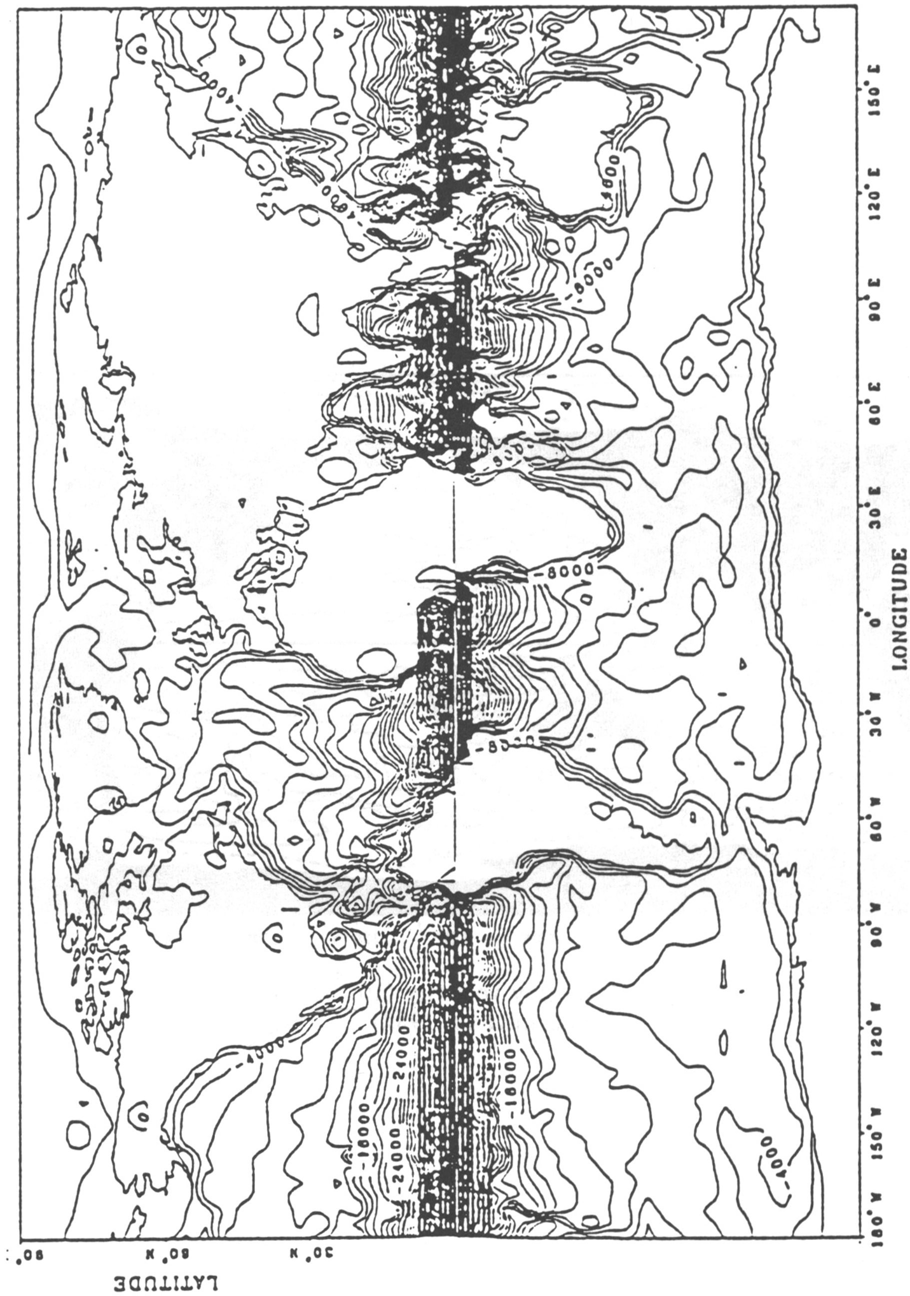

Fig. 2b. Isolines h/cos θ = const (m).

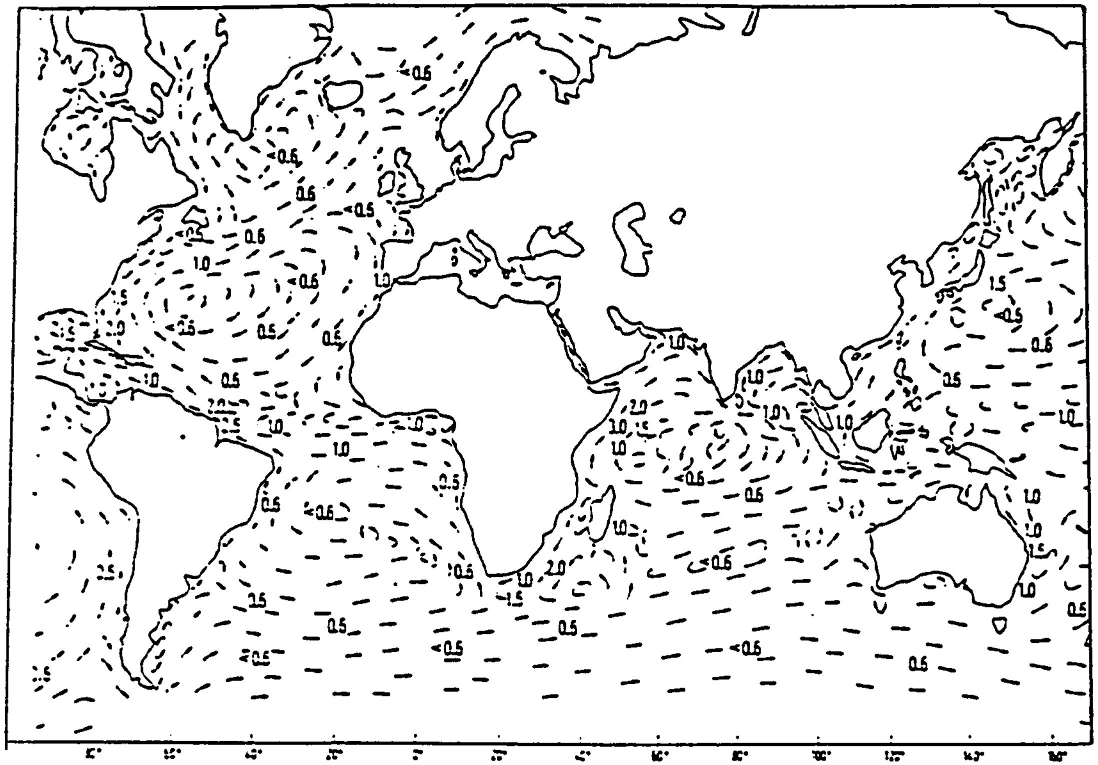

Fig. 3 .World ocean surface currents, in knots, compiled from ship drift estimates for northern summer. From USA Navy (1974, 1976–1979).

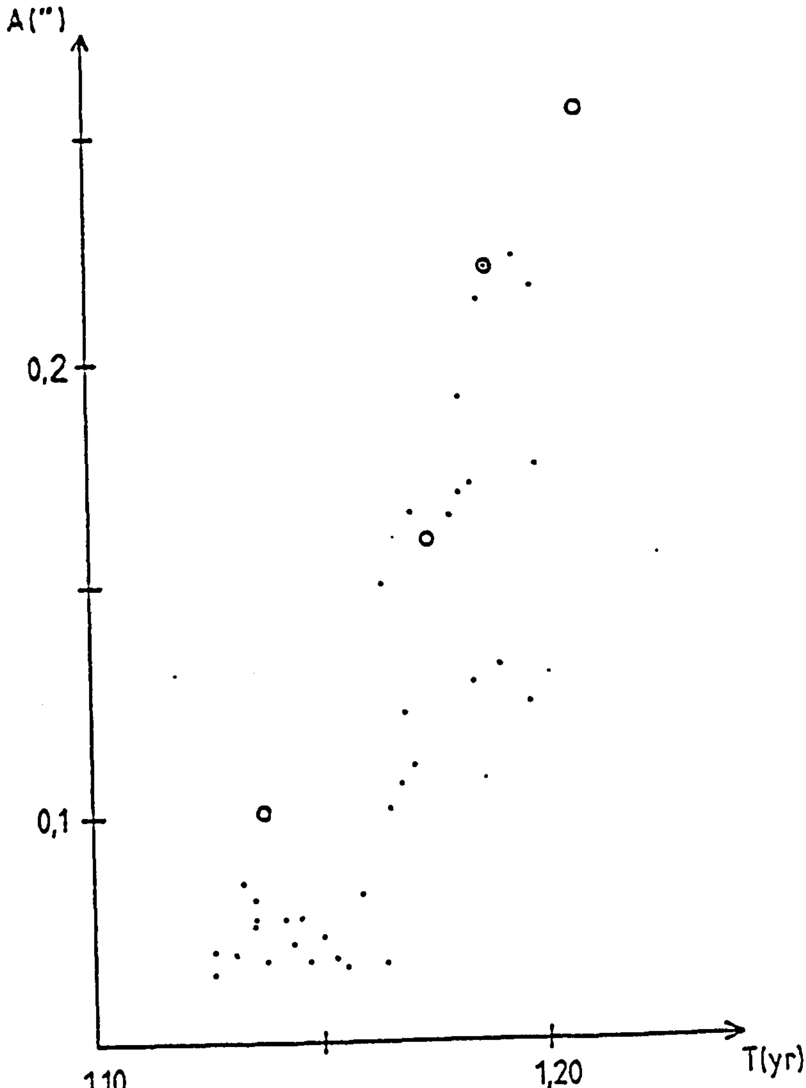

Fig. 4

The correlation between the amplitudes and periods of the Chandler wobble (the small second peak in the spectrum for the period 1940-1960 (Lenhardt, Groten, 1985) is excluded)