1 Basic Concepts

1.1 Introduction

In the description of any physical phenomenon, there always arises the question of effectively separating the few relevant degrees of freedom from the myriad irrelevant ones. One could even state that the very existence of physics as an exact science directly hinges on the *possibility* of such a separation because, at any time, only a limited amount of information about a given physical system is available.

Consequently the question arises, which criteria should be used to achieve such a separation? A universal criterion is based on the comparison of the length (momentum) scales that are inherent to any complex system. In the presence of multiple scales, the physics at a low momentum (large-distance) scale is insensitive to the dynamics at high momenta (short distances). This is called *scale separation* and should be considered as of one the cornerstones of the concept of effective field theory. A trivial example is provided by Newtonian mechanics. In order to describe the free fall of a stone, the knowledge of the structure of the stone (molecules, atoms, quarks and gluons, etc.) is not needed. Another example is provided by the well-known multipole expansion in electrodynamics. The electrostatic potential produced by an arbitrary static distribution of charges that are located in a small area near the origin (see Fig. 1.1) is given *at large distances* by

$$V(\mathbf{r}) = \sum_{i=1}^{N} \frac{q_i}{|\mathbf{r}-\mathbf{d}_i|} = \sum_{i=1}^{N} \frac{q_i}{r} + \sum_{i=1}^{N} \frac{q_i\,(\mathbf{d}_i\cdot\mathbf{r})}{r^3} + \sum_{i=1}^{N} \frac{q_i\left[3(\mathbf{d}_i\cdot\mathbf{r})^2 - \mathbf{d}_i^2\mathbf{r}^2\right]}{2r^5} + \cdots . \tag{1.1}$$

Here, $r_i = |\mathbf{r}-\mathbf{d}_i|$ is the distance between the ith charge and the observer located at the point P. Introducing the total charge, the dipole moment and the quadrupole moment, in order,

$$Q = \sum_{i=1}^{N} q_i, \qquad \mathbf{P} = \sum_{i=1}^{N} q_i \mathbf{d}_i, \qquad Q_{\alpha\beta} = \frac{1}{2}\sum_{i=1}^{N} q_i \left[3 d_{i\alpha} d_{i\beta} - \delta_{\alpha\beta}\, \mathbf{d}_i^2\right], \tag{1.2}$$

one obtains

$$V(\mathbf{r}) = \frac{Q}{r} + \frac{\mathbf{P}\cdot\mathbf{r}}{r^3} + \frac{Q_{\alpha\beta} r_\alpha r_\beta}{r^5} + \cdots . \tag{1.3}$$

This expansion converges if the distance r between the observer and the center of the charge distribution is much larger than the size of the charge distribution itself, that is, $|\mathbf{d}_i| \ll r$.

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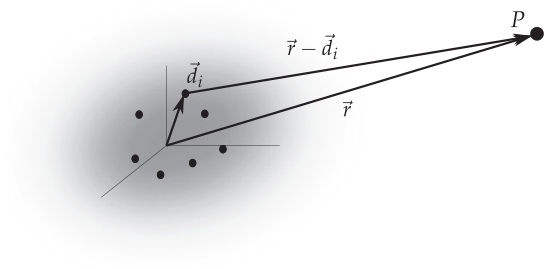

Figure 1.1 An electrostatic potential produced by an arbitrary localized static distribution of charges. The observer is located at point P far away from the charge distribution. Figure courtesy of Serdar Elhatisari.

Equation (1.3) demonstrates that by choosing the appropriate degrees of freedom, or variables, for describing the problem at large distances (i.e., choosing $\mathbf{r}$ instead of the individual distances $\mathbf{r}_i$), the solution of the problem is considerably simplified and can be described by a few parameters, here $Q, \mathbf{P}, Q_{\alpha\beta}$. These characterize the system as a whole rather than its individual components. Equation (1.2) can be considered as a *matching condition,* giving the expressions of these parameters in terms of the underlying physics at short distances.

This separation of scales is encountered in any field of physics. In this chapter, we consider the application of this idea in quantum field theory and demonstrate how the physics at the heavy scales (at short distances) can be consistently integrated out from the theory, leading to an effective theory, which contains the light degrees of freedom only.

1.2 Warm-up: Effective Theory for Scattering on the Potential Well

1.2.1 Effective Range Expansion

Before addressing effective field theories, we would like to start from a more familiar example and consider constructing an effective theory for quantum-mechanical scattering on a short-range potential. This allows us to explain many fundamental concepts and notions of effective field theories in an intuitive and transparent fashion.[1] Namely, we shall consider a spherical potential well, depicted in Fig. 1.2. The potential of the well is given by

$$U(r) = \begin{cases} -U_0 & \text{for} \quad r \le b\,, \\ 0 & \text{for} \quad r > b\,. \end{cases} \tag{1.4}$$

[1] A similar problem has been addressed, for example, in the beautiful lectures given by Lepage [1], which we strongly recommend for further reading.

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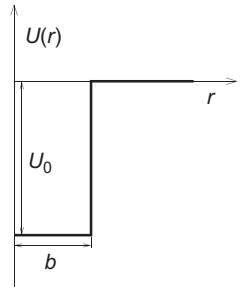

Figure 1.2 The spherical potential well. U_0 and b are the depth and the range of the potential, respectively.

This choice has the advantage that the Schrödinger equation (note that throughout we work in natural units $\hbar = c = 1$),

$$\left(-\frac{1}{2m}\nabla^2 + U(r)\right)\psi(\mathbf{r}) = E\,\psi(\mathbf{r}), \tag{1.5}$$

is exactly solvable, thus rendering our arguments explicit. Here, m denotes the mass of the particle that moves in the potential $U(r)$. Note that any short-ranged potential of an arbitrary form, but with the same range b, should lead to a similar behavior of the wave function at large distances $r \gg b$, since the details of the potential at short range should not matter.

The solution of this quantum-mechanical problem is well known from textbooks, and we display only those results here, which will be needed in the following. Due to rotational symmetry, the wave function in Eq. (1.5) can be factorized into the radial part $R_\ell(r)$ and the angular part, given by the spherical harmonics with the angular momentum ℓ and projection m:

$$\psi(\mathbf{r}) = R_\ell(r)Y_{\ell m}(\theta,\phi)\,, \tag{1.6}$$

with θ and ϕ the polar and the azimuthal angle, respectively. For large values of $r \gg b$, the asymptotic form of the solution with $E > 0$ is given by

$$R_\ell(r) \to \frac{A_\ell}{r}\,\sin\left(kr - \frac{\pi\ell}{2} + \delta_\ell(k)\right). \tag{1.7}$$

Here, A_ℓ is a constant, k is the wave vector, related to the energy by $E = k^2/(2m)$, and $\delta_\ell(k)$ is the *scattering phase shift,* which encodes all information about the behavior of the wave function at asymptotically large distances. It is given by the expression

$$\tan\delta_\ell(k) = \frac{k j'_\ell(kb)\,j_\ell(Kb) - K\,j_\ell(kb)\,j'_\ell(Kb)}{k n'_\ell(kb)\,j_\ell(Kb) - K n_\ell(kb)\,j'_\ell(Kb)}\,, \tag{1.8}$$

where $j_\ell(x)$ and $n_\ell(x)$ denote the spherical Bessel functions of the first and second kind, respectively, and $K^2 = k^2 + 2mU_0$. Further, the prime denotes differentiation with respect to the argument.

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In the following, we restrict ourselves to S-wave scattering with $\ell = 0$ and thus drop the index ℓ. The S-wave scattering phase is given by

$$\tan\delta(k) = \frac{k\tan(Kb) - K\tan(kb)}{K + k\tan(kb)\tan(Kb)}\,. \tag{1.9}$$

Using this expression, we may write down the *effective-range expansion* (ERE) for the phase shift:

$$k\cot\delta(k) = -\frac{1}{a} + \frac{1}{2}rk^2 + v_4k^4 + O(k^6)\,. \tag{1.10}$$

Here, a is the *scattering length,* r is called *effective range,* and the higher coefficients $v_4, v_6, \ldots$ are known under the name of *shape parameters.* Generally, $a, r, v_4, v_6, \ldots$ are referred to as *effective-range parameters.* The explicit expressions for these parameters are obtained by Taylor-expanding Eq. (1.9). It is convenient to express the results in terms of b and the dimensionless parameter $x = b\sqrt{2mU_0}$:

$$\begin{aligned} a &= bf_0(x)\,,\\ r &= bf_2(x)\,,\\ v_{2n} &= b^{2n-1}f_{2n}(x)\,. \end{aligned} \tag{1.11}$$

Next, we display the first two coefficients explicitly:

$$\begin{aligned} f_0(x) &= 1 - \frac{\tan x}{x}\,,\\ f_2(x) &= \frac{3\tan x - 3x + 3x\tan^2 x - 6x^2\tan x + 2x^3}{3x(x-\tan x)^2}\,, \end{aligned} \tag{1.12}$$

and so on.

Next, let us consider the limit $x \to \pi/2 + \pi n$. We can easily convince ourselves that $f_2(x), f_4(x), \ldots$ stay finite in this limit. Concerning the first coefficient, matters are, however, different. As seen from Eq. (1.12), $f_0(x) \to \infty$ as $\tan x \to \infty$. Thus, we have two distinct possibilities (we consider the magnitudes of the various parameters or scales):

1. All effective-range parameters are of *natural size,* which is determined by the interaction range b. Namely, $a \sim b$, $r \sim b$, $v_{2n} \sim b^{2n-1}$.
2. We have an *unnaturally large scattering length,* namely, $a \gg b$. All other parameters are of natural size, that is, we still have $r \sim b$, $v_{2n} \sim b^{2n-1}$.

Note also that the convergence of the effective range expansion is in both cases controlled by the parameter b. In other words, the effective range expansion converges when $kb \ll 1$. Physically, this means that for the large distances $r \sim 1/k \gg b$, the scattering on the short-ranged potential irrespective of its shape can be parameterized by the first few coefficients in this expansion (just as any static charge distribution in classical electrodynamics at large distances can be characterized by the first few coefficients in the multipole expansion of the electric field). At distances $r \sim 1/k \sim b$ this expansion does not make sense anymore, since all terms become equally important.

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Finally, let us find out what the condition $x \to \pi/2 + \pi n$ means. To this end, note that the bound-state spectrum in the potential well is determined by the equation

$$\sqrt{2mU_0 + \kappa^2}\cot(b\sqrt{2mU_0 + \kappa^2}) = \kappa\,, \tag{1.13}$$

where κ denotes the bound-state momentum, which is related to the energy by $E = -\kappa^2/(2m)$. When the bound state emerges exactly at threshold (i.e., at $\kappa = 0$), from Eq. (1.13) one gets $\cot(b\sqrt{2mU_0}) = 0$, that is, $b\sqrt{2mU_0} = x = \pi/2 + \pi n$. This shows that the scenario with the unnaturally large scattering length is realized when the parameters of the potential are fine-tuned so that a very shallow bound state emerges. The existence of a such zero-energy bound state does not affect the other effective range parameters.

1.2.2 Construction of the Effective Theory

At large distances $r \gg b$ (or, equivalently, at small momenta $kb \ll 1$), one cannot resolve the details of the potential. All short-ranged potentials at this distance should look pretty much the same and similar to the potential with zero range. In the first approximation, one can replace the exact potential with a local δ-function potential, that is,

$$U(r) \to C_0\delta^{(3)}(\mathbf{r})\,, \tag{1.14}$$

and adjust the single available coupling C_0, so that the lowest-order term in the effective-range expansion of the scattering phase (the scattering length) is the same in the original theory and in the effective theory. Since the effective range expansion encodes all physical information about the system at low energies, it is then intuitively clear that the zero-range potential in Eq. (1.14) – in the lowest-order approximation – is equivalent to the initial potential $U(r)$ with a finite range at low energies. The procedure of adjusting C_0 goes under the name of *matching*.

The issues, considered in what follows, have been addressed in the literature; see, for example, Refs. [2, 3]. In order to find the scattering phase shift in the effective theory, one writes down the Lippmann–Schwinger equation for the S-wave scattering amplitude $T(p,k)$:

$$T(p,k) = V(p,k) + \frac{2}{\pi}\int \frac{q^2dq}{q^2 - k^2 - i\varepsilon}V(p,q)T(q,k)\,,\ \ \varepsilon \to 0^+\,, \tag{1.15}$$

with $V(p,k)$ the potential in momentum space. Further, k and p denote the magnitudes of the incoming and outgoing relative three-momenta, respectively. The on-shell amplitude $T(k,k) \doteq T(k)$ obeys elastic unitarity,

$$\mathrm{Im}\,T(k) = k|T(k)|^2\,, \tag{1.16}$$

and can be expressed through the phase shift

$$T(k) = \frac{1}{k}e^{i\delta(k)}\sin\delta(k)\,. \tag{1.17}$$

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We may also introduce the scattering R-matrix, which obeys the Lippmann–Schwinger equation,

$$R(p,k) = V(p,k) + \frac{2}{\pi}\,\mathrm{P.V.}\!\int \frac{q^2 dq}{q^2-k^2} V(p,q)R(q,k)\,. \tag{1.18}$$

Here, unlike Eq. (1.15), the integral is equipped with the principal value (P.V.) prescription. The T- and R-matrices are related. On shell, this relation takes the form

$$T(k) = \frac{R(k)}{1-ikR(k)}\,, \qquad R(k) = \frac{1}{k}\tan\delta(k)\,. \tag{1.19}$$

In the following, we prefer to work with the R-matrix. The Fourier transform of the potential in Eq. (1.14) is given by

$$V(p,k) = C_0\,. \tag{1.20}$$

The solution of the Lippmann–Schwinger equation, (1.18), is

$$R(k) = \frac{C_0}{1-C_0 I_2(k^2)}\,, \qquad I_2(k^2) = \frac{2}{\pi}\mathrm{P.V.}\!\int \frac{q^2 dq}{q^2-k^2}\,. \tag{1.21}$$

Here, we encounter the problem of an ultraviolet (UV) divergence. Since the δ-function potential is singular at short distances, the integral I_2 diverges at the upper limit and should be regularized. The most straightforward way to do this is to introduce a momentum cutoff, Λ. Then, the integral is equal to

$$I_2(k^2) = \frac{2}{\pi}\mathrm{P.V.}\!\int^{\Lambda} \frac{q^2 dq}{q^2-k^2} = \frac{2}{\pi}\Lambda + O(1/\Lambda)\,. \tag{1.22}$$

In the following, the terms of order $1/\Lambda$ are always neglected and never displayed explicitly. The R-matrix at leading order, which is given by Eq. (1.21), turns out to be constant. The matching condition then reads

$$R(0) = -a\,, \qquad C_0 = -\frac{a}{1-aI_2(0)}\,. \tag{1.23}$$

It follows from Eq. (1.23) that C_0 should be Λ-dependent in order to ensure that the observable (the scattering length a) does not depend on the cutoff.

In order to illustrate the matching, let us do a simple numerical exercise. We arbitrarily choose the parameters of the square well as $b = 1$ and $x = b\sqrt{U_0} = \pi/4$. In Fig. 1.3, we display the exact phase shift, given by Eq. (1.9), as well as the phase shift, obtained in the effective field theory with a zero-range potential given in Eq. (1.14), where the parameter C_0 is determined from the matching to the exact scattering length. The parameter Λ is set equal to $1/b$ (i.e., to the inverse of the short-range scale of the model). It is seen that, up to the momenta $k^2/\Lambda^2 \leq 0.5$, the phase shift is reproduced in the effective theory rather well.

We are not going to stop here, however: we ask ourselves whether it is possible to *systematically* improve the description of the phase shift. To this end, note that, using the leading-order potential (with no derivatives), it is possible to adjust only the scattering length in the effective theory. The higher-order coefficients of the effective-range

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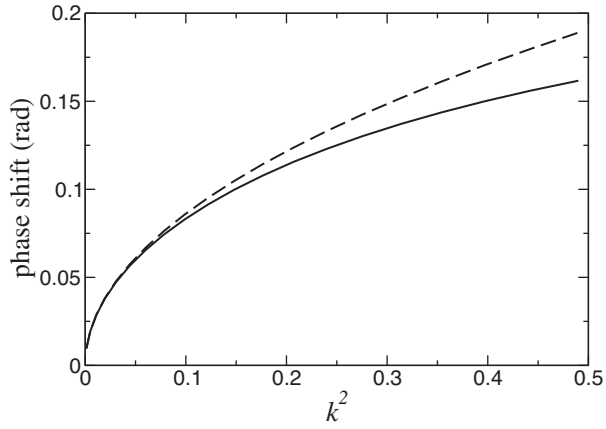

Figure 1.3 The phase shift obtained at leading order in the effective theory (dashed line). For comparison, we plot the exact solution for the potential well given by Eq. (1.9) (solid line).

expansion at this order are all zero (because the R-matrix is constant at this order). To reproduce these as well, we need more adjustable parameters. For example, the effective range also can be tuned, if we add a term with two derivatives to the potential, and so on.

Let us now explicitly demonstrate how this can be done. We modify the potential in Eq. (1.14):

$$U(r) \to C_0 \delta^{(3)}(\mathbf{r}) + C_2 \nabla^2 \delta^{(3)}(\mathbf{r})\,. \tag{1.24}$$

We note that the first term in this expansion is referred to as leading order (LO), whereas the second gives the next-to-leading order (NLO) contribution. Calculating the Fourier transform, we get

$$\begin{aligned} V(\mathbf{p}-\mathbf{k}) &= \int d^3\mathbf{r} e^{-i(\mathbf{p}-\mathbf{k})\mathbf{r}} \left(C_0 \delta^{(3)}(\mathbf{r}) + C_2 \nabla^2 \delta^{(3)}(\mathbf{r})\right) \\ &= C_0 - C_2 (\mathbf{p}-\mathbf{k})^2\,. \end{aligned} \tag{1.25}$$

Projecting onto the S-wave gives

$$\begin{aligned} V(p,k) &= \frac{1}{2}\int_{-1}^{+1} d\cos\theta\, V(\mathbf{p}-\mathbf{k}) \\ &= C_0 - C_2(p^2+k^2) = \sum_{i,j=1}^{2} v_i(p) C_{ij} v_j(k)\,, \end{aligned} \tag{1.26}$$

where

$$v_1(k) = 1\,, \quad v_2(k) = k^2\,, \qquad C = \begin{pmatrix} C_0 & -C_2 \\ -C_2 & 0 \end{pmatrix}. \tag{1.27}$$

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We look for a solution of the Lippmann–Schwinger equation using the ansatz:

$$R(p,k) = \sum_{i,j=1}^{2} v_i(p) D_{ij}(k^2) v_j(k)\,. \tag{1.28}$$

The matrix D obeys the following equations:

$$D(k^2) = C + C\Gamma(k^2) D(k^2)\,, \qquad \Gamma(k^2) = \begin{pmatrix} I_2(k^2) & I_4(k^2) \\ I_4(k^2) & I_6(k^2) \end{pmatrix},$$

$$I_{2n}(k^2) = \frac{2}{\pi}\mathrm{P.V.}\!\int \frac{q^{2n}dq}{q^2-k^2}\,. \tag{1.29}$$

$I_2(k^2)$ is given by Eq. (1.22). The remaining integrals are equal to

$$I_4(k^2) = \frac{2}{\pi}\mathrm{P.V.}\!\int^{\Lambda} \frac{q^4 dq}{q^2-k^2} = \frac{2}{\pi}\left(\frac{1}{3}\Lambda^3 + \Lambda k^2\right),$$

$$I_6(k^2) = \frac{2}{\pi}\mathrm{P.V.}\!\int^{\Lambda} \frac{q^6 dq}{q^2-k^2} = \frac{2}{\pi}\left(\frac{1}{5}\Lambda^5 + \frac{1}{3}\Lambda^3 k^2 + \Lambda k^4\right). \tag{1.30}$$

The solution for the R-matrix is then given by

$$R(k) = \frac{C_0 + C_2^2 I_6(k^2) - 2k^2 C_2(1 + C_2 I_4(k^2)) + k^4 C_2^2 I_2(k^2)}{(1 + C_2 I_4(k^2))^2 - I_2(k^2)(C_0 + C_2^2 I_6(k^2))}\,. \tag{1.31}$$

This expression can be expanded in powers of k^2, and the first two coefficients can be matched to the effective-range expansion for the scattering phase. Introducing the dimensionless variables,

$$x_0 = \frac{\Lambda C_0}{\pi}\,, \qquad x_2 = \frac{\Lambda^3 C_2}{\pi}\,, \tag{1.32}$$

we obtain two coupled equations for x_0, x_2:

$$h_0 \doteq \frac{\Lambda a}{9\pi} = \frac{5x_0 + 2x_2^2}{18(5x_0 + 2x_2^2) - 5(3 + 2x_2)^2}\,,$$

$$h_2 \doteq \frac{27\pi\Lambda r}{100} = \frac{x_2(3 + x_2)(3 + 2x_2)^2}{(5x_0 + 2x_2^2)^2}\,. \tag{1.33}$$

It is possible to find an explicit solution to these equations:

$$x_0 = \frac{1}{5}(\alpha y - 2x_2^2)\,,\; x_2 = \frac{1}{2}(\sqrt{y} - 3)\,,\; y = \frac{9}{1 - 4\alpha^2 h_2}\,,\; \alpha = -\frac{5h_0}{1 - 18h_0}\,. \tag{1.34}$$

Substituting this solution in Eq. (1.32), one arrives at the values of C_0 and C_2 that reproduce the first two terms in the effective-range expansion. It is clear that the description can be improved systematically, adding terms with more derivatives to the potential. All this makes sense for $k^2 \ll 1/b^2$, for which the effective range expansion is justified.

In Fig. 1.4, the numerical solution for the phase shift at next-to-leading order is depicted. It is seen that there is a systematic improvement as compared to the leading order. The next-to-leading order phase shift almost follows the exact curve. Here, it

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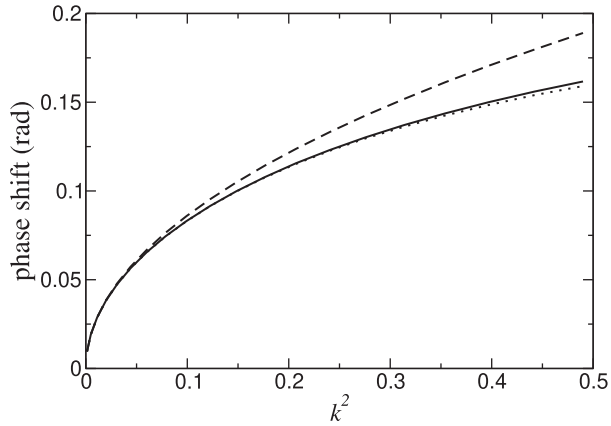

Figure 1.4 The same as in Fig. 1.3. Shown is the phase shift in the effective theory obtained at the leading (dashed line) and the next-to-leading order (dotted line), respectively, in comparison to the exact solution (solid line).

is worth mentioning that the effective range expansion is convergent very fast in the simple case considered here: keeping only the first two terms in this expansion, one obtains a phase shift which is indistinguishable from the exact solution by bare eye (in the region where the effective theories are applicable).

1.2.3 Regularization

Let us now address the question how the results depend on the regularization. Up to now, we have used cutoff regularization to tame the ultraviolet divergences. Here, we consider the use of dimensional regularization [4, 5], where the expressions look particularly simple.[2] This happens because all "no-scale" integrals vanish by definition in this scheme, that is,

$$\int d^d\mathbf{q}\,(\mathbf{q}^2)^\alpha = 0 \quad \text{for all } \alpha\,, \tag{1.35}$$

with d the number of space dimensions, which is set to three after the integral has been evaluated. It is then immediately seen that in this regularization $I_2 = I_4 = I_6 = 0$ and, hence,

$$R(k) = C_0 - 2C_2k^2\,. \tag{1.36}$$

Matching to the effective-range expansion is then straightforward:

$$C_0 = -a\,, \qquad C_2 = \frac{1}{4}ra^2\,, \tag{1.37}$$

and so on. It is seen that, in this scheme, the couplings $C_0, C_2, \ldots$ do not depend on any scale, except the one implicitly present in the effective-range expansion parameters, that is, the scale b.

[2] A good introduction to this method is given in the textbook [6].

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There has been discussion in the literature, whether certain *physical* results may depend on the choice of the regularization in a *non-perturbative* setting, which we are considering here; see, for example, Refs. [2, 3, 7, 8]. Of course, all results obtained within perturbation theory should be strictly regularization independent. To understand the argument, it suffices to look at Eq. (1.34). From this equation, it is clear that the limit $\Lambda \to \infty$ exists only in the case $h_2 \leq 0$ (and, hence, the effective range $r < 0$), whereas there is no restriction on the sign of the effective range in dimensional regularization, and in nature this quantity obviously may have either sign. Further, it has been shown in Refs. [9, 10] that this restriction can be obtained from the so-called Wigner bound on the scattering phase shift [11], assuming that the interaction has zero range.

However, putting the argument differently, one could state that there is no justification to consider the limit $\Lambda \to \infty$ in the preceding expansion, since the physical cutoff scale for the system is given by $1/b$. It is then seen that no problem emerges, if the low-energy expansion is carried out in the presence of a finite cutoff on the order of $1/b$. We may also conclude that the physical results indeed do not depend on the regularization chosen, as long as the regularization parameter is chosen within a reasonable range.

1.2.4 Counting Rules

From the very beginning, our method is aimed at a systematic improvement of the description of the scattering phase (any physical observables, in general) in the low momentum region. This means that, for instance, the contributions to the phase shift from the terms with more derivatives, which could be added to Eq. (1.24), will be suppressed by a factor $(kb)^2 \ll 1$, where k is a small momentum. Let us ask, however, what is the meaning of the effective potential in the cutoff regularization. Of course, since all couplings depend on the cutoff, one has to consider the potential at a single cutoff. Following our intuition, we expect that there exists a string of effective couplings, determining the effective potential to all orders, so that adding more terms affects the observables less and less. The situation is, however, more subtle. If one adds higher-order terms, one has also to readjust the lower-order terms because this is required by matching. For example, comparing Eq. (1.34) to the solution at the lowest order $x_0 = -9h_0$ and $x_2 = 0$, we see that the change in the values of the dimensionless parameters x_0, x_2 is of order one, since h_0, h_2 are of order one for $\Lambda \sim 1/b$. The same pattern holds at higher orders as well. Consequently, albeit a systematic improvement of the precision in the description of physical observables can be achieved, the terms in the effective potential, strictly speaking, cannot be ordered according to their relative importance. Adding a formally higher-order term leads to a renormalization in the lower-order terms as well. In other words, no consistent *power counting* scheme can be defined in this case.

The situation is different in the dimensional regularization scheme where, for example, C_0 can be fixed through the scattering length and stays put for all orders. The same is true for the higher-order couplings. The question about the *size* of these couplings is, however, more subtle. If the scattering length is of natural size, we have, as expected,

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$C_0 = O(b)$, $C_2 = O(b^3)$ and so on, so that the Taylor expansion for $R(k)$ converges for $k^2 \ll 1/b^2$. As seen from Eq. (1.37), this is no more true in the case of the unnaturally large scattering length, which renders the radius of convergence of this expansion smaller.

The reason for the difference between the cutoff and dimensional regularization is easy to understand. Consider, for simplicity, the case of natural-sized couplings. In dimensional regularization, there is only one short-distance parameter b, the various couplings have different dimensions (they are proportional to different powers of b) and do not talk to each other. In contradistinction to this, there exist two short-distance parameters in the cutoff regularization, and a dimensionless variable Λb can be constructed. The couplings at different orders can then mix in the matching condition, and this mixing differs with increasing orders. An analog to this phenomenon emerges when a heavy particle is present in the loops of the effective theory (e.g., relativistic baryon Chiral Perturbation Theory and the breakdown of the counting rules). We shall address this issue in the following.

We postpone the discussion of other regularization schemes to subsequent chapters.

1.2.5 Error Estimates

An important issue for any theoretical prediction is the precision that can be achieved. Stated differently, similar to experimental measurements, theoretical predictions also carry an uncertainty, also called the *theoretical error*. One of the main advantages of effective theories is the possibility of estimating this uncertainty. Here, we briefly discuss the so-called *naive dimensional analysis* (NDA) and refer to later sections for more precise and refined methods to deal with error estimates. Consider some observable $\mathcal{O}$ that is expanded in a small parameter Q, with Q the ratio of small (soft) momenta to the hard scale Λ (or a collection of small parameters as encountered in later sections),

$$\mathcal{O} = \mathcal{O}_{\rm LO} + \mathcal{O}_{\rm NLO} + \cdots = \sum_{i=0} c_i Q^i \,, \tag{1.38}$$

where the coefficients c_i are assumed to be of order one, which is called *naturalness*. Here, LO, NLO, …, denote the leading, next-to-leading, … order in the expansion in Q. Note that the lowest order can start with some other power of Q, but this does not invalidate the following considerations. If, for example, $Q = 0.1$, we expect that the corrections at NLO to be on the order of 10% and at NNLO on the order of 1%. Two remarks are relevant here. First, such NDA estimates have always to be taken with a grain of salt. Often one encounters the situation that at a given order, some symmetry might suppress the contributions or in a very different case, there can be a remarkable enhancement of a given order due to some close-by state not accounted for explicitly (like, e.g., the Delta resonance in pion–nucleon scattering). In such cases, the NDA obviously will fail. Second, NDA does not allow us to estimate the sign of the corrections. Note further that sometimes only even powers appear in the expansion; the model just discussed is an expansion in powers of k^2/Λ^2, where the hard scale was

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identified with $1/b$, and the observable under consideration is $k\cot\delta$. This important issue will be taken up later in various places.

1.2.6 Renormalization Group Equations

As mentioned already, the couplings in the cutoff scheme should run with the cutoff parameter, Λ, in order to ensure that the observables are cutoff-independent. The renormalization group (RG) equations at leading order can be easily obtained by differentiating Eq. (1.23) with respect to Λ:

$$\Lambda\frac{d}{d\Lambda}C_0(\Lambda) = -\frac{2}{\pi}\Lambda C_0(\Lambda)^2\,. \tag{1.39}$$

The equations at next-to-leading order can be obtained by differentiating the solutions given in Eq. (1.34) with respect to the scale Λ and taking into account the fact that a and r are Λ-independent. The right-hand side of Eq. (1.39) gives the so-called β-function that determines the running of the coupling $C_0(\Lambda)$ with respect to Λ. Recalling the discussion in the previous section, we may expect that the beta functions in cutoff regularization at different orders are different merely by a quantity of order one.

1.2.7 What Did We Learn from This Example?

- At low momenta ($k \ll 1/b$) a scattering process can be characterized by a small set of effective range expansion parameters.
- The interaction range is implicitly encoded in the size of the effective range expansion parameters. Namely, if the scattering length is of natural size, then we have $a \sim b$, $r \sim b$, $v_4 \sim b^3$ and so on. In case of an unnaturally large scattering length, only the first of these relations is not valid.
- An unnaturally large scattering length is related to the formation of a near-threshold bound state (or a virtual state).
- One may construct a low-energy effective theory, approximating the square well potential by a series of the δ-function potential and derivatives thereof. The couplings in front of these potentials are adjustable parameters and are used to reproduce the effective-range expansion parameters order by order. This procedure goes under the name of matching.
- Albeit the matching conditions may look different in different regularizations, the resulting scattering amplitude, expressed in terms of the effective range parameters, is the same in all regularizations up to terms of higher orders.
- Last but not least, it is interesting to mention that the matching fixes not only the scattering amplitude at small momenta, but the spectrum of the shallow bound states as well. To see this, it suffices to note that, according to Eq. (1.19), the poles of the T-matrix (corresponding to the bound states) emerge for purely imaginary values of k, corresponding to the solution of the equation

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$$R^{-1}(k) - ik = -\frac{1}{a} + \frac{1}{2} rk^2 + \cdots - ik = 0\,. \tag{1.40}$$

If the effective theory reproduces the values of $a, r, \ldots$, then the solution of this equation will also be reproduced up to higher-order terms.

- Theoretical uncertainties can and should be estimated. A first estimate can be based on naive dimensional analysis, but in general more sophisticated methods should be used.

1.3 Integrating out a Heavy Scale: a Model at Tree Level

1.3.1 Matching at Tree Level

After this warm-up example, let us proceed with the construction of an effective theory in a simple field-theoretical model. This model is described by the following Lagrangian,[3]

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\Phi)^2 - \frac{m^2}{2}\phi^2 - \frac{M^2}{2}\Phi^2 - \frac{g}{2}\phi^2\Phi\,, \tag{1.41}$$

with ϕ, Φ denoting the light and heavy scalar fields with masses m, M, respectively, and $m \ll M$. Further, the coupling constant g has dimension [mass] and ∂ is a shorthand notation for ∂_μ, where μ enumerates the space-time indices.

Let us consider processes involving light particles only (that means we only have external legs made from the light particle species), with energies E much smaller that the heavy mass M (i.e., assume that $E \sim m \ll M$). For such energies we expect that the presence of the heavy particle cannot lead to any observable consequence and the system can be described by an effective Lagrangian containing the light degrees of freedom only.[4] The inverse of the heavy mass $1/M$ plays the role of the short-range scale b in our case of the square well potential, and in the limit $M \to \infty$, a local potential emerges.

Consider, in particular, the $2 \to 2$ scattering process $\phi\phi \to \phi\phi$ at low energies. The momenta of the initial (final) particles are p_1 and p_2 (p_3 and p_4). The scattering amplitude in perturbation theory is a series in the coupling constant g, so that the matching to the effective field theory can be performed at each order independently. At tree level, the scattering amplitude is given by the diagrams depicted in Fig. 1.5 and is equal to

[3] In general, there will be an additional linear term $c\Phi$ present in the Lagrangian in Eq. (1.41), which is needed to cancel tadpole diagrams with one external Φ-leg. Here, however, we work in tree approximation, where it is possible to put $c = 0$.

[4] There is a well-known example, which exactly follows the path outlined in this toy model. Namely, in the Standard Model, the interactions between left-handed charged currents are mediated by the $W^\pm$ bosons with a mass $M_W \simeq 80$ GeV. If the momentum transfer in a process is much smaller than M_W, the flavor-changing weak interactions are described by the local four-fermion Fermi Lagrangian. The Fermi coupling G_F, which appears in the effective theory at tree level, is inversely proportional to M_W^2.

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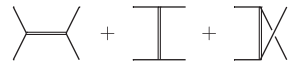

Figure 1.5 The tree-level scattering amplitude for the process $\phi\phi \to \phi\phi$ in the model described by the Lagrangian given in Eq. (1.41). Single and double lines correspond to the light and heavy fields, respectively. Shown are the s-, t- and u-channel contributions, in order.

$$T_{\text{tree}} = \frac{g^2}{M^2 - s} + \frac{g^2}{M^2 - t} + \frac{g^2}{M^2 - u}, \tag{1.42}$$

where $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ are the usual Mandelstam variables. On the mass shell, these variables obey the relation $s + t + u = 4m^2$.

In the limit of a large mass M, the amplitude in Eq. (1.42) can be expanded in a Taylor series:

$$T_{\text{tree}} = \frac{3g^2}{M^2} + \frac{g^2}{M^4}(s + t + u) + \frac{g^2}{M^6}(s^2 + t^2 + u^2) + \cdots. \tag{1.43}$$

At low energies, each subsequent term in this expansion is suppressed by a factor E^2/M^2 with respect to the previous one, where E is the characteristic energy of the light particles.

Our aim is to find a Lagrangian that contains only ϕ-fields, and which reproduces the expansion of the amplitude in Eq. (1.43). In general, such an *effective* Lagrangian must contain an infinite tower of quartic terms in the field ϕ. By analogy with Eq. (1.24) we may try to use the Lagrangian of the following form:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - C_0\phi^4 - C_1\phi^2\Box\phi^2 - C_2\phi^2\Box^2\phi^2 + \cdots, \tag{1.44}$$

with $\Box = \partial^\mu\partial_\mu = \partial\partial$. Note that at tree level the mass parameters in both the underlying and effective Lagrangians are equal. As we shall see, this is no more the case at one loop order.

The tree-level amplitude, obtained from this Lagrangian, takes the form

$$T^{\text{eff}}_{\text{tree}} = -24C_0 + 8C_1(s + t + u) - 8C_2(s^2 + t^2 + u^2) + \cdots. \tag{1.45}$$

This amplitude is shown in Fig. 1.6. Demanding $T^{\text{eff}}_{\text{tree}} = T_{\text{tree}}$ leads to *matching conditions* which enable one to express the couplings of the effective theory $C_0, C_1, C_2, \ldots$ in terms of the parameters of the underlying theory g, m and M.

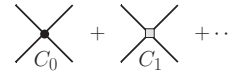

Figure 1.6 The tree-level scattering amplitude for the process $\phi\phi \to \phi\phi$ in the effective theory described by the Lagrangian given in Eq. (1.44). This amplitude can be obtained from the amplitude shown in Fig. 1.5 by contracting all heavy lines to a point.

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1.3.2 Equations of Motion

The matching condition is imposed on *observables*, that is, in our case, on the scattering amplitude defined on shell, $p_i^2 = m^2$. As is known, the Mandelstam variables on shell obey the constraint

$$s + t + u = 4m^2 \,, \tag{1.46}$$

and the tree-level amplitudes in the full theory and in the effective theory are given by the expressions

$$T_{\text{tree}} = \frac{3g^2}{M^2} + \frac{4g^2m^2}{M^4} + \frac{g^2}{M^6}\,(s^2 + t^2 + u^2) + \cdots \tag{1.47}$$

and

$$T_{\text{tree}}^{\text{eff}} = -24C_0 + 32m^2C_1 - 8C_2(s^2 + t^2 + u^2) + \cdots \,. \tag{1.48}$$

The matching conditions, which enable one to express the couplings of the effective theory in terms of the parameters of the underlying theory, take the form

$$-24C_0 + 32m^2C_1 = \frac{3g^2}{M^2} + \frac{4g^2m^2}{M^4} \,, \quad -8C_2 = \frac{g^2}{M^6} \,, \tag{1.49}$$

and so on.

Note that the mass-shell matching does not allow one to determine the couplings C_0 and C_1 separately. According to Eq. (1.49), only the combination $-24C_0 + 32m^2C_1$ can be determined from the matching condition. This is related to the fact that (accidentally in this model) all second-order terms can be eliminated by using the equations of motion (EOM). In order to prove this, note that on the one hand

$$\phi^2 \Box \phi^2 = 2\phi^3(\Box + m^2)\phi - 2m^2\phi^4 + 2\phi^2(\partial\phi)^2 \,, \tag{1.50}$$

and on the other hand,

$$\phi^2(\partial\phi)^2 = \underbrace{\frac{1}{3}\,\partial^\mu(\phi^3\partial_\mu\phi)}_{=\text{total derivative}} - \frac{1}{3}\phi^3(\Box + m^2)\phi + \frac{m^2}{3}\,\phi^4 \,. \tag{1.51}$$

Using the EOM

$$(\Box + m^2)\phi = -4C_0\phi^3 + \cdots \,, \tag{1.52}$$

it is seen that $\phi^3(\Box + m^2)\phi$ is transformed into a sum of operators containing more than four fields and, therefore, does not contribute to the tree-level amplitude. Finally, the term proportional to ϕ^4 can be lumped together with the similar term in the Lagrangian. To summarize, the second-order terms can be completelyeliminated from the Lagrangian. Thus, without losing generality, one may set $C_1 = 0$ everywhere.

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1.3.3 Unitarity Bound

We have constructed an effective theory that is equivalent to the underlying theory at tree level. However, one does not stop at tree level. The effective field theory is a full-fledged field theory, so one has to consider loop diagrams, generated by the Lagrangian in Eq. (1.44) as well. Here a question arises naturally: The underlying theory is a superrenormalizable theory (the single coupling constant g has the dimension of mass), whereas the resulting effective theory contains a tower of non-renormalizable vertices. How should one deal with these divergences? Or stated differently, how can one interpret the equivalence of these two theories beyond the tree level?

Moreover, it can be seen that the tree-level amplitude in the effective field theory necessarily violates unitarity. In order to see this, it is convenient to define the partial-wave amplitudes

$$T_\ell^{\text{eff}}(s) = \frac{1}{32\pi\sqrt{s}} \int_{-1}^{+1} d\cos\theta\, T^{\text{eff}}(s,\cos\theta)\, P_\ell(\cos\theta)\,,$$

$$T^{\text{eff}}(s,\cos\theta) = 16\pi\sqrt{s} \sum_{\ell=0}^{\infty} (2\ell+1) T_\ell^{\text{eff}}(s) P_\ell(\cos\theta)\,, \tag{1.53}$$

where $P_\ell(\cos\theta)$ denote the conventional Legendre polynomials and θ is the scattering angle in the center-of-mass system. The unitarity relation for the partial-wave amplitudes gives

$$\text{Im}\, T_\ell^{\text{eff}}(s) \geq p\, |T_\ell^{\text{eff}}(s)|^2\,, \qquad p = \sqrt{\frac{s}{4} - m^2}\,, \tag{1.54}$$

where the inequality turns into the equality below the first inelastic threshold, $s_{\text{thr}} = (4m)^2$, where processes like $\phi\phi \to \phi\phi\phi\phi$ are not allowed energetically.

Transforming Eq. (1.54), we get

$$p(\text{Re}\, T_\ell^{\text{eff}}(s))^2 + p\left(\text{Im}\, T_\ell^{\text{eff}}(s) - \frac{1}{2p}\right)^2 - \frac{1}{4p} \leq 0\,. \tag{1.55}$$

Now, it is immediately seen that the real part of the amplitude obeys the so-called *unitarity bound*:

$$|\text{Re}\, T_\ell^{\text{eff}}(s)| \leq \frac{1}{2p}\,. \tag{1.56}$$

This bound is violated by the tree amplitude given in Eq. (1.45). For example, in the partial wave with $\ell = 0$ the tree-level amplitude is equal to

$$\text{Re}\, T_0^{\text{eff}}(s) = \frac{1}{16\pi\sqrt{s}} \left(-24C_0 + 32m^2 C_1 - 8C_2 \left(\frac{2}{3}(s-4m^2)^2 + s^2 \right) + \cdots \right). \tag{1.57}$$

Substituting this expression into Eq. (1.56), it is seen that the left-hand side grows with increasing s, whereas the right-hand side decreases. Using the values of the coupling

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constants, determined by the matching condition given in Eq. (1.49), and assuming $s \gg m^2$, it is seen that the unitarity bound is saturated at

$$s_M = M^2 \sqrt{\frac{16\pi - 3\tilde{g}^2}{5\tilde{g}^2/3}} + O(1), \qquad \tilde{g} = \frac{g}{M}. \tag{1.58}$$

Note that the large-M limit in the underlying theory is performed so that the dimensionless quantity $\tilde{g}$ stays finite. Otherwise, the leading coupling C_0 could not be finite. Consequently, the quantity s_M is of order of M^2. If $s > s_M$, loops are necessary in order to render the tree-level amplitude unitary. In turn, this means that the loops must be of the same order of magnitude as the tree amplitude, heralding trouble in the perturbative expansion.

In reality, if s is on the order of $s_M \sim M^2$, the effective theory cannot be applied any more, and one should resort to a perturbative expansion in the underlying theory, which is superrenormalizable and where the amplitude decreases as s^{-1} at large values of s. It is said that the underlying theory provides a *Wilsonian ultraviolet (UV) completion* of the effective theory at scales of order M.

1.4 The Model at Tree Level: Path-Integral Formalism

Consider the generating functional of the theory described by the Lagrangian in Eq. (1.41):

$$Z(j,J) = \int d\phi d\Phi \exp\left\{ i \int d^4x (\mathcal{L}(\phi,\Phi) + j\phi + J\Phi) \right\}, \tag{1.59}$$

where $j(x), J(x)$ denote classical external sources for the fields $\phi(x)$ and $\Phi(x)$, respectively. The Green's functions are obtained by functional differentiation of Z with respect to these sources (once per each external leg) and by putting $j = J = 0$ at the end.

Since we are interested here in the Green's functions of the light field only, we may put $J = 0$ and consider the quantity $Z(j) \doteq Z(j, J = 0)$. Performing a shift of the integration variable,

$$\Phi \to \Phi - \frac{g}{2} (\Box + M^2)^{-1} \phi^2, \tag{1.60}$$

it is possible to rewrite the generating functional in the following form:

$$Z(j) = \int d\phi d\Phi \exp\left\{ i \int d^4x \left(-\frac{1}{2} \Phi(\Box + M^2)\Phi + \frac{g^2}{8} \phi^2 (\Box + M^2)^{-1} \phi^2 \right.\right.$$
$$\left.\left. - \frac{1}{2} \phi(\Box + m^2)\phi + j\phi \right) \right\}. \tag{1.61}$$

The integration over the variable Φ in the first term gives a constant that can be included into the normalization. Expanding now the second term in the exponential, we get

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$$\frac{g^2}{8}\,\phi^2(\Box+M^2)^{-1}\phi^2=\frac{g^2}{8M^2}\left(\phi^4-\phi^2\frac{\Box}{M^2}\,\phi^2+\phi^2\frac{\Box^2}{M^4}\,\phi^2+\cdots\right). \tag{1.62}$$

Comparing this expansion with Eq. (1.44), we may immediately read off

$$C_0=-\frac{g^2}{8M^2}\,,\qquad C_1=\frac{g^2}{8M^4}\,,\qquad C_2=-\frac{g^2}{8M^6}\,,\qquad \cdots, \tag{1.63}$$

and the result in Eq. (1.49) is reproduced. Of course, as we already know, C_0 and C_1 are not independent, as on the mass shell only a linear combination thereof survives. One could use the EOM in Eq. (1.62) in order to reduce the number of the independent matching conditions. There is, however, nothing wrong in using an overcomplete set of independent couplings.

It is legitimate to ask why this result is valid only at tree level, even if no approximation has been made so far. The answer to this question is that the Taylor expansion of the integrand in the path integral is not justified, since the value of the integral changes as a result of this expansion. On the other hand, at tree level, the path integral is equal just to the value of the integrand along the classical trajectory. Consequently, in this case, the expansion is justified, since an integration over ϕ is no longer performed.

A final remark is in order. It is easy to see that before Taylor-expanding, the theory with the effective Lagrangian, which contains only ϕ fields, is formally equivalent to the underlying theory to all orders in perturbation theory. The effective theory contains a vertex, $\phi^2(\Box+M^2)^{-1}\phi^2$, and is thus nonlocal. Its high-energy behavior is, however, damped by the inverse D'Alembertian and corresponds to that of the original superrenormalizable theory. The expansion makes a local effective Lagrangian out of a nonlocal one, but at the cost of a worse behavior at high momenta. It is clear that the expansion breaks down at momenta of the order of M, and we are back to the underlying theory.

1.5 Equations of Motion and Field Redefinitions

In the previous sections, those terms in the Lagrangian, which vanish by using the EOM, have been dropped. In what follows, we shall prove that these terms do not contribute to the S-matrix and thus to physical observables, and hence the procedure is justified. Moreover, we shall prove that the two Lagrangians, which differ from each other by field redefinition, lead to the same S-matrix and thus the theories described by these Lagrangians are equivalent.[5]

In the framework of field theory, the S-matrix for a generic process $n\to m$ is obtained by using the well-known Lehmann–Symanzik–Zimmermann (LSZ) rule for the Green's function with $n+m$ external legs.[6] (For simplicity, we consider here the case of a real scalar field with a mass m, but the argument applies without modifications

[5] In what follows, we mainly follow the arguments given in Ref. [12].
[6] The LSZ formalism is considered in detail in most of the field theory textbooks; see, e.g., Ref. [13].

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to other cases as well.) The S-matrix is related to the T-matrix through $S = 1 + iT$. If none of the external momenta are equal, the relation between the T-matrix element and the Green's function is given by

$$iT(p_1,\cdots,p_m;q_1,\cdots,q_n) = (iZ^{-1/2})^{n+m}$$
$$\times \lim_{p_i^2,q_j^2\to m^2} \prod_{i=1}^{m} \theta(p_i^0)(m^2-p_i^2) \prod_{j=1}^{n} \theta(q_j^0)(m^2-q_j^2)\, G(p_1,\cdots,p_m;q_1,\cdots,q_n)\,, \qquad (1.64)$$

where G is the Fourier transform of the $n+m$-point Green's function:

$$(2\pi)^4\delta^{(4)}(p_1+\cdots+p_m-q_1-\cdots-q_n)G(p_1,\cdots,p_m;q_1,\cdots,q_n)$$
$$= \int \prod_{i=1}^{m} d^4x_i e^{ip_ix_i} \prod_{j=1}^{n} d^4y_j e^{-iq_jy_j} \langle 0|T\phi(x_1)\cdots\phi(x_m)\phi(y_1)\cdots\phi(y_m)|0\rangle\,. \qquad (1.65)$$

Here, the symbol "T" denotes the conventional time-ordering, $TA(x)B(y) = \theta(x^0-y^0)A(x)B(y) + \theta(y^0-x^0)B(y)A(x)$, and Z stands for the wave function renormalization constant which is given by the residue of the two-point function at the one-particle pole:

$$D(p^2) = i\int d^4x e^{ipx}\langle 0|T\phi(x)\phi(0)|0\rangle\,,$$
$$D(p^2) \to \frac{Z}{m^2-p^2}\,, \quad \text{as } p^2\to m^2\,. \qquad (1.66)$$

In other words, the Green's function contains poles in all external momenta, when the latter approach the mass shell. The generic S-matrix element is obtained from the Green's function by extracting the residue on the mass shell and multiplying by a factor $iZ^{-1/2}$ for each external leg.

The amputated Green's function Γ is defined as

$$G(p_1,\cdots,p_m;q_1,\cdots,q_n) = \prod_{i=1}^{m} D(p_i^2) \prod_{j=1}^{n} D(q_j^2)\Gamma(p_1,\cdots,p_m;q_1,\cdots,q_n)\,, \qquad (1.67)$$

and the T-matrix element can be determined from the amputated function as

$$iT(p_1,\cdots,p_m;q_1,\cdots,q_n) = (iZ^{1/2})^{n+m} \lim_{p_i^2,q_j^2\to m^2} \Gamma(p_1,\cdots,p_m;q_1,\cdots,q_n). \qquad (1.68)$$

Below, we shall demonstrate that the T-matrix element does not depend on the choice of the interpolating field. To this end, let us consider a general nonlinear local field transformation of the type

$$\phi'(x) = F[\phi(x)] = \phi(x) + a_2\Box\phi(x) + \cdots + b_0\phi^2(x) + b_1\partial_\mu\phi(x)\partial^\mu\phi(x)$$
$$+ b_2\phi(x)\Box\phi(x) + \cdots + c_0\phi^3(x) + \cdots\,. \qquad (1.69)$$

The single requirement is that the matrix element of the field ϕ' between the vacuum and the one-particle state $\langle 0|\phi'(x)|p\rangle$ is different from zero. Note also that, to ease the notations, the coefficient in front of $\phi(x)$ in the r.h.s. of Eq. (1.69) is set equal to 1, since

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an arbitrary constant coefficient can be removed by a mere rescaling of the field that does not change the S-matrix elements.

It is clear that the Green's functions for the fields ϕ and ϕ' differ off shell. In order to extract the T-matrix element, however, we need the behavior only in the vicinity of the mass shell. Only the diagrams that are one-particle reducible in all external particles can possess poles in all external momenta and therefore contribute to the T-matrix elements. Diagrammatically, this corresponds to the situation when the particle described by the field ϕ escapes the connected part of the diagram and then turns into ϕ' without interacting with other external legs. Hence, the $n+m$ point Green's function of the fields ϕ' is given by

$$\begin{aligned} &G'(p_1,\cdots,p_m;q_1,\cdots,q_n) \\ &=\prod_{i=1}^{m}\Pi(p_i^2)D(p_i^2)\prod_{j=1}^{n}\Pi(q_j^2)D(q_j^2)\Gamma(p_1,\cdots,p_m;q_1,\cdots,q_n)+\cdots . \end{aligned} \tag{1.70}$$

Here, $\Pi(p^2)$ denotes the sum of all one-particle irreducible diagrams, which describe the transition of ϕ into ϕ' (see Fig. 1.7b), and the ellipses denote the regular terms. (These terms do not contain one-particle reducible diagrams in at least one of the external lines.) These regular terms do not contribute to the T-matrix, and we shall consistently omit them in the following. The key point is that the amputated Green's function Γ is the same in the two cases, simply because it contains the same set of diagrams in both cases (cf. Figs. 1.7a and 1.7b).

Moreover, the two-point function of field ϕ' is given by

$$D'(p^2)=\Pi^2(p^2)D(p^2)+\cdots , \tag{1.71}$$

where, again, the ellipsis denotes the regular terms.

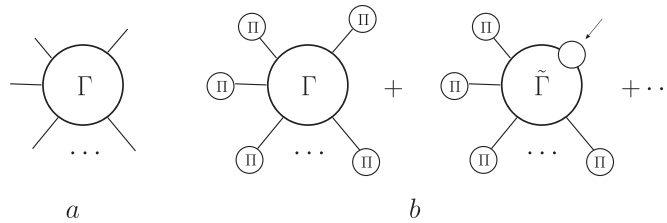

Figure 1.7 (a) The representation of the $n+m$ point Green's function of the fields ϕ through the amputated Green's function Γ. Solid lines denote the dressed propagator of a single particle. (b) The same for the $n+m$ point Green's function of the fields ϕ'. The singular part is given by a set of the diagrams, where the single particle lines corresponding to the field ϕ are emanating from the connected part of the diagram and then turn into ϕ' without interacting with other lines. The blocks denoted by Π contain the set of all one-particle irreducible diagrams that describe this transition. An example of the regular part that does not contribute to the T-matrix also is shown. In this part the vertex indicated by the arrow cannot be separated from the rest of the diagram by cutting just one internal line. The amputated function Γ is the same in both cases, and $\tilde{\Gamma}$ does not contribute on shell.

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Now, taking into account the fact that $\Pi(p^2)$ is regular in the variable p^2 because it is a sum of the one-particle irreducible diagrams, we obtain the relation between the wave function renormalization constants of the fields ϕ and ϕ':

$$Z' = Z\Pi^2(m^2)\,. \tag{1.72}$$

Finally, using Eq. (1.70), we see that the T-matrix elements, obtained from the Green's functions of the fields ϕ and ϕ' with the use of the LSZ formula, are exactly the same, since the factors $\Pi(m^2)$, obtained for each external leg, are canceled by the same factor emerging in $(Z')^{-1/2}$. This proves the statement that the T-matrix elements do not depend on the choice of the interpolating field.

Based on this result, we can now show that the field redefinitions leave the S-matrix elements invariant. The generating functional for the Green's functions in the path integral formulation is given by

$$Z(J) = \int d\phi \exp\left(i\int d^4x\left[\mathcal{L}(\phi(x)) + J(x)\phi(x)\right]\right). \tag{1.73}$$

The Green's functions are obtained by differentiating this expression with respect to the external sources $J(x)$ and setting them to zero at the end.

Next, within this path integral, let us perform the field transformation given in Eq. (1.69). The Jacobian of this transformation is

$$\left|\frac{d\phi'}{d\phi}\right| = \exp\left\{\mathrm{Tr}\ln\left(\frac{d\phi'}{d\phi}\right)\right\}. \tag{1.74}$$

Here, in order to simplify the notations, we shall carry out the calculation of the determinant, when the field transformation has the following form (cf. Eq. (1.69)),

$$\phi'(x) = \phi(x) + a_2\Box\phi(x) + b_0\phi^2(x)\,, \tag{1.75}$$

albeit the treatment is, of course, completely general. Then,

$$\frac{d\phi'}{d\phi} = \delta^{(D)}(x-y) + a_2\Box\delta^{(D)}(x-y) + 2b_0\phi(x)\delta^{(D)}(x-y) \doteq 1 + r\,. \tag{1.76}$$

Here, we have anticipated that an UV regularization will be needed to calculate the determinant and write down the expression in D dimensions, setting $D \to 4$ at the end of calculations. Using $\ln(1+r) = r - r^2/2 + \ldots$, we obtain

$$\begin{aligned}
\mathrm{Tr}\ln\left(\frac{d\phi'}{d\phi}\right) &= \int d^Dx\, r(x,x) - \frac{1}{2}\int d^Dx d^Dy\, r(x,y)r(y,x) + \cdots\,,\\
\int d^Dx\, r(x,x) &= \int d^Dx\left(a_2\Box\delta^{(D)}(0) + 2b_0\phi(x)\delta^{(D)}(0)\right) = 0\,,\\
\int d^Dx d^Dy\, r(x,y)r(y,x) &= \int d^Dx\left(a_2^2\Box^2\delta^{(D)}(0) + 4b_0^2\phi(x)^2\delta^{(D)}(0)\right.\\
&\quad \left. + 4a_2b_0\phi(x)\Box\delta^{(D)}(0)\right) = 0\,,
\end{aligned} \tag{1.77}$$

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and so on. Here, we have used the fact that, in dimensional regularization, $\delta^{(D)}(0) = \partial_\mu \delta^{(D)}(0) = \ldots = 0$. Hence, the Jacobian of the transformation is equal to unity in dimensional regularization. Of course, the physical results do not depend on the regularization. Using another regularization can be accounted for by using a different renormalization prescription.

The rest is then straightforward. Under the field transformation the generating functional turns into

$$Z(J) = \int d\phi \exp\left(i \int d^4x \left[\mathcal{L}(F[\phi(x)]) + J(x) F[\phi(x)] \right] \right), \tag{1.78}$$

where we already took into account the fact that the Jacobian is equal to unity. This generating functional produces the Green's functions of the operator $F[\phi(x)]$ in a theory that is described by the Lagrangian $\mathcal{L}(F[\phi(x)])$. However, since the S-matrix elements do not depend on the choice of the interpolating field, the same S-matrix elements will be obtained from the generating functional

$$\tilde{Z}(J) = \int d\phi \exp\left(i \int d^4x \left[\mathcal{L}(F[\phi(x)]) + J(x) \phi(x) \right] \right). \tag{1.79}$$

Comparing this with the original expression in Eq. (1.73), we may conclude that the theories, described by the Lagrangians $\mathcal{L}(\phi(x))$ and $\mathcal{L}(F[\phi(x)])$ (before and after the field transformations), lead to the same S-matrix and are thus equivalent to each other.

Finally, we can easily show that in the Lagrangian it is possible to consistently drop the terms that vanish by using the EOM [12]. For illustration, let us consider a theory described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4} \phi^4 . \tag{1.80}$$

The classical EOM for this theory is

$$E[\phi(x)] = \frac{\delta S}{\delta \phi(x)} = -(\Box + m^2)\phi(x) - \lambda \phi^3(x) = 0 , \tag{1.81}$$

where S denotes the action functional.

Let us now consider an arbitrary local functional $H[\phi]$ and amend the initial Lagrangian:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \varepsilon H[\phi(x)] E[\phi(x)] . \tag{1.82}$$

In other words, the additional term vanishes for the solutions of the classical EOM. We shall now demonstrate that the amended theory leads to the same S-matrix as the original one. In order to do this, note that the infinitesimal transformation

$$\phi(x) \to \phi(x) + \varepsilon H[\phi(x)] \tag{1.83}$$

leads to the new Lagrangian:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \varepsilon H[\phi(x)] \frac{\delta S}{\delta \phi(x)} + O(\varepsilon^2) = \mathcal{L}(x) + \varepsilon H[\phi(x)] E[\phi(x)] + O(\varepsilon^2) . \tag{1.84}$$

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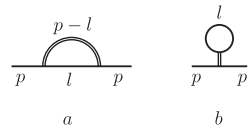

Figure 1.8 The self-energy of the light particle at one loop in the model described by the Lagrangian given in Eq. (1.41). Single and double lines correspond to the light and heavy fields, respectively.

Thus, the Lagrangians $\mathcal{L}(x)$ and $\mathcal{L}(x)+\varepsilon H[\phi(x)]E[\phi(x)]$ differ by the field transformation and hence lead to the same S-matrix. Using this procedure, we may eliminate all terms in the Lagrangian that vanish as solutions of the classical EOM.

1.6 Light Particle Mass at One Loop

Let us start the loop calculations for the simplest case of the two-point function of the light field, which is described by the Lagrangian given in (1.41). This two-point function can be written down in the following form:

$$D(p^2) = i\int d^4x e^{ipx}\langle 0|T\phi(x)\phi(0)|0\rangle = \frac{1}{m^2-p^2-\Sigma(p^2)}\,. \tag{1.85}$$

The self-energy of the light particle, $\Sigma(p^2)$, in the underlying theory at one loop is described by two diagrams shown in Fig. 1.8. We shall calculate these using dimensional regularization. The contribution of the diagram in Fig. 1.8a is given by

$$\Sigma_a(p^2) = g^2\int \frac{d^Dl}{(2\pi)^Di}\,\frac{1}{m^2-l^2}\,\frac{1}{M^2-(p-l)^2}\,. \tag{1.86}$$

In (1.86) D denotes the number of space-time dimensions. In addition, in all propagators the usual causal prescription $\text{mass}^2 \to \text{mass}^2 - i\varepsilon$, with $\varepsilon \to 0^+$, is implicit.

Performing the integral with the help of the Feynman trick,

$$\frac{1}{AB} = \int_0^1 \frac{dx}{(xA+(1-x)B)^2}\,, \tag{1.87}$$

as $D \to 4$, we obtain

$$\Sigma_a(p^2) = -2g^2L - \frac{g^2}{16\pi^2}\int_0^1 dx \ln\frac{xm^2+(1-x)M^2-x(1-x)p^2}{\mu^2}\,, \tag{1.88}$$

where μ denotes the scale of dimensional regularization, and

$$L = \frac{\mu^{D-4}}{16\pi^2}\left(\frac{1}{D-4}-\frac{1}{2}\left(\Gamma'(1)+\ln 4\pi\right)\right). \tag{1.89}$$

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Here, $\Gamma(z)$ is the Γ-function and $\Gamma'(1) = -\gamma$, where $\gamma = 0.577215665\ldots$ denotes Euler's constant. Integrating over the variable x, we obtain

$$\Sigma_a(p^2) = -2g^2 L - \frac{g^2}{16\pi^2}\left\{\frac{1}{2}\left(1-\frac{M^2-m^2}{p^2}\right)\ln\frac{m^2}{\mu^2} + \frac{1}{2}\left(1+\frac{M^2-m^2}{p^2}\right)\ln\frac{M^2}{\mu^2}\right.$$
$$\left. -\frac{\lambda^{1/2}}{2p^2}\left(\ln\frac{\frac{1}{2}\left(1-\frac{M^2-m^2}{p^2}\right)-\frac{\lambda^{1/2}}{2p^2}}{\frac{1}{2}\left(1-\frac{M^2-m^2}{p^2}\right)+\frac{\lambda^{1/2}}{2p^2}} - \ln\frac{-\frac{1}{2}\left(1+\frac{M^2-m^2}{p^2}\right)-\frac{\lambda^{1/2}}{2p^2}}{-\frac{1}{2}\left(1+\frac{M^2-m^2}{p^2}\right)+\frac{\lambda^{1/2}}{2p^2}} - 2\right)\right\}, \quad (1.90)$$

where

$$\lambda \doteq \lambda(p^2, m^2, M^2), \qquad \lambda(x,y,z) = x^2+y^2+z^2-2xy-2yz-2zx \quad (1.91)$$

denotes the Källén triangle function. Expanding this expression for large M, we obtain

$$\Sigma_a(p^2) = -2g^2 L - \frac{g^2}{16\pi^2}\left(-1+\ln\frac{M^2}{\mu^2}\right)$$
$$-\frac{g^2}{16\pi^2 M^2}\left(m^2\ln\frac{M^2}{\mu^2} - m^2\ln\frac{m^2}{\mu^2} - \frac{p^2}{2}\right) + O(M^{-4}), \quad (1.92)$$

where the notation $O(M^{-4})$ includes also the terms of the form $O(M^{-4}\ln^k M^2)$. We shall consistently adhere to this notation in the following.

The calculations in case of the diagram in Fig. 1.8b (the "tadpole") can be done analogously. The result is given by

$$\Sigma_b(p^2) = \frac{g^2}{2M^2}\int\frac{d^D l}{(2\pi)^D i}\frac{1}{m^2-l^2} = \frac{g^2m^2}{M^2}L - \frac{g^2m^2}{32\pi^2M^2}\left(1-\ln\frac{m^2}{\mu^2}\right). \quad (1.93)$$

Adding these two expressions, we finally obtain

$$\Sigma_a(p^2)+\Sigma_b(p^2) = -2g^2L - \frac{g^2}{16\pi^2}\left(-1+\ln\frac{M^2}{\mu^2}\right) + \frac{g^2m^2}{M^2}L$$
$$-\frac{g^2}{16\pi^2M^2}\left(m^2\ln\frac{M^2}{\mu^2} - \frac{3m^2}{2}\ln\frac{m^2}{\mu^2} - \frac{1}{2}(p^2-m^2)\right) + O(M^{-4}). \quad (1.94)$$

Next, let us answer the following question. We know that the effective Lagrangian in Eq. (1.44) reproduces all Green's functions of the underlying theory in the tree approximation. Are the results of the *loop calculations* in the underlying theory also reproduced by the loops in the effective theory, if one is using the same Lagrangian? The answer to this question is *no*, as will become clear from our calculations at one loop using the effective Lagrangian given in Eq. (1.44). Note also that, since we are calculating the Green's function and not the S-matrix element, one cannot use the EOM and eliminate the operator, which is multiplied by the coupling C_1.

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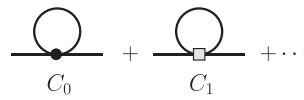

Figure 1.9 The self-energy of the light particle in the effective theory described by the Lagrangian given in Eq. (1.44). C_0 multiplies the vertex with no derivatives, C_1 the vertex with two derivatives and so on.

In the effective theory up to and including $O(M^{-2})$, only the first diagram in Fig. 1.9, which is proportional to $C_0 = -g^2/(8M^2)$, contributes. The result is given by

$$\begin{aligned}\Sigma_{\text{eff}}(p^2) &= -12C_0 \int \frac{d^D l}{(2\pi)^D i} \frac{1}{m^2 - l^2} + O(M^{-4}) \\ &= -24C_0 m^2 L_{\text{eff}} + \frac{3C_0 m^2}{4\pi^2}\left(1 - \ln\frac{m^2}{\mu_{\text{eff}}^2}\right) + O(M^{-4})\,. \end{aligned} \tag{1.95}$$

In this expression, μ_{eff} denotes the scale of dimensional regularization in the effective theory, which need not be the same as the one in the underlying theory, and L_{eff} is determined from Eq. (1.89) with the replacement $\mu \to \mu_{\text{eff}}$.

As clearly seen from the preceding equations, $\Sigma_a + \Sigma_b \neq \Sigma_{\text{eff}}$ at one loop. One may now ask the question, how could one modify the effective theory so that the Green's functions are the same? It can be seen that for the two-point Green's function, expanded up to the terms of order M^{-4}, this can be achieved by supplementing the effective Lagrangian by counterterms that correspond to mass and wave function renormalization:

$$\begin{aligned}\mathcal{L}_{\text{eff}} &\to \mathcal{L}_{\text{eff}} + \frac{A}{2}(\partial\phi)^2 - \frac{B}{2}\phi^2\,, \\ A &= \frac{g^2}{32\pi^2 M^2} + O(M^{-4})\,, \\ B &= g^2\left(2L_{\text{eff}} + \frac{1}{16\pi^2}\left(\ln\frac{M^2}{\mu_{\text{eff}}^2} - 1\right)\right) \\ &\quad + \frac{g^2 m^2}{M^2}\left(2L_{\text{eff}} + \frac{1}{16\pi^2}\left(\ln\frac{M^2}{\mu_{\text{eff}}^2} - 1\right)\right) + O(M^{-4})\,. \end{aligned} \tag{1.96}$$

Note that A and B do not depend on the scale μ_{eff}, as well as the quantities $\Sigma_a, \Sigma_b, \Sigma_{\text{eff}}$.

The quantities of interest are, however, not the (ultraviolet divergent) coefficients of the Lagrangian, but the renormalized quantities. Using explicit expressions for the two-point function, we are now in a position to perform the matching of the renormalized masses. In order to do this, let us note that the physical mass m_P of the light particle, which is determined by the position of the pole in the two-point Green's function, should be the same in both theories. In the underlying theory, at one loop, the pole is located at

$$m^2 - m_P^2 - (\Sigma_a(m_P^2) + \Sigma_b(m_P^2)) = 0\,. \tag{1.97}$$

From this equation we obtain, to lowest order in g,

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$$m_P^2 = m_r^2 + \frac{g^2}{16\pi^2}\left(-1 + \ln\frac{M^2}{\mu^2}\right) + \frac{g^2 m_r^2}{16\pi^2 M^2}\left(\ln\frac{M^2}{\mu^2} - \frac{3}{2}\ln\frac{m_r^2}{\mu^2}\right) + O(M^{-4})\,, \tag{1.98}$$

where m_r denotes the *running (or renormalized) mass* in the underlying theory in the modified minimal subtraction ($\overline{\text{MS}}$) scheme, which is defined through the subtraction of the divergent piece proportional to L:

$$m_r^2(\mu) = m^2 + 2g^2 L - \frac{g^2 m^2}{M^2} L\,. \tag{1.99}$$

Here, m denotes the bare mass in the underlying theory. Note that, at this order, it is still not necessary to consider the loop corrections of other parameters of the theory.

Since we have modified the effective Lagrangian to ensure that the Green's functions in the underlying and the effective theories coincide, the poles in both theories will be at the same place. The physical mass calculated in the effective theory is given by the solution of the following equation:

$$m^2 + B - (1+A)m_P^2 - \Sigma_{\text{eff}}(m_P^2) = 0\,, \tag{1.100}$$

and takes the form

$$m_P^2 = m_{r,\text{eff}}^2 + \frac{3g^2 m_{r,\text{eff}}^2}{32\pi^2 M^2}\left(1 - \ln\frac{m_{r,\text{eff}}^2}{\mu_{\text{eff}}^2}\right) + O(M^{-4})\,, \tag{1.101}$$

where we used the matching condition for the constant C_0 at tree level. Further, $m_{r,\text{eff}}$ denotes the running mass in the effective field theory, which is related to the bare mass in the following manner:

$$m_{r,\text{eff}}^2(\mu_{\text{eff}}) = m_{\text{eff}}^2 - \frac{3g^2 m_{\text{eff}}^2}{M^2} L_{\text{eff}}\,. \tag{1.102}$$

The bare mass in the effective theory, m_{eff}, which appears in Eq. (1.102), can be read off from the effective Lagrangian,

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 + \frac{A}{2}(\partial\phi)^2 - \frac{B}{2}\phi^2 + \text{quartic terms} \\ &= \frac{1}{2}Z_{\text{eff}}(\partial\phi)^2 - \frac{m_{\text{eff}}^2}{2}Z_{\text{eff}}\phi^2 + \text{quartic terms}\,, \\ Z_{\text{eff}} &= 1 + A\,, \qquad m_{\text{eff}}^2 = \frac{m^2 + B}{1+A}\,.\end{aligned} \tag{1.103}$$

Since observables (here the physical mass) should be the same in the underlying theory and in the effective theory, this finally gives the relation between the running mass in both theories:

$$\begin{aligned}m_{r,\text{eff}}^2(\mu_{\text{eff}}) =& m_r^2(\mu) + \frac{g^2}{16\pi^2}\left(-1 + \ln\frac{M^2}{\mu^2}\right) \\ &+ \frac{g^2 m_r^2(\mu)}{16\pi^2 M^2}\left(\ln\frac{M^2}{\mu^2} - \frac{3}{2}\left(1 + \ln\frac{\mu_{\text{eff}}^2}{\mu^2}\right)\right) + O(M^{-4})\,.\end{aligned} \tag{1.104}$$

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As we see, the *running masses* in both theories are not the same beyond tree approximation.[7] Moreover, these masses *run differently* with respect to the scale variations:

$$\mu \frac{dm_r^2(\mu)}{d\mu} = \frac{g^2}{8\pi^2} - \frac{g^2 m_r^2(\mu)}{16\pi^2 M^2}$$

$$\mu_{\text{eff}} \frac{dm_{r,\text{eff}}^2(\mu_{\text{eff}})}{d\mu_{\text{eff}}} = \frac{3g^2 m_{r,\text{eff}}^2(\mu_{\text{eff}})}{16\pi^2 M^2} + O(M^{-4}) . \tag{1.105}$$

The above RG equations can be obtained by differentiating the expression for the physical mass with respect to the scale and setting this derivative to zero, because the physical mass does not depend on the scale. Moreover, it should be pointed out that even if the scale μ is present in Eq. (1.104), the running mass in the effective theory, $m_{r,\text{eff}}(\mu_{\text{eff}})$, in fact, does not depend on this scale. This statement can be straightforwardly checked by using the first of the equations in Eq. (1.105). This happens because Eq. (1.104) was obtained from the matching to the physical observable, which has to be scale-independent.

A few concluding remarks:

i) As we have seen, matching the two Lagrangians at tree level does not mean that the loops calculated with these Lagrangians also will match. The difference, however, can be taken away completely by renormalization. This means that both theories are physically equivalent. This is a particular case of the decoupling theorem [14], as detailed in what follows.

ii) Matching enables us to express the parameters of the effective theory in terms of the parameters of the underlying theory. What makes sense is the relation between the finite quantities, for example, between the running masses and the couplings.

iii) Both sets of the running parameters depend on their own scales (μ and μ_{eff}, respectively). The parameters of the effective theory do not depend on the underlying scale μ, if they can be determined from the matching to physical observables.

iv) Note that in the relation given by Eq. (1.104), all logarithms containing the light mass cancel. This is the manifestation of the general pattern, which states that the couplings of the effective theory do not have a nonanalytic behavior that emerges at the light scales. All of this nonanalytic behavior has to be reproduced by the loops in the effective theory. On the contrary, the parameters of the effective theory encode the short-distance dynamics and thus depend on the light mass, at most, in a polynomial form. For consistency, here we assume that the scales μ, μ_{eff} are also "hard." On the other hand, reducing μ_{eff} down to the "light" scale, the couplings will no more depend analytically on this scale. We shall observe this phenomenon explicitly in Chiral Perturbation Theory (ChPT).

[7] Strictly speaking, only the matching of *observables* in two theories (i.e., the masses and the S-matrix elements) is required. The two-point function is not an observable. So, in principle, one could leave the wave function renormalization constant Z_{eff} free. However, not much will change in our discussion of the physical mass if we lift the restriction on this constant.

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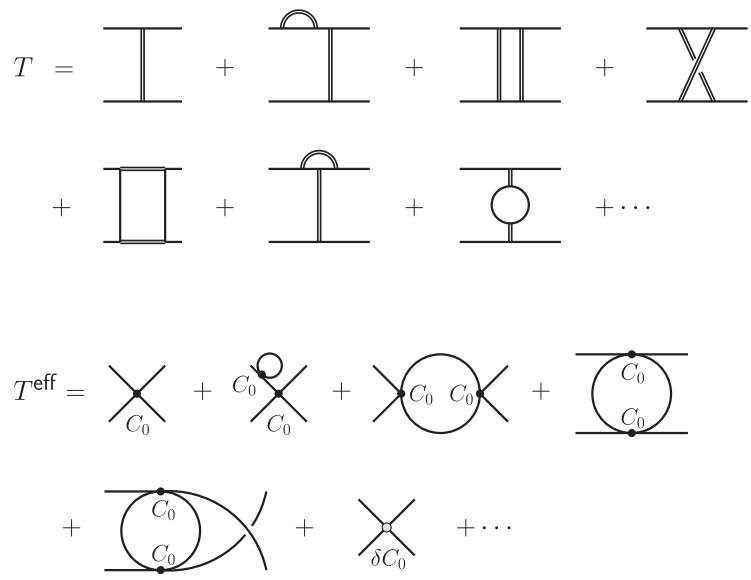

Figure 1.10 Representative set of the diagrams that contribute to the $\phi\phi \to \phi\phi$ amplitudes in the underlying (upper panel) and the effective theory (lower panel).

v) As we know, the dimensionful coupling constant g is on the order of the heavy mass M in the large-M limit. As one sees from Eq. (1.104), the running mass in the underlying theory is not protected from large loop corrections,[8] and it is driven up to the heavy scale, unless some fine-tuning is enforced. This phenomenon is closely related to the *hierarchy problem* in the Standard Model.

1.7 Matching of the Quartic Coupling at One Loop

After matching the two-point function, we turn to the Green's functions with more external legs. Matching of the $\phi\phi \to \phi\phi$ scattering amplitudes at one loop proceeds analogously. First of all, we have to calculate the scattering amplitude in the underlying theory and in the effective theory. A representative set of the diagrams is shown in Fig. 1.10. The matching condition is

$$T = T^{\text{eff}} \,. \tag{1.106}$$

It is seen that as a result of this matching condition, the quartic couplings in the tree-level effective Lagrangian, given by Eq. (1.44), are modified according to $C_i \to C_i + \delta C_i$. This is shown schematically in Fig. 1.10.

[8] The protection might arise due to the symmetries, e.g., the chiral symmetry in case of fermions. However, in the case that we are considering, there are no such symmetries.

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Since the one-loop contributions to the scattering amplitude in the effective theory (see Fig. 1.10) are divergent, the modified C_i should also contain divergent parts,

$$C_i = \nu_i L_{\text{eff}} + C_i^r(\mu_{\text{eff}}), \tag{1.107}$$

where the coefficients ν_i determine the running of the renormalized couplings $C_i^r(\mu_{\text{eff}})$ with respect to the scale μ_{eff}:

$$\mu_{\text{eff}} \frac{dC_i^r(\mu_{\text{eff}})}{d\mu_{\text{eff}}} = -\frac{\nu_i}{16\pi^2}, \tag{1.108}$$

where the ν_i are the pertinent β-functions whose explicit values are not needed here. Matching enables us to express the *renormalized* couplings $C_i^r(\mu_{\text{eff}})$ in terms of the fundamental parameters of the underlying theory. Comparing with Eq. (1.63), which contains matching at tree level, and using the fact that g has dimension of mass, we get

$$C_i^r(\mu_{\text{eff}}) = (-)^{i+1} \frac{g^2}{8M_r^{2(i+1)}(\mu_{\text{eff}})} \left\{ 1 + \kappa_i \frac{g^2}{16\pi^2 M_r^2(\mu_{\text{eff}})} \right\}, \tag{1.109}$$

where $M_r(\mu_{\text{eff}})$ is the renormalized heavy mass in the underlying theory, and the dimensionless constants κ_i can depend only on the dimensionless arguments m_r/M_r and μ_{eff}/M_r. (Without loss of generality and in order to ease the notation, we used here $\mu = \mu_{\text{eff}}$.) In Eq. (1.109) we further took into account the fact that in the underlying (superrenormalizable) theory the coupling g is not renormalized, and we used g instead of g_r everywhere. Moreover, as became clear from the discussion in Section 1.6, the coupling constants determined from the matching cannot contain infrared singularities at $m_r \to 0$, since these singularities are the same in the underlying and in the effective theory, canceling each other in the matching condition. An example of this is the cancellation of all $\ln(m_r^2)$-terms in the matching of the two-point functions; see Section 1.6. Consequently, the κ_i are a polynomial in the variable m_r^2/M_r^2, and the dependence on this variable can be traded for the derivative terms by using the EOM in the Lagrangian;[9] see Section 1.3. On the contrary, the dependence on the second variable μ_{eff}/M_r is nonanalytic, as in perturbation theory logarithms $\ln(\mu_{\text{eff}}/M_r)$ usually appear.

Carrying out the matching at one loop is straightforward but not very enlightening, since a large number of Feynman diagrams have to be calculated. In what follows, we shall demonstrate how the same goal can be achieved within the path-integral formalism with considerably less effort. To this end, we evaluate the generating functional given in Eq. (1.61) at one loop by using the saddle-point technique. In the beginning, we carry out the integration over the field Φ. (This integration gives an uninteresting constant, which can be included in the normalization of the path integral.) Further, we expand the action functional in this integral around the *classical* solution for the

[9] Using the EOM is justified, since the S-matrix elements, which are used in the matching condition, do not change. One should bear in mind, however, that the off-shell behavior of the Green's function changes, if the EOM are used.

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field ϕ, setting $\phi = \phi_c + \xi$. Here, the field ξ denotes a quantum fluctuation around the classical solution ϕ_c, which obeys the following EOM:

$$\begin{aligned} 0 &= (\Box + m^2)\phi_c(x) + j(x) + \frac{g^2}{2}\int d^4y\, \phi_c(x) D_M(x-y)\phi_c^2(y) \\ &= (\Box + m^2)\phi_c(x) + j(x) + \frac{g^2}{2M^2}\,\phi_c^3(x) + \cdots, \end{aligned} \tag{1.110}$$

with

$$\begin{aligned} D_M(x-y) &= \langle x|(\Box + M^2)^{-1}|y\rangle = \int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip(x-y)}}{M^2 - p^2} \\ &= \frac{1}{M^2}\int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\left(1 + \frac{p^2}{M^2} + \cdots\right) \\ &= \frac{1}{M^2}\,\delta^{(4)}(x-y) - \frac{1}{M^4}\,\Box\delta^{(4)}(x-y) + \cdots. \end{aligned} \tag{1.111}$$

Retaining terms up to second order in the expansion over ξ, and taking into account the fact that $d\phi = d\xi$, the generating functional in Eq. (1.61) can be rewritten as

$$\begin{aligned} Z(j) = \int d\xi \exp\left\{ i\int d^4x \left(-\frac{1}{2}\,\phi_c(\Box + m^2)\phi_c + \frac{g^2}{8}\,\phi_c^2(\Box + M^2)^{-1}\phi_c^2 + j\phi_c \right) \right\} \\ \times \exp\left\{ i\int d^4x d^4y \left(-\frac{1}{2}\,\xi(x) H(x-y)\xi(y) + O(\xi^3) \right) \right\}, \end{aligned} \tag{1.112}$$

with

$$\begin{aligned} H(x-y) &= (\Box + m^2 + S(x))\delta^{(4)}(x-y) - \Lambda(x-y), \\ S(x) &= -\frac{g^2}{2}\,(\Box + M^2)^{-1}\phi_c^2(x), \\ \Lambda(x-y) &= g^2\phi_c(x)\langle x|(\Box + M^2)^{-1}|y\rangle\phi_c(y). \end{aligned} \tag{1.113}$$

Note that there are no terms linear in ξ, because ϕ_c is the solution of the EOM that makes the action functional stationary.

Evaluating the Gaussian integral over ξ in a standard manner, we obtain

$$\begin{aligned} Z(j) = \exp\left\{ i\int d^4x \left(-\frac{1}{2}\,\phi_c(\Box + m^2)\phi_c \right.\right. \\ \left.\left. + \frac{g^2}{8}\,\phi_c^2(\Box + M^2)^{-1}\phi_c^2 + j\phi_c \right) + iS_{\text{eff}} \right\}, \end{aligned} \tag{1.114}$$

where

$$\begin{aligned} S_{\text{eff}} = &\frac{i}{2}\,\text{Tr}\,\ln\left((\Box + m^2 + S) - \Lambda\right) = \frac{i}{2}\,\text{Tr}\,\ln(\Box + m^2) + \frac{i}{2}\,\text{Tr}\left((\Box + m^2)^{-1}S\right) \\ &- \frac{i}{4}\,\text{Tr}\left((\Box + m^2)^{-1}S(\Box + m^2)^{-1}S\right) - \frac{i}{2}\,\text{Tr}\left((\Box + m^2)^{-1}\Lambda\right) \end{aligned}$$

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$$+\frac{i}{2}\,\mathrm{Tr}\left((\Box+m^2)^{-1}S(\Box+m^2)^{-1}\Lambda\right)-\frac{i}{4}\,\mathrm{Tr}\left((\Box+m^2)^{-1}\Lambda(\Box+m^2)^{-1}\Lambda\right)+\cdots$$

$$=T_0+T_1+T_2+T_3+T_4+T_5+O(g^6)\,. \tag{1.115}$$

Here, "Tr" denotes the trace of an operator in coordinate space, that is,

$$\mathrm{Tr}A=\int d^4x\,\langle x|A|x\rangle\,. \tag{1.116}$$

Note that T_0 is an uninteresting constant, which can be included in the normalization of the path integral. T_1 and T_3 are quadratic in the field ϕ_c and contribute to the renormalization of the two-point function of the light field. We have studied this issue in detail in Section 1.6. The remaining terms T_2, T_4 and T_5, which contribute to the renormalization of the quartic couplings, can be rewritten as

$$T_2=\frac{g^4}{16}\int d^4xd^4yd^4ud^4v\,(-iD(u-v)D(v-u)D_M(u-x)D_M(v-y))\,\phi_c^2(x)\phi_c^2(y)\,,$$

$$T_4=\frac{g^4}{4}\int d^4xd^4yd^4ud^4v(-iD(v-u)D(u-y)D_M(y-v)D_M(u-x))$$
$$\times\phi_c^2(x)\phi_c(y)\phi_c(v)\,,$$

$$T_5=\frac{g^4}{4}\int d^4xd^4yd^4ud^4v\,(-iD(v-u)D_M(u-x)D(x-y)D_M(y-v))$$
$$\times\phi_c(x)\phi_c(y)\phi_c(u)\phi_c(v)\,, \tag{1.117}$$

where $D(x-y)$ is a light scalar propagator with a mass m. Schematically, the three quantities T_2, T_4, T_5 are depicted in Fig. 1.11.

Let us now consider the strategy for matching at one loop. First, we recall that the matching condition is altered by loop corrections, because the heavy particles are present in the loops and the Taylor expansion in the inverse powers of the heavy mass cannot be straightforwardly carried out in the Feynman integrals. Namely, let us denote $T_M\{T_i\},\ i=2,4,5$, the quantities T_i, evaluated from the same diagrams shown in Fig. 1.11, but with the Taylor-expanded heavy particle propagator,

$$\frac{1}{M^2-l^2}\to\frac{1}{M^2}+\frac{l^2}{M^4}+\cdots\,. \tag{1.118}$$

Here and in what follows, the symbol "T_M" stands for the procedure of Taylor-expanding in inverse powers of the heavy mass.[10] Then, the difference,

$$\Delta T=\sum_{i=2,4,5}(T_i-T_M\{T_i\}), \tag{1.119}$$

should be compensated by adjusting the quartic coupling constants. This gives us the desired matching condition for these couplings.

[10] Note that, graphically, the operation T_M amounts to contracting the heavy propagators to one point. Consequently the diagrams describing $\phi\phi\to\phi\phi$ scattering in the effective theory arise from the diagrams T_2, T_4 and T_5, shown in Fig. 1.11.

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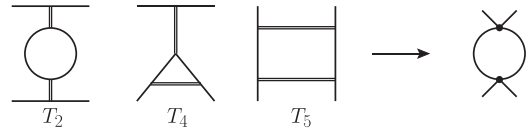

Figure 1.11 A schematic representation of T_2, T_4 and T_5. The solid and double lines denote the light and heavy fields, respectively. The arrow points toward the one loop graph in the effective theory that is obtained from T_2, T_4, T_5 by contracting the heavy propagators.

From Fig. 1.11 we immediately conclude that T_2 will not affect the matching condition, because it does not contain heavy particles in the loops. Consequently,

$$T_2 - T_M\{T_2\} = 0\,. \tag{1.120}$$

T_4 and T_5 will, however, affect the matching condition. Let us start with the quantity T_4. The vertex diagram, which is part of T_4 (see Fig. 1.11), is given by

$$\begin{aligned} &-iD(v-u)D(u-y)D_M(y-v) \\ &= \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\, e^{-ip_1(v-u)-ip_2(u-y)}\Gamma_v(p_1,p_2)\,, \end{aligned} \tag{1.121}$$

with

$$\Gamma_v(p_1,p_2) = \int \frac{d^Dl}{(2\pi)^D i}\,\frac{1}{(m^2-(p_1+l)^2)}\,\frac{1}{(m^2-(p_2+l)^2)}\,\frac{1}{(M^2-l^2)}\,. \tag{1.122}$$

Note that the second heavy propagator $D_M(u-x)$, which is outside the loop, can be expanded in inverse powers of M without much ado.

We are interested in the quantity $R_v(p_1,p_2) = \Gamma_v(p_1,p_2) - T_M\{\Gamma_v(p_1,p_2)\}$. Since the quantity $R_v(p_1,p_2)$ should be a low-energy polynomial in the small momenta p_1, p_2, one may expand it in a Taylor series:

$$\begin{aligned} R_v(p_1,p_2) = R_v(0,0) &+ p_1^\mu \frac{\partial}{\partial p_1^\mu} R_v(p_1,p_2)\Big|_{p_1,p_2=0} \\ &+ p_2^\mu \frac{\partial}{\partial p_2^\mu} R_v(p_1,p_2)\Big|_{p_1,p_2=0} + \cdots\,. \end{aligned} \tag{1.123}$$

Note that in the effective Lagrangian this expansion translates into the derivative expansion in the light fields. In order to perform matching at lowest order in the inverse heavy mass M, it suffices to retain the first term in this expansion. Generalization to higher orders is straightforward.

Calculating $\Gamma_v(0,0)$, we get, on the one hand,

$$\begin{aligned} \Gamma_v(0,0) &= \int \frac{d^Dl}{(2\pi)^D i}\,\frac{1}{(m^2-l^2)^2}\,\frac{1}{M^2-l^2} = \frac{1}{16\pi^2}\int_0^1 \frac{dx\,x}{xm^2+(1-x)M^2} \\ &= \frac{1}{16\pi^2}\left(\frac{1}{m^2-M^2} - \frac{M^2}{(m^2-M^2)^2}\ln\frac{m^2}{M^2}\right) \\ &= -\frac{1}{16\pi^2 M^2}\left(1+\ln\frac{m^2}{M^2}\right) + O(M^{-4})\,. \end{aligned} \tag{1.124}$$

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On the other hand,

$$T_M\{\Gamma_v(0,0)\} = \int \frac{d^D l}{(2\pi)^D i}\,\frac{1}{(m^2-l^2)^2}\left\{\frac{1}{M^2}+O(M^{-4})\right\}$$
$$= \frac{2}{M^2}L_{\text{eff}} - \frac{1}{16\pi^2 M^2}\ln\frac{m^2}{\mu_{\text{eff}}^2} + O(M^{-4})\,. \tag{1.125}$$

Subtracting these two expressions gives

$$R_v(0,0) = \frac{2}{M^2}L_{\text{eff}} - \frac{1}{16\pi^2 M^2}\left(1+\ln\frac{\mu_{\text{eff}}^2}{M^2}\right) + O(M^{-4})\,. \tag{1.126}$$

As expected, the nonanalytic terms proportional to $\ln m^2$ cancel in this difference. Substituting now this expression into Eqs. (1.121) and (1.117), we finally obtain

$$T_4 - T_M\{T_4\} = \frac{g^4}{4M^2}R_v(0,0)\int d^4x\,\phi_c^4(x) + O(M^{-6})\,. \tag{1.127}$$

The quantity T_5 can be treated analogously. Here, we need the expression of the box integral at zero momenta (see Fig. 1.11):

$$\Gamma_b(0,0) = \int \frac{d^D l}{(2\pi)^D i}\,\frac{1}{(m^2-l^2)^2}\,\frac{1}{(M^2-l^2)^2} = \frac{1}{16\pi^2}\int_0^1 \frac{dx\,x(1-x)}{(xm^2+(1-x)M^2)^2}$$
$$= \frac{1}{16\pi^2}\,\frac{-2(M^2-m^2)+(M^2+m^2)\ln(M^2/m^2)}{(M^2-m^2)^3}$$
$$= \frac{1}{16\pi^2 M^4}\left(-2+\ln\frac{M^2}{m^2}\right) + O(M^{-6})\,. \tag{1.128}$$

The same integral, with the Taylor-expanded heavy propagator, is equal to

$$T_M\{\Gamma_b(0,0)\} = \int \frac{d^D l}{(2\pi)^D i}\,\frac{1}{(m^2-l^2)^2}\,\frac{1}{(M^2)^2} = -\frac{2}{M^4}L_{\text{eff}} - \frac{1}{16\pi^2 M^4}\ln\frac{m^2}{\mu_{\text{eff}}^2}\,. \tag{1.129}$$

From these equations we obtain

$$R_b(0,0) = \Gamma_b(0,0) - T_M\{\Gamma_b(0,0)\}$$
$$= \frac{2}{M^4}L_{\text{eff}} + \frac{1}{16\pi^2 M^4}\left(-2-\ln\frac{\mu_{\text{eff}}^2}{M^2}\right) + O(M^{-6})\,. \tag{1.130}$$

Finally, from Eq. (1.117) we have

$$T_5 - T_M\{T_5\} = \frac{g^4}{4}R_b(0,0)\int d^4x\,\phi_c^4(x)\,. \tag{1.131}$$

Eqs. (1.127) and (1.131) allow one to read off the matching of the low-energy constant C_0 at one loop:

$$C_0 = -\frac{g^2}{8M^2} - \frac{g^4}{4M^2}R_v(0,0) - \frac{g^4}{4}R_b(0,0) + O(M^{-6})$$
$$= -\frac{g^2}{8M^2} - \frac{g^4}{M^4}L_{\text{eff}} + \frac{g^4}{64\pi^2 M^4}\left(3+2\ln\frac{\mu_{\text{eff}}^2}{M^2}\right) + O(M^{-6})\,. \tag{1.132}$$

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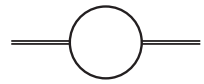

Figure 1.12 Renormalization of the heavy mass at one loop. The solid and double lines denote the light and heavy fields, respectively.

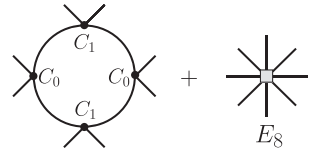

Figure 1.13 The renormalization of the Green's function with eight external legs in the effective theory. In order to cancel the divergence, a local term with eight ϕ-fields is introduced in the Lagrangian. The pertinent coupling is denoted by E_8.

In order to arrive at the final result, one has to express everything in Eq. (1.132) in terms of the renormalized couplings. As already mentioned, g is not renormalized. The quantity M^2 should, however, be renormalized (see Fig. 1.12):

$$M^2 = M_r^2(\mu_{\text{eff}}) - g^2 L_{\text{eff}}\,, \tag{1.133}$$

where, without loss of generality, one may assume that the scales in the underlying and effective theories coincide, $\mu = \mu_{\text{eff}}$.

Substituting this expression into Eq. (1.132), we finally obtain

$$\begin{aligned} C_0 &= -\frac{g^2}{8M_r^2} - \frac{9g^4}{8M_r^4} L_{\text{eff}} + \frac{g^4}{64\pi^2 M_r^4}\left(3 + 2\ln\frac{\mu_{\text{eff}}^2}{M_r^2}\right) + O(M_r^{-6}) \\ &= \nu_0 L_{\text{eff}} + C_0^r(\mu_{\text{eff}})\,. \end{aligned} \tag{1.134}$$

It is seen that $C_0^r(\mu_{\text{eff}})$ can be written in the form of Eq. (1.109). Reading off the coefficient κ_0, we get

$$\kappa_0 = -6 - 4\ln\frac{M^2}{\mu_{\text{eff}}^2} + O(M_r^{-2})\,. \tag{1.135}$$

The coefficient κ_0 does not depend on the light mass m at this order. This is, however, not true in general, that is, to all orders in the expansion in the inverse powers of M_r, unless the EOMs are used.

Finally, from Eq. (1.134) we can straightforwardly ensure that the renormalized coupling constant at this order obeys the well-known RG equation in the ϕ^4 theory:

$$\mu_{\text{eff}}\frac{dC_0^r}{d\mu_{\text{eff}}} = \frac{9}{2\pi^2}(C_0^r)^2\,. \tag{1.136}$$

Last but not least, it should be noted that the effective Lagrangian beyond tree level contains terms with 6,8,... ϕ-fields as well. These are needed, in particular, to cancel the divergences in the loop diagrams of the effective theory of the type shown in Fig. 1.13. Such terms emerge as a result of using the EOM in the quartic terms as well.

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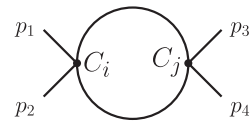

Figure 1.14 Insertion of two irrelevant vertices into the one-loop diagram. The incoming and outgoing momenta are p_1, p_2 and p_3, p_4, respectively. The total momentum is $P = p_1 + p_2 = p_3 + p_4$.

1.8 Dependence of the Effective Couplings on the Heavy Mass

In the simple model considered in the previous sections, the heavy mass M sets the hard scale at which the structure of the theory changes. For this reason, it is interesting to find out how the parameters of the low-energy theory depend on the heavy mass. First, let us consider the effective Lagrangian at tree level. We can judge about the leading behavior of these couplings in the limit $M \to \infty$ on the basis of the mass dimensions of these couplings alone. Only the effective mass of the light particle has a positive mass dimension. The coupling C_0 is dimensionless, and the couplings C_i with $i > 0$ have negative mass dimension. On dimensional grounds, the leading behavior in M in the latter couplings should be proportional to $g^2/M^{2(i+1)} \propto M^{-2i}$. Consequently, the couplings $C_i,\ i > 0$, fall off as negative powers of M in the limit $M \to \infty$. The dimensionless couplings are defined as $\tilde{C}_i = C_i M^{2i}$. The couplings $\tilde{C}_i$ are said to be of *natural size* if they are of order one. The dimension of the operators in the Lagrangian, which are multiplied by these couplings, is correlated with the preceding counting, in order to ensure that the Lagrangian has the correct mass dimension.

According to the mass dimension, the couplings are referred to as *relevant* (positive mass dimension), *marginal* (dimensionless) and *irrelevant* (negative mass dimension). It is seen that at low energies corresponding to the limit of a very large M, the contribution from the irrelevant couplings to the Green's functions is suppressed by powers of the large mass M.

Does the situation change beyond the tree level? Let us consider the insertion of irrelevant couplings in the loops. For simplicity, consider one loop in the effective theory with the insertion of two irrelevant vertices multiplied by the couplings C_i and C_j; see Fig. 1.14. The product of these two couplings falls off as $M^{-2(i+j)}$. Further, the mass dimension of the diagram in Fig. 1.14 is equal to zero. So, in order to obtain the required mass dimension, the preceding factor should be multiplied by $mass^{2(i+j)}$, where *mass* denotes any available mass scale in the effective theory: external momenta, effective mass or the regulator mass in the loops.

The discussion is particularly simple in dimensional regularization. The diagram in Fig. 1.14 is given by the expression

$$I_{ij} = \frac{\tilde{C}_i \tilde{C}_j}{M^{2(i+j)}} \int \frac{d^D l}{(2\pi)^D i} \frac{N(l; p_1, p_2, p_3, p_4)}{(m^2 - l^2)(m^2 - (P-l)^2)}, \tag{1.137}$$

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where the tree-level couplings $\tilde{C}_i = C_i M^{2i}$ are dimensionless and stay finite as $M \to \infty$. Further, $p_1, \ldots, p_4$ are the external momenta with $P = p_1 + p_2$, and the numerator N, which has the mass dimension $2(i+j)$, depends on the integration momentum l, the external momenta and the light mass m (at this order, one may replace the running effective mass of the light particle $m_{r,\mathrm{eff}}$ by m). After integration, the dependence on the scale μ_{eff} appears. However, in dimensional regularization the dependence on the scale μ_{eff} is logarithmic and thus safe (i.e., the power of M in front of the integral is not changed through the multiplication by a logarithm). Loops with insertions of the irrelevant couplings are also irrelevant in the limit $M \to \infty$. Thus, irrelevant couplings can be eliminated from the theory at one loop level as well. Moreover, as seen in this example, the naive power counting is respected in dimensional regularization. The insertion of two couplings that scale as M^{-i} and M^{-j} yields a result that scales like $M^{-(i+j)}$.

The argumentation is a bit more subtle in arbitrary regularization (say, cutoff regularization), where the powers of the large regulator scale Λ_{cut} can appear. This situation also emerges if we have a multi-scale problem, with heavy particles appearing in the effective field theory loops together with light particles (one prominent example being pion–nucleon scattering in Chiral Perturbation Theory). According to the dimensional counting, the maximal power of Λ_{cut} is contained in the maximally UV-divergent piece of the integral I_{ij} in Eq. (1.137), which does not depend on the external momenta $p_1, \cdots p_4$. Denoting this maximally divergent piece by $\tilde{I}_{ij}$, we have

$$\tilde{I}_{ij} = \frac{\tilde{C}_i \tilde{C}_j}{M^{2(i+j)}} \int^{\Lambda_{\mathsf{cut}}} \frac{d^4 l}{(2\pi)^4 i} \frac{l^{2(i+j)}}{(m^2 - l^2)^2} \sim \frac{\tilde{C}_i \tilde{C}_j \Lambda_{\mathsf{cut}}^{2(i+j)}}{M^{2(i+j)}} \,. \tag{1.138}$$

In other words, this term is no more suppressed since $\Lambda_{\mathsf{cut}} \sim M$. Note, however, that the above contribution does not depend on the external momenta and has exactly the same form as the contribution coming at tree level from the marginal vertex with the coupling C_0. Consequently, the whole contribution $\tilde{I}_{ij}$ can be removed by renormalization of C_0, which we are free to perform. We arrive at the same conclusion as earlier within dimensional regularization: the contributions from the irrelevant couplings are irrelevant at one loop as well. Thus, our results, as expected, do not depend on the regularization used, only the naive power counting holds no more in the cutoff regularization (cf. the effective theory of the potential well, Section 1.2.4).

The above arguments can be readily generalized for any number of insertions in diagrams with an arbitrary number of loops and external legs. The picture particularly simplifies in the limit $M \to \infty$, where the contributions from the irrelevant operators vanish. One arrives at the so-called decoupling theorem by Appelquist and Carazzone [14], which, loosely spoken, states that the whole contribution of the infinitely heavy degrees of freedom can be included in the renormalization of the masses and coupling constants, and the wave function renormalization constants of the light degrees of

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freedom.[11] In the context of the simple model considered in this section, the statement amounts to the claim that arbitrary renormalized Green's functions in the underlying and the effective theories in the limit $M_r \to \infty$ are related according to

$$G_r(p_1, \cdots, p_n; m_r, M_r, g, \mu) = Z^{n/2} G_{r,\text{eff}}(p_1, \cdots, p_n; m_{r,\text{eff}}, C_0^r, \mu) + \cdots, \quad (1.139)$$

where, for simplicity, we took $\mu_{\text{eff}} = \mu$, Z denotes the wave function renormalization constant and the ellipses stand for the terms that vanish in the limit $M_r \to \infty$.

To summarize: at low energies the contributions from the irrelevant operators in the Lagrangian to all physical observables are suppressed by inverse powers of the large scale and are thus small. Therefore, the physics at low energies is governed by a few operators with nonpositive mass dimension. This is obviously true at tree level but, as we have seen, holds as well when insertion of effective vertices in the loops is considered. This property of the effective theories has important implications when we try to establish the limits of existing theories and look for physics beyond them. As a simple example, consider QED, which is described by a Lagrangian containing only the relevant and marginal operators. Whatever the physics beyond QED is, it can be described by a string of the effective interactions in the Lagrangian, constructed with the use of the electron and photon fields. The operators with the smallest mass dimension will be least suppressed and, hence, more important at low momenta. Recalling the engineering dimensions of the fermion and photon fields, $[\psi] = [\bar{\psi}] = \frac{3}{2}$ and $[\mathscr{A}_\mu] = 1$, one may conclude that, at leading order, the effective Lagrangian can contain a single operator of dimension 5, the well-known Pauli term,

$$\delta\mathcal{L} = \frac{e}{M}\, \bar{\psi}\sigma_{\mu\nu}\mathscr{F}^{\mu\nu}\psi\,, \quad (1.140)$$

where e is the electric charge, $\mathscr{F}^{\mu\nu} = \partial^\mu \mathscr{A}^\nu - \partial^\nu \mathscr{A}^\mu$ denotes the electromagnetic field strength tensor, and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. According to dimensional counting, the operator of dimension 5 in the Lagrangian should be accompanied by an inverse power of a large scale M and thus suppressed at low energies. Further, note that the operator given in Eq. (1.140) contributes to the anomalous magnetic moment of the electron, and this contribution is not contained in QED. Consequently, measuring the anomalous magnetic moment very precisely in the experiment and confronting the result with the sufficiently accurate calculations in QED, one obtains a lower bound on the scale M of the physics beyond QED; see, for example, Ref. [15]. The search of new physics at the precision frontier generally proceeds along similar patterns.

[11] Note that in the theories with spontaneous breaking of the symmetry, the masses of some particles are equal to the product of the vacuum expectation value of the Higgs field and the coupling constant of a given particle with the Higgs field (a nice example is the Standard Model). The heavy mass limit in such theories can mean: a) the vacuum expectation value becomes large, or b) some of the couplings become large. The decoupling theorem applies in the first case only, whereas in the second case we are dealing with the theory in the strong coupling limit, where the perturbative arguments cannot be used.

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Figure 1.15 The dressed photon propagator by summing up the self-energy insertions. The wiggly and solid lines denote photons and electrons, respectively.

1.9 Decoupling in Different Renormalization Schemes

Next, we shall discuss the choice of the renormalization scheme that will prove to be very instructive. As we have seen in the previous sections, the decoupling of a heavy scale in the theory proceeds differently, if different regularizations and renormalization schemes are used (e.g., the $\overline{\text{MS}}$ scheme in dimensional regularization versus cutoff regularization). In this section, we wish to elaborate on this issue.

Consider the well-known example of charge renormalization in QED at one loop. Summing up all self-energy insertions in the photon propagator (see Fig. 1.15), it is seen that the latter obeys the Dyson–Schwinger equation:

$$D_{\mu\nu}(p) = D^0_{\mu\nu}(p) + iD^0_{\mu\lambda}(p)e^2\Pi^{\lambda\rho}(p)D_{\rho\nu}(p)\,. \tag{1.141}$$

For simplicity, we choose the Feynman gauge, where the free photon propagator is given by $D^0_{\mu\nu}(p) = -g_{\mu\nu}/p^2$. Furthermore, the one-loop self-energy is given by

$$\begin{aligned} e^2\Pi_{\mu\nu}(p) &= i(p_\mu p_\nu - p^2 g_{\mu\nu})e^2\Pi(p^2) \\ &= \frac{ie^2}{2\pi^2}(p_\mu p_\nu - p^2 g_{\mu\nu})\left\{-\frac{16\pi^2}{3}L - \int_0^1 dx\, x(1-x)\ln\frac{m_e^2 - p^2x(1-x)}{\mu^2}\right\}, \end{aligned} \tag{1.142}$$

where m_e and e denote the electron mass and charge, respectively, and μ is the scale of dimensional regularization. The UV-divergent quantity L is defined in Eq. (1.89).

The solution of the Dyson–Schwinger equation takes the form

$$D_{\mu\nu}(p) = -\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2(1+e^2\Pi(p^2))} - \frac{p_\mu p_\nu}{p^4}\,. \tag{1.143}$$

As seen from Eq. (1.142), the quantity $\Pi(p^2)$ is ultraviolet-divergent. This divergence has to be "eaten up" by the charge renormalization. Namely, the bare charge e is also divergent. The renormalized charge and renormalized self-energy are defined by requiring that

$$\frac{e^2}{1+e^2\Pi(p^2)} = \frac{e^2_{\text{ren}}}{1+e^2_{\text{ren}}\Pi_{\text{ren}}(p^2)}\,. \tag{1.144}$$

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The method to remove the divergences in $\Pi_{\text{ren}}(p^2)$ and, hence, the definition of e_{ren}, depends on the renormalization prescription used. In what follows we shall compare two different renormalization schemes. (The indices "r" and "R" are used to distinguish between these schemes):

a) *The $\overline{MS}$ scheme:* In this scheme, the renormalized self-energy is obtained by just dropping the term proportional to L:

$$e^2\Pi_r(p^2;\mu^2) = -\frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\frac{m_e^2 - p^2x(1-x)}{\mu^2}\,. \tag{1.145}$$

b) *The MOM scheme:* In this scheme, the renormalized self-energy is obtained by a subtraction at $p^2 = \mu_0^2$ with $\mu_0^2 < 0$:

$$e^2\Pi_R(p^2;\mu_0^2) = e^2\Pi(p^2) - e^2\Pi(\mu_0^2) = -\frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\frac{m_e^2 - p^2x(1-x)}{m_e^2 - \mu_0^2x(1-x)}\,. \tag{1.146}$$

In both cases, the divergent quantity, which is subtracted, is a constant independent of p^2. Writing down $\Pi(p^2) = \Pi_{\text{div}} + \Pi_{\text{ren}}(p^2)$, we get

$$e_{\text{ren}}^2 = e^2(1 - e^2\Pi_{\text{div}} + O(e^4))\,. \tag{1.147}$$

Furthermore, Π_{div} and $\Pi_{\text{ren}}(p^2)$ both depend on the renormalization scale (μ in the $\overline{\text{MS}}$ scheme, μ_0 in the MOM scheme), whereas their sum does not. Hence, e_{ren} also depends on this scale, since the bare parameter, e, does not. The dependence on the scale is described by the RG equations:

(a) *$\overline{MS}$ scheme:*

$$\mu\frac{de_r(\mu)}{d\mu} = \beta_r(e_r(\mu))\,,$$

$$\beta_r(e) = \frac{e}{2}\,\mu\frac{d}{d\mu}\,\Pi_r(p^2;\mu^2) = \frac{e^3}{12\pi^2}\,. \tag{1.148}$$

(b) *MOM scheme:*

$$\mu_0\frac{de_R(\mu_0)}{d\mu_0} = \beta_R(e_R(\mu_0))\,,$$

$$\beta_R(e) = \frac{e}{2}\,\mu_0\frac{d}{d\mu_0}\,\Pi_R(p^2;\mu_0^2) = -\frac{e^3}{4\pi^2}\int_0^1 \frac{dx\,x^2(1-x)^2\mu_0^2}{m_e^2 - \mu_0^2x(1-x)}\,. \tag{1.149}$$

It is instructive to study two limiting cases, with m_e much larger and much smaller than the renormalization scale. We have:

$$\beta_r(e) = \frac{e^3}{12\pi^2}\,, \quad \text{all values of } m_e\,,$$

$$\beta_R(e) = \frac{e^3}{12\pi^2}\,, \quad \text{if } m_e \ll \mu_0\,, \quad \text{the same result as in the } \overline{\text{MS}} \text{ scheme}\,,$$

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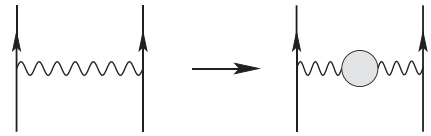

Figure 1.16 Vacuum polarization correction to the one-photon exchange diagram in electron–electron scattering. The shaded circle represents the full propagator.

$$\beta_R(e) = -\frac{e^3\mu_0^2}{60\pi^2 m_e^2} \to 0\,, \qquad \text{if } m_e \gg \mu_0\,, \quad \text{decoupling, approaches a free theory}\,. \tag{1.150}$$

Does this result mean that the decoupling of the heavy scale occurs only within the MOM renormalization scheme? Of course not, as can be seen from the discussion in Section 1.6. The lesson to be learned here is different. The decoupling is explicit, if everything is expressed in terms of *low-energy quantities*. Such a low-energy quantity is, for example, the physical charge, which can be defined as follows. Consider the scattering amplitude of two electrons, with the one-photon exchange diagram modified by an electron loop;[12] see Fig. 1.16:

$$T_{ee\to ee} = \bar{u}(p_1', s_1')\gamma^\mu u(p_1, s_1)\,\frac{e_{\text{ren}}^2}{-q^2(1+e_{\text{ren}}^2\Pi_{\text{ren}}(q^2))}\,\bar{u}(p_2', s_2')\gamma_\mu u(p_2, s_2)\,, \tag{1.151}$$

where p_i, s_i and p_i', s_i' $(i = 1,2)$ denote the momenta and the spins of the electrons in the initial and in the final state, respectively, and $q = p_1' - p_1 = p_2' - p_2$. To simplify the discussion, we do not consider the second diagram, which is obtained by a permutation of the two electrons in the initial or in the final state.

Consider now these expressions at very low momenta, $\mathbf{p}_i^2 \ll m_e^2$ and $\mathbf{p}_i'^2 \ll m_e^2$. It is easily seen that

$$p_i^0 = \sqrt{m_e^2 + \mathbf{p}_i^2} = m_e + \frac{\mathbf{p}_i^2}{2m_e} + \cdots = m_e + O(m_e^{-1})\,. \tag{1.152}$$

Similar relations hold for $p_i'^0$. Further, $q^2 = (p_i'^0 - p_i^0)^2 - (\mathbf{p}_i' - \mathbf{p}_i)^2 = -(\mathbf{p}_i' - \mathbf{p}_i)^2 + O(m_e^{-1})$. The nonrelativistic reduction of the Dirac spinors takes the form

$$\begin{aligned}\bar{u}(p', s')\gamma^\mu u(p, s) &= \bar{u}(0, s')\,\frac{\not{p}' + m_e}{\sqrt{p'^0 + m_e}}\,\gamma^\mu\,\frac{\not{p} + m_e}{\sqrt{p^0 + m_e}}\,u(0, s) \\ &= (2m_e)g^{\mu 0}\delta_{s's}(1 + O(m_e^{-1}))\,.\end{aligned} \tag{1.153}$$

According to this, the amplitude at low momenta is given by

$$T_{ee\to ee} = (2m_e)^2\delta_{s_1's_1}\delta_{s_2's_2}\,\frac{e_{\text{ren}}^2}{\mathbf{q}^2(1+e_{\text{ren}}^2\Pi_{\text{ren}}(-\mathbf{q}^2))}(1 + O(m_e^{-1}))\,. \tag{1.154}$$

[12] According to the Ward identity, only vacuum polarization contributes to charge renormalization.

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On the other hand, in the Born approximation, this amplitude is proportional to the interaction potential between two electrons,[13] which takes into account the vacuum polarization effect. Dropping the overall normalization factor and spin indices, we get

$$V(r) = \int \frac{d^3\mathbf{q}}{(2\pi)^3}\, e^{i\mathbf{qr}}\, \frac{e^2_{\text{ren}}}{\mathbf{q}^2(1+e^2_{\text{ren}}\Pi_{\text{ren}}(-\mathbf{q}^2))}\,. \tag{1.155}$$

Neglecting the correction in the denominator, the static Coulomb potential is obtained from this expression.

We fix the parameters of the theory (here, the electric charge) at large distances, that is, by measuring the force acting on small charged oil droplets (Millikan-type experiment). The elementary charge measured in this manner corresponds to $\alpha = e^2_{\text{phys}}/(4\pi) \simeq 1/137$, the fine-structure constant. Since the distances in such an experiment are much larger than the Compton wavelength of the electron, in momentum space we are focusing on the region $\mathbf{q}^2 \to 0$. In this region, the modified potential (1.155) asymptotically coincides with the Coulomb potential at large distances. This gives

$$V(r) = \int \frac{d^3\mathbf{q}}{(2\pi)^3}\, e^{i\mathbf{qr}}\, \frac{e^2_{\text{phys}}}{\mathbf{q}^2(1+e^2_{\text{phys}}(\Pi_{\text{ren}}(-\mathbf{q}^2)-\Pi_{\text{ren}}(0)))}\,,$$

$$e^2_{\text{phys}} = \frac{e^2_{\text{ren}}}{(1+e^2_{\text{ren}}\Pi_{\text{ren}}(0))}\,. \tag{1.156}$$

For the different renormalization schemes we have the following:

(a) *$\overline{MS}$ scheme:*

$$e^2_r(\mu) = \frac{e^2_{\text{phys}}}{1+\dfrac{e^2_{\text{phys}}}{12\pi^2}\ln\dfrac{m_e^2}{\mu^2}}\,. \tag{1.157}$$

(b) *MOM scheme:*

$$e^2_R(\mu_0) = \frac{e^2_{\text{phys}}}{1+\dfrac{e^2_{\text{phys}}}{2\pi^2}\displaystyle\int_0^1 dx\, x(1-x)\ln\left(1-\frac{\mu_0^2}{m_e^2}x(1-x)\right)}\,. \tag{1.158}$$

Differentiating $e(\mu)$ and $e(\mu_0)$ with respect to μ and μ_0, respectively, and taking into account that the quantity e_{phys} is a physical observable that is scale-independent, we again arrive at the RG equations, (1.148) and (1.149). Finally, expressing everything in terms of e_{phys}, the modified Coulomb potential takes the form

$$V(r) = \int \frac{d^3\mathbf{q}}{(2\pi)^3}\, e^{i\mathbf{qr}}\, \frac{e^2_{\text{phys}}}{\mathbf{q}^2(1+e^2_{\text{phys}}F(\mathbf{q}^2))}\,,$$

[13] Strictly speaking, this is the potential energy of two electrons and not the static electromagnetic potential. We shall, however, use this short term in the following and hope that it does not lead to any confusion.

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$$F(\mathbf{q}^2) = \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left(1+\frac{\mathbf{q}^2}{m_e^2}x(1-x)\right). \tag{1.159}$$

Note that the quantity $F(\mathbf{q}^2)$ is scale-independent and is the same in both regularizations. The decoupling is explicit as $F(\mathbf{q}^2)\to 0$ for $m_e^2\to\infty$. Thus, the whole difference between the two regularizations is hidden in Eqs. (1.157) and (1.158), which describe how the renormalized charge $e_r(\mu)$ and $e_R(\mu_0)$ behave at $m_e\to\infty$, when e_{phys} is fixed. This behavior is different. Namely, $e_R(\mu_0)\to e_{\mathsf{phys}}$, meaning that $e_R(\mu_0)$ stays a perfect low-energy quantity in this limit. On the contrary, the limit $m_e\to\infty$ cannot be performed at a fixed e_{phys} and μ in the quantity $e_r(\mu)$, because of the large logarithms $\ln(m_e^2/\mu^2)$ in perturbation theory. In order to suppress these logarithms, one has to take $\mu\sim m_e$, meaning that one is fixing the charge at a scale of order m_e. Thus, $e_r(\mu)$ is not a quantity defined at low energy, and the decoupling is not explicit if the expressions are written in terms of $e_r(\mu)$ instead of e_{phys}.

1.10 Floating Cutoff

In Section 1.8 we gave arguments in favor of the conclusion that at low energies, only superrenormalizable and renormalizable interactions, described by relevant and marginal operators in the Lagrangian, survive, whereas the contributions from irrelevant operators are suppressed by the powers of a large mass. In this section, we address this issue from a different point of view, using a method that is based on the ideas of Wilson's renormalization group [16]. In this method the high-frequency modes in the generating functional of the theory are systematically integrated out. The discussion here closely follows Polchinski's original paper [17]; see also [18].

We do not want to focus on any particular model. To this end we shall interpret M merely as some hard scale of the theory, after which the unknown physics starts, be this a new particle with a mass M, nonlocal effects, or whatever. Further, in order to make the arguments maximally transparent, here we shall use a momentum cutoff instead of dimensional regularization. Consider, for simplicity, a theory with a single scalar field ϕ. The Euclidean generating functional for the *renormalized Green's functions* in momentum space is given by

$$Z(J) = \int [d\phi]_M \exp\left\{S(\phi, C_i(M)) + J\phi\right\}, \tag{1.160}$$

where $[d\phi]_M$ denotes the path integral measure with a cutoff on the high-frequency modes with $p\sim M$. This shorthand notation should be interpreted as follows: calculating Eq. (1.160) in perturbation theory, a cutoff at the momentum scale of order M is introduced in all Feynman graphs. Note that the explicit form of the cutoff does not play a role. Further, $S(\phi, C_i(M))$ is the action functional that contains the *bare* coupling constants $C_i(M)$ corresponding to the cutoff at a scale M. In addition, in a renormalizable theory (in a conventional sense), it is possible to choose $C_i(M)$ so that the generating functional remains finite as $M\to\infty$. This is not possible in a

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non-renormalizable theory, but everything is perfectly well defined if the cutoff stays finite.

Now, let us ask the question how the action depends on the cutoff when the latter varies from M to some lower value Λ_{eff}, where our effective field theory is defined. This scale obeys the inequality $m \ll \Lambda_{\text{eff}} \ll M$. (Here, m denotes the light scale in the theory, for example, the mass of a light particle described by the field ϕ.) More precisely, we consider a smooth change of a cutoff from M to Λ_{eff}, defining a scale $\Lambda_{\text{eff}} \leq \Lambda \leq M$. The Euclidean path integral is given by Eq. (1.160), with M replaced by Λ.

Here, we are only interested in the low-frequency modes; thus, we assume that

$$J(p) = 0 \qquad \text{for} \quad p^2 > \Lambda_{\text{eff}}^2 \,. \tag{1.161}$$

The crucial point is that, in order to ensure that the renormalized Green's functions do not depend on Λ, the effective action $S(\phi, C_i(\Lambda))$ should obey certain *RG flow equations*. In other words, the effective couplings that enter $S(\phi, C_i(\Lambda))$ should depend on Λ in a manner that compensates the explicit Λ-dependence coming from the cutoff. For example, the mass parameters at two scales in the theory with the interaction Lagrangian $\mathcal{L}_I = C_0\phi^4$ at one loop are related by

$$\begin{aligned} m^2(\Lambda) &= m^2(M) - 12C_0 \int^M \frac{d^D l}{(2\pi)^D} \frac{1}{m^2 + l^2}\bigg|_{\text{Eucl.}} \\ &\quad + 12C_0 \int^\Lambda \frac{d^D l}{(2\pi)^D} \frac{1}{m^2 + l^2}\bigg|_{\text{Eucl.}} \\ &= m^2(M) - \frac{3C_0}{4\pi^2}\left(M^2 - \Lambda^2 - m^2 \ln\frac{M^2}{\Lambda^2} + \cdots\right), \\ \Lambda\frac{d}{d\Lambda} m^2(\Lambda) &= \frac{3C_0}{2\pi^2}\Lambda^2\left(1 + O\left(\frac{m^2}{\Lambda^2}\right)\right), \end{aligned} \tag{1.162}$$

where the momentum integrals are evaluated in Euclidean space. Note that, in order to obtain these equations, the tadpole diagram (the first diagram in Fig. 1.9) has been evaluated. In general, at a scale Λ, the effective Lagrangian includes the contributions from all momenta $\Lambda < p < M$, which emerge through the loops. Thus, the Lagrangian at lower scales necessarily contains all derivative vertices, even if the theory did not contain non-renormalizable operators at $\Lambda = M$ at all. The RG flow equation tells us that these will be generated at lower scales. For dimensional reasons these operators $\phi^2\Box\phi^2$, $\phi^2\Box^2\phi^2, \cdots$ will be suppressed by the respective powers of Λ if $m \ll \Lambda$ still holds. The first-order differential equations, analogous to one given in Eq. (1.162), emerge for the couplings $C_i(\Lambda)$. These are nothing but the conventional RG equations.

In the following it will be useful to consider the mass as one of the couplings $C_i(\Lambda)$. These couplings define a point $C = \{C_i(\Lambda)\}$ in the (infinite-dimensional) parameter space, which moves along some trajectory when Λ changes from M to Λ_{eff}. This trajectory is defined by the RG equations and the boundary conditions at $\Lambda = M$.

It is convenient to define the dimensionless couplings, $\tilde{C}_i(\Lambda)$, according to

$$C_i(\Lambda) = \Lambda^{\Delta_i}\tilde{C}_i(\Lambda) \,. \tag{1.163}$$

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If Δ_i is positive, zero or negative, the pertinent operators are referred to as relevant, marginal and irrelevant couplings, respectively.

The RG equations for the dimensionless couplings are written as follows:

$$\Lambda\frac{d}{d\Lambda}\tilde{C}_i(\Lambda)+\Delta_i\tilde{C}_i(\Lambda)=\beta_i(\tilde{C}(\Lambda))\,, \tag{1.164}$$

where the β_i denote the pertinent β-functions. In what follows, we shall demonstrate that, if the couplings are sufficiently small (i.e., allowing for a perturbative treatment), for a completely arbitrary boundary condition at $\Lambda = M$, the system at a lower scale $\Lambda = \Lambda_{\text{eff}}$ lives on a surface with a dimension given by a total number of the relevant and marginal operators only. In other words, systems with very different values of the irrelevant couplings at a high scale behave similarly at lower scales. Stated differently, the information about the high scale is erased during the RG evolution. Only the relevant and marginal couplings survive.

In order to understand how this happens, consider a toy example with two couplings: a marginal one, $C_0(\Lambda)$, and an irrelevant one, $C_1(\Lambda)$, with a mass dimension equal to -2. We set, for simplicity, all other couplings to zero. The RG equations for the couplings C_0, C_2 take the form

$$\begin{aligned}\Lambda\frac{d\tilde{C}_0}{d\Lambda}&=\beta_0(\tilde{C}_0,\tilde{C}_1)\,,\\ \Lambda\frac{d\tilde{C}_1}{d\Lambda}-2\tilde{C}_1&=\beta_1(\tilde{C}_0,\tilde{C}_1)\,.\end{aligned} \tag{1.165}$$

The boundary conditions are fixed at $\Lambda = M$:

$$\tilde{C}_i(\Lambda)\big|_{\Lambda=M}=\tilde{C}_i^{(0)}\,,\qquad i=0,1\,. \tag{1.166}$$

Let us now assume that the pair $(\bar{C}_0,\bar{C}_1)$ is some solution of the preceding equations. Consider a small deviation $\tilde{C}_i^{(0)}\to\tilde{C}_i^{(0)}+\delta\tilde{C}_i^{(0)}$ in the boundary conditions. Then, the solutions will also change. We denote $\varepsilon_i=\tilde{C}_i-\bar{C}_i$ and note that the RG equations linearize for ε_i:

$$\begin{aligned}\Lambda\frac{d\varepsilon_0}{d\Lambda}&=\overline{\frac{\partial\beta_0}{\partial\tilde{C}_0}}\varepsilon_0+\overline{\frac{\partial\beta_0}{\partial\tilde{C}_1}}\varepsilon_1\,,\\ \Lambda\frac{d\varepsilon_1}{d\Lambda}-2\varepsilon_1&=\overline{\frac{\partial\beta_1}{\partial\tilde{C}_0}}\varepsilon_0+\overline{\frac{\partial\beta_1}{\partial\tilde{C}_1}}\varepsilon_1\,,\end{aligned} \tag{1.167}$$

where the bar means that the partial derivatives are evaluated at $(\bar{C}_0,\bar{C}_1)$. The term $-2\varepsilon_1$ in Eq. (1.167) is crucial in our discussion, since it will cause a damping of the variation of the deviations in the ε_1-direction.

The equations for $\varepsilon_0,\varepsilon_1$ can be decoupled by introducing a new variable,

$$\zeta_1=\varepsilon_1-\varepsilon_0\frac{d\bar{C}_1/d\Lambda}{d\bar{C}_0/d\Lambda}\,. \tag{1.168}$$

The physical meaning of the variable ζ_1 is the distance between two neighboring trajectories in the $\tilde{C}_0,\tilde{C}_1$ plane along the $\tilde{C}_1$ axis; see Fig. 1.17. As already mentioned, the

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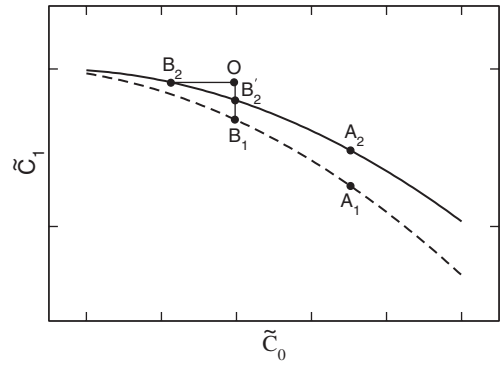

Figure 1.17 Two neighboring trajectories in the $\tilde{C}_0, \tilde{C}_1$ plane. Let the points A_1, A_2 correspond to a pair $(\tilde{C}_0, \tilde{C}_1)$ at some Λ. The distance between the curves is just the length of $|A_1A_2|$. When $\Lambda \to \Lambda + \Delta\Lambda$, then A_1 moves to B_1 and A_2 to B_2. The vertical distance between two curves in this case is, however, given by the length of $|B_1B_2'| = |OB_1| - |OB_2'|$, which coincides with variable ζ_1 defined in Eq. (1.168).

RG equation for this quantity decouples. Indeed, using Eqs. (1.165) and (1.167), it can be straightforwardly shown that ζ_1 obeys the following equation:

$$\Lambda \frac{d\zeta_1}{d\Lambda} - 2\zeta_1 = \left(\overline{\frac{\partial \beta_1}{\partial \tilde{C}_1}} + \overline{\frac{\partial \beta_0}{\partial \tilde{C}_0}} - \Lambda \frac{d}{d\Lambda} \ln \bar{\beta}_0 \right) \zeta_1 \,. \tag{1.169}$$

In perturbation theory, the r.h.s. of this equation is small, and the RG evolution of ζ_1 is governed by the term $-2\zeta_1$ in the l.h.s. of the equation, determined by the mass dimension of the irrelevant coupling. Hence, the solution of the equation at $\Lambda = \Lambda_{\text{eff}}$ to lowest order in perturbation theory is given by

$$\zeta_1(\Lambda_{\text{eff}}) = \zeta_1(M) \left(\frac{\Lambda_{\text{eff}}^2}{M^2} \right) . \tag{1.170}$$

This means that all RG trajectories approach each other in the infrared, and there is one essential parameter left instead of two: The value of $\tilde{C}_1$ is predicted, given the value of $\tilde{C}_0$. Moreover, the value of the parameter $\tilde{C}_0$ is also not independent, it just marks the place where we are on a single trajectory in the infrared. In other words, $\tilde{C}_0$ can be traded for the scale Λ_{eff}. Let us stress once more that the validity of this result is restricted to the perturbative regime, where the small corrections are supposed not to change the leading-order behavior. Strong non-perturbative effects might invalidate some arguments used in the proof and lead to the modification of the results.[14]

Imagine now that we arbitrarily change input values of couplings $\tilde{C}_0, \tilde{C}_1$ at the scale $\Lambda = M$. The curves will still converge in the infrared. Moreover, the resulting curve will be defined by the same RG equation for a single coupling $\tilde{C}_0$, since $\tilde{C}_1$ is a function of $\tilde{C}_0$ in the infrared limit. Thus, adjusting $\tilde{C}_0(M), \tilde{C}_1(M)$ so that $\tilde{C}_0(\Lambda)$ at $\Lambda = \Lambda_{\text{eff}}$ takes the

[14] A nice example for this is provided by the Thirring model; see, e.g., the discussion in Ref. [19].

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same value as before,[15] one ends up with the same curve below Λ_{eff}. Stated differently, different theories at the scale M converge to the same theory with a single coupling $\tilde{C}_0(\Lambda_{\text{eff}})$ at a lower scale, which means that some information, which was present at the scale M, gets erased at the scale Λ_{eff}. This is exactly the essence of decoupling. Furthermore, note that $C_1(\Lambda_{\text{eff}}) = \tilde{C}_1(\Lambda_{\text{eff}})/\Lambda_{\text{eff}}^2$ is still power-suppressed in the low-energy region, if the cutoff Λ_{eff} is taken much larger than the light particle mass and the characteristic momenta.

1.11 Emergent Symmetries

The fact that only a few couplings survive in the infrared region can have far-reaching implications. Namely, if we set all couplings that become irrelevant at low energies to zero, the resulting Lagrangian might have a higher symmetry than the original one, containing, for example, a whole string of operators with increasing mass dimension. Consequently, one speaks of an *emergent symmetry*. This means that the theory at low energies exhibits a symmetry that was not present in the original Lagrangian at high energies. We note that the content of this section is somewhat speculative and it might be skipped upon first reading.

Below, we will demonstrate this effect within a simple model for a global symmetry. We shall follow the perturbative approach outlined in Refs. [20, 21]. The Lagrangian of the model is given by

$$\mathcal{L} = \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 - \frac{m^2}{2}(\phi_1^2+\phi_2^2) - \frac{\lambda}{4}(\phi_1^4+\phi_2^4) - \frac{g}{2}\phi_1^2\phi_2^2 + \text{counter terms}\,. \tag{1.171}$$

Here, $\phi_{1,2}$ are real scalar fields. The preceding Lagrangian possesses a discrete symmetry with respect to $\phi_{1,2} \to -\phi_{1,2}$ and $\phi_1 \leftrightarrow \phi_2$. If $\lambda = g$, however, the theory becomes invariant under the continuous $O(2)$ group

$$\phi_1 \to \phi_1\cos\theta - \phi_2\sin\theta\,, \qquad \phi_2 \to \phi_1\sin\theta + \phi_2\cos\theta\,, \tag{1.172}$$

where θ is an angle. Let us now consider the renormalization group running for the couplings of the Lagrangian. To this end, we shall first calculate the pertinent β-functions at one loop, using the saddle-point method in the path integral. Expanding the action functional in the vicinity of the classical solution of the EOM,

$$\phi_{1,2}(x) = \phi_{1,2}^c(x) + \xi_{1,2}(x)\,, \tag{1.173}$$

we get

$$\int d^4x\mathcal{L}(x) = \int d^4x\left(\mathcal{L}^c(x) + \frac{1}{2}\Xi^T\hat{D}\Xi + O(\xi^3)\right), \tag{1.174}$$

[15] We have two couplings $\tilde{C}_0(M), \tilde{C}_1(M)$ at our disposal, to adjust a single coupling $\tilde{C}_0(\Lambda_{\text{eff}})$. It is evident that some residual freedom at the scale M remains after matching, performed at a lower scale.

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where $\mathcal{L}^c(x)$ is the Lagrangian evaluated with the classical solution $\phi^c_{1,2}(x)$, $\Xi^T = (\xi_1, \xi_2)$, and

$$\hat{D} = \begin{pmatrix} -(\Box + m^2) - 3\lambda(\phi_1^c)^2 - g(\phi_2^c)^2 & -2g\phi_1^c\phi_2^c \\ -2g\phi_1^c\phi_2^c & -(\Box + m^2) - 3\lambda(\phi_2^c)^2 - g(\phi_1^c)^2 \end{pmatrix}$$
$$= \hat{D}_0 + \hat{D}_1\,, \quad \hat{D}_0 = \begin{pmatrix} -(\Box + m^2) & 0 \\ 0 & -(\Box + m^2) \end{pmatrix}. \tag{1.175}$$

Note also that the linear terms in $\xi_{1,2}$ are absent in Eq. (1.174) because the $\phi^c_{1,2}(x)$ obey the EOM. The path integral can now be evaluated in the semiclassical approximation:

$$\int d\phi_1 d\phi_2 \exp\left[i\int d^4x \mathcal{L}(x)\right] = \int d\xi_1 d\xi_2 \exp\left[i\int d^4x\left(\mathcal{L}^c(x) + \frac{1}{2}\Xi^T\hat{D}\Xi\right)\right]$$
$$= (\det\hat{D})^{-1/2}\exp\left[i\int d^4x\mathcal{L}^c(x)\right]$$
$$= \exp\left[i\left(\int d^4x\mathcal{L}^c(x) + \frac{i}{2}\,\mathrm{tr}\,\ln\hat{D}\right)\right]. \tag{1.176}$$

Expanding the logarithm in powers of the fields $\phi^c_{1,2}$,

$$\frac{i}{2}\,\mathrm{tr}\,\ln\hat{D} = \frac{i}{2}\,\mathrm{tr}\,\ln(\hat{D}_0 + \hat{D}_1)$$
$$= \frac{i}{2}\,\mathrm{tr}\,\ln\hat{D}_0 + \frac{i}{2}\,\mathrm{tr}\,(\hat{D}_0^{-1}\hat{D}_1) - \frac{i}{4}\,\mathrm{tr}\,(\hat{D}_0^{-1}\hat{D}_1\hat{D}_0^{-1}\hat{D}_1) + \cdots, \tag{1.177}$$

it is immediately seen that the renormalization of the quartic couplings can be read off from the last term in this equation. This term can be rewritten in the following form:

$$-\frac{i}{4}\,\mathrm{tr}\,(\hat{D}_0^{-1}\hat{D}_1\hat{D}_0^{-1}\hat{D}_1) = -\frac{i}{4}\int d^4x d^4y D(x-y)D(y-x)\mathrm{tr}\,(\hat{D}_1(y)\hat{D}_1(x))\,, \tag{1.178}$$

with

$$D(x-y) = \langle x|(\Box + m^2)|y\rangle = \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik(x-y)}}{m^2 - k^2}\,. \tag{1.179}$$

Then, using translational invariance, one may write

$$D(x-y)D(y-x) = i\int \frac{d^4k_1}{(2\pi)^4}\,e^{-ik_1(x-y)}\int \frac{d^4k_2}{(2\pi)^4 i}\,\frac{1}{(m^2 - (k_1+k_2)^2)(m^2 - k_2^2)}\,. \tag{1.180}$$

The integral over k_2 is UV-divergent. For clarity, let us use cutoff regularization. Then,

$$\int \frac{d^4k_2}{(2\pi)^4 i}\,\frac{1}{(m^2 - (k_1+k_2)^2)(m^2 - k_2^2)} = \frac{1}{16\pi^2}\,\ln\frac{\Lambda^2}{m^2} + \cdots, \tag{1.181}$$

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where the ellipsis denotes the terms that do not depend on Λ in the limit of large cutoff $\Lambda \gg m$. Hence,

$$D(x-y)D(y-x) = i\delta^{(4)}(x-y)\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m^2} + \cdots, \tag{1.182}$$

and

$$-\frac{i}{4}\,\mathrm{tr}\,(\hat{D}_0^{-1}\hat{D}_1\hat{D}_0^{-1}\hat{D}_1) = \frac{1}{4}\,\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m^2}\int d^4x\,\mathrm{tr}\,(\hat{D}_1(x)\hat{D}_1(x))$$
$$= \frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m^2}\int d^4x\left(\frac{9\lambda^2+g^2}{4}\left((\phi_1^c)^4+(\phi_2^c)^4\right)+(3\lambda g+2g^2)(\phi_1^c)^2(\phi_2^c)^2\right). \tag{1.183}$$

The coefficients in front of the operators $(\phi_1^c)^4+(\phi_2^c)^4$ and $(\phi_1^c)^2(\phi_2^c)^2$ in the effective action (that includes $\mathcal{L}^c$) should not depend on the cutoff Λ. This condition gives

$$0 = \Lambda\frac{d}{d\Lambda}\left(-\frac{1}{4}\lambda+\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m^2}\,\frac{9\lambda^2+g^2}{4}\right),$$
$$0 = \Lambda\frac{d}{d\Lambda}\left(-\frac{1}{2}g+\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m^2}\,(3\lambda g+2g^2)\right). \tag{1.184}$$

Hence, the RG equations at one loop take the following form:

$$\Lambda\frac{d}{d\Lambda}\lambda = \frac{3}{8\pi^2}\left(3\lambda^2+\frac{1}{3}g^2\right),$$
$$\Lambda\frac{d}{d\Lambda}g = \frac{3}{8\pi^2}\left(2\lambda g+\frac{4}{3}g^2\right). \tag{1.185}$$

The quantity of interest is $\eta = g/\lambda$. The RG equation for this quantity is readily obtained from Eq. (1.185):

$$\Lambda\frac{d}{d\Lambda}\eta = \frac{3}{8\pi^2}(-\lambda\eta)\left(\frac{1}{3}\eta^2-\frac{4}{3}\eta+1\right). \tag{1.186}$$

As seen, the β-function for the coupling η vanishes at $\eta = 0,1,3$; these are *fixed points*. The point $\eta = 0$ (i.e., $g = 0$) corresponds to two mutually noninteracting scalar fields. The case $\eta = 3$ (i.e., $g = 3\lambda$) is reduced to the former by the substitution $\phi_1 = (\psi_1 + \psi_2)/\sqrt{2}$ and $\phi_2 = (\psi_1 - \psi_2)/\sqrt{2}$. Finally, the case $\eta = 1$ (i.e., $g = \lambda$) corresponds to the $O(2)$ symmetric case. Note also that the derivative of the β-function with respect to η is positive for $\eta = 1$ (infrared attractor) and negative for $\eta = 0,3$ (infrared repulsor). Here, we assume that λ is positive so that the Hamiltonian is bound from below.

Let us now consider the running of the coupling η in the vicinity of the fixed points. If $\eta \simeq 1$, the RG equation linearizes:

$$\Lambda\frac{d}{d\Lambda}\eta \simeq c(\eta-1), \qquad c = \frac{3\lambda}{8\pi^2} > 0. \tag{1.187}$$

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The solution of this differential equation takes the form

$$\eta(\Lambda) - 1 = (\eta(M) - 1)\left(\frac{\Lambda}{M}\right)^c , \tag{1.188}$$

where Λ and M are two different scales with $M \gg \Lambda$. Now, it is seen that starting from some high scale M and evolving down to the scale Λ, the coupling $\eta(\Lambda)$ tends to one irrespective of the value $\eta(M)$ at the high scale. In other words, the $O(2)$ global symmetry emerges at low energies, even if at high energies there was no such symmetry and the couplings λ and g were independent. Note also that with the fixed points $\eta = 0, 3$ the situation is the opposite: the sign of the pertinent c's is now negative, and the coupling $\eta(\Lambda)$ is repelled from the fixed points as $\Lambda \ll M$.

Next, we turn to the case of *local* symmetries. As is well known, theories with (massive) vector fields are, in general, non-renormalizable. Only when these fields describe gauge bosons of some local symmetry group (unbroken or broken) can the theory be made renormalizable. On the other hand, as we have learned from the previous discussions, the non-renormalizable interactions that are described by the operators with higher dimensions become irrelevant at low energies, that is, only renormalizable interactions survive. This raises the intriguing question of whether gauge symmetries are also emergent at low energies and need not be postulated from the beginning. In a loose language, one could even speak of "the emergence of light from chaos" (i.e., massless photons out of massive vector fields) at low energies.

The preceding question is difficult. In Refs. [20, 21], a one-loop perturbative study has been carried out in different models, including those with supersymmetry. The results are inconclusive. In some cases, we encounter infrared attractors, whereas in other cases we do not. On the other hand, in Ref. [22], very plausible *non-perturbative* arguments were given in favor of the statement that gauge theories should emerge in the infrared limit. In what follows, we shall illustrate these arguments in a simple example.

Let us start from a theory that contains a vector field $G_\mu = -iG^a_\mu T^a$ that belongs to the adjoint representation of some *compact* group $\mathcal{G}$. The quantities T^a denote the generators of the group $\mathcal{G}$, normalized as $\mathrm{tr}\,(T^a T^b) = \delta^{ab}/2$. Except G_μ, there can be other fields (matter fields) in the theory, which transform as basis vectors of some irreducible representations of $\mathcal{G}$. In the following, we do not need to specify these fields explicitly. Note that the symmetry considered here is *global.*

The most prominent signature of an (unbroken) gauge symmetry is the emergence of a vector particle with zero mass. This manifests itself as the pole in the two-point function of two vector fields, located at $p^2 = 0$. It is, however, more convenient to consider the two-point function of currents $J_\mu = -iJ^a_\mu T^a$, which couple to the field. For example, if $\mathcal{G}$ is the group $SU(N)$, and the vector field G_μ interacts with the fermions in the fundamental representation, ψ, then a convenient choice for J^a_μ by analogy with QED is

$$J^a_\mu(x) = \bar{\psi}(x)\gamma_\mu T^a \psi(x)\,. \tag{1.189}$$

The reason for considering currents instead of fields will become clear in what follows.

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Next, using the invariance under global transformations, it can be seen immediately that the two-point function

$$\Delta_{\mu\nu}(x-y) = \int dG_\mu \cdots \mathrm{tr}\big(J_\mu(x)J_\nu(y)\big) \exp\left(i \int d^4x \mathcal{L}\big[G_\mu, \cdots\big]\right) \tag{1.190}$$

develops a pole at the physical mass of the vector particle. Here, the ellipses stand for the matter fields, which are not shown explicitly.[16]

Up to now, the theory did not possess gauge invariance at all, as $\mathcal{G}$ is a global group. In Ref. [22], Förster, Nielsen and Ninomiya have proposed an elegant trick to formally elevate the global symmetry to a local one. To this end, one *defines* the gauge transformation of G_μ in a standard manner:

$$G_\mu(x) \mapsto G^\Lambda_\mu(x) = \Lambda(x) G_\mu(x) \Lambda(x)^{-1} - \partial_\mu \Lambda(x) \Lambda(x)^{-1}\,, \tag{1.191}$$

where $\Lambda(x)$ is an element of a *local* group $\mathcal{G}$. The matter fields transform under the gauge group in the standard manner as well. Note also that the current, which was introduced in Eq. (1.189), under the gauge transformation transforms as

$$J_\mu(x) \mapsto \Lambda(x) J_\mu(x) \Lambda(x)^{-1}\,, \tag{1.192}$$

so that the trace in Eq. (1.190) is invariant under the transformations. This is the rationale for considering the two-point function of currents instead of the fields G_μ. Further, the path integral measure is invariant under local transformations. Performing the transformations on the integration variables, one may write

$$\begin{aligned}\Delta_{\mu\nu}(x-y) &= \int d\Lambda \int dG_\mu \cdots \mathrm{tr}\big(J_\mu(x)J_\nu(y)\big) \exp\left(i \int d^4x \mathcal{L}\big[G_\mu, \cdots\big]\right) \\ &= \int d\Lambda \int dG_\mu \cdots \mathrm{tr}\big(J_\mu(x)J_\nu(y)\big) \exp\left(i \int d^4x \mathcal{L}\big[G^\Lambda_\mu, \cdots\big]\right).\end{aligned} \tag{1.193}$$

The action functional in Eq. (1.193) consists of two parts: the one that preserves gauge invariance, and the one that does not; that is,

$$S\big[G^\Lambda_\mu, \cdots\big] = \int d^4x \mathcal{L}\big[G^\Lambda_\mu, \cdots\big] = S_0\big[G_\mu, \cdots\big] + S_1\big[G^\Lambda_\mu, \cdots\big]\,. \tag{1.194}$$

Defining next

$$\exp\big(i\delta S\big[G_\mu, \cdots\big]\big) = \int d\Lambda \exp\big(iS_1\big[G^\Lambda_\mu, \cdots\big]\big)\,, \tag{1.195}$$

we finally get

$$\Delta_{\mu\nu}(x-y) = \int dG_\mu \cdots \mathrm{tr}\big(J_\mu(x)J_\nu(y)\big) \exp\big(iS_0\big[G_\mu, \cdots\big] + i\delta S\big[G_\mu, \cdots\big]\big)\,. \tag{1.196}$$

[16] Two disclaimers should be immediately made here. First, we implicitly assume that the theory is regularized on a finite space-time lattice (see the following), and hence there is no need for adding a gauge-fixing term. Second, for simplicity, we assume that there is no confinement of the vector field; otherwise, the physical mass cannot be defined.

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At first glance, a paradoxical result emerges. Starting from a gauge non-invariant expression, we obtain an expression where everything is written in terms of gauge-invariant quantities. Performing the gauge transformation $G_\mu \to G_\mu^\Lambda, \Lambda \in \mathcal{G}$ in Eq. (1.195), it can be easily seen that the quantity $\delta S[G_\mu, \cdots]$ is explicitly gauge-invariant. However, the gauge invariance of the latter does not yet guarantee that it is given in form of the integral over a local Lagrangian, and thus one gets a gauge-invariant effective theory at the end. In fact, the variables Λ in Eq. (1.195) can be considered as *dynamical fields,* living on a compact group manifold and interacting with *external* fields $G_\mu, \ldots$. An analog is a system of spins that are localized on the sites of a regular lattice in space and that interact with an external magnetic field. It is known that, for varying parameters of the model, two different phases can be realized:

- High-temperature phase, no spontaneous magnetization:

  In this case, the variables $\Lambda(x)$ are not correlated at distances larger than the inverse UV cutoff. This corresponds to the situation where S_1 is "small" and the path integral can be calculated by expanding the exponent $\exp(iS_1) = 1 + iS_1 + \ldots$. In this case, δS can be written as

  $$\delta S\big[G_\mu, \cdots\big] = \int d^4x \sum_i C_i \mathcal{O}_i(x)\,, \tag{1.197}$$

  where the $\mathcal{O}_i(x)$ are *local, gauge-invariant operators* and the C_i are couplings. At low energies, only relevant and marginal operators survive and we end up with a gauge theory with massless gauge bosons.[17] So, light from chaos indeed emerges.[18]
- Low-temperature phase, spontaneous magnetization:

  In this case, the correlation length of variables Λ is large and a local, gauge-invariant effective Lagrangian cannot be derived. The field G_μ is in general massive.

A nontrivial part of the statement is that there exists a region in the parameter space where the transition to the disordered phase takes place and, hence, an effective low-energy gauge-invariant theory emerges from the theory, which was not gauge-invariant originally. The statement has been backed, for example, by the numerical results of Monte-Carlo simulations in the two-dimensional XY model on the lattice [23]. Even being tempting, we do not claim here that the fundamental gauge interactions in Nature necessarily emerge from a theory that does not exhibit gauge symmetry at high energies. Although it is an interesting option to entertain, a further discussion goes beyond the scope of this book.

[17] These can still acquire mass through the Higgs mechanism in the interactions with matter fields.

[18] A gauge theory must be compact in order to be dynamically stable. A simple counter-example is provided by a single free massive vector field, described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}\,(\partial_\mu G_\nu - \partial_\nu G_\mu)^2 + \frac{m^2}{2}\,G_\mu G^\mu\,. \tag{1.198}$$

This is a free theory, and the vector meson there has a nonzero mass [22].

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1.12 Triviality of the ϕ^4-Theory

Let us consider a theory that is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4, \tag{1.199}$$

where ϕ denotes a one-component real scalar field. It is well known that this theory is perturbatively renormalizable in a strict sense. To all orders in perturbation theory, the divergences arising in the Green's functions can be removed by the renormalization of the mass m, the coupling constant λ and the multiplicative renormalization of the field $\phi \to Z^{1/2}\phi$, where Z is the wave function renormalization constant. For a detailed discussion, see, for example, the textbook [24].

In this section, we use cutoff regularization and denote the cutoff scale by Λ. Although dimensional regularization is more simple and elegant, it lacks physical transparency. In the following, m and λ are the bare parameters. The renormalized parameters are denoted by m_R and λ_R, respectively.

Suppose that one calculates the Green's function of the fields ϕ with n external legs, the n-point function. The unrenormalized Green's function is given by

$$\begin{aligned}(2\pi)^4\delta^{(4)}\left(\sum_{i=1}^{n} p_i\right) G(p_1,\cdots,p_n;m,\lambda,\Lambda)\\ = \int \prod_{i=1}^{n}\left(dx_i\, e^{ip_i x_i}\right)\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle\,.\end{aligned} \tag{1.200}$$

The renormalized Green's function follows as

$$G_R(p_1,\cdots,p_n;m_R,\lambda_R,\Lambda) = Z^{-n/2}(m_R,\lambda_R,\Lambda)G(p_1,\cdots,p_n;m,\lambda,\Lambda)\,, \tag{1.201}$$

with m_R, λ_R functions of m, λ and Λ and vice versa. Renormalizability means that, for fixed values of m_R, λ_R, one may choose the dependence on Λ in the parameters m, λ and Z so that all Green's functions $G_R(p_1,\cdots,p_n;m_R,\lambda_R,\Lambda)$ stay finite in the limit $\Lambda \to \infty$. If the Green's functions are expanded up to k loops in perturbation theory, then

$$G_R(p_1,\cdots,p_n;m_R,\lambda_R,\Lambda) = G_R(p_1,\cdots,p_n;m_R,\lambda_R) + O(\Lambda^{-2}\ln^k\Lambda)\,, \tag{1.202}$$

where $G_R(p_1,\cdots,p_n;m_R,\lambda_R)$ is the renormalized Green's function in the limit $\Lambda \to \infty$, and the remainder describes the so-called *scaling violation.*

In order to set the framework unambiguously, one in addition should fix the renormalization prescription, that is, one should define how the quantities m_R, λ_R, Z are expressed through the pertinent bare quantities. In the following, we choose the renormalization at vanishing external momenta (a particular choice within the so-called MOM scheme). Consider first the two-point Green's function:

$$\begin{aligned}G_R(p,-p;m_R,\lambda_R,\Lambda) &= Z^{-1}(m_R,\lambda_R,\Lambda)G(p,-p;m,\lambda,\Lambda)\\ &\doteq -i\left[\Gamma_R^{(2)}(p^2;m_R,\lambda_R,\Lambda)\right]^{-1}.\end{aligned} \tag{1.203}$$

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In order to fix m_R and Z, we may require

$$\Gamma_R^{(2)}(0;m_R,\lambda_R,\Lambda) = m_R^2\,, \tag{1.204}$$

and

$$\frac{d}{dp^2}\Gamma_R^{(2)}(p^2;m_R,\lambda_R,\Lambda)\bigg|_{p^2=0} = -1\,. \tag{1.205}$$

Note that m_R is *not* equal to the physical mass denoted by m_P. Also, m_R is different from the renormalized mass m_r in the $\overline{\text{MS}}$ scheme, which was considered in the previous sections.

In order to fix the coupling constant λ_R, one considers the four-point function:

$$\begin{aligned} &T_R(p_1,p_2,p_3,p_4;m_R,\lambda_R,\Lambda) \\ &= -iZ^{-2}(m_R,\lambda_R,\Lambda)\prod_{i=1}^{4}(m_P^2-p_i^2)G(p_1,p_2,p_3,p_4;m,\lambda,\Lambda)\,. \end{aligned} \tag{1.206}$$

The renormalized coupling λ_R is defined as

$$T_R(0,0,0,0;m_R,\lambda_R,\Lambda) = -6\lambda_R\,. \tag{1.207}$$

The preceding three relations indeed determine the bare quantities m,λ,Z in terms of the renormalized ones, m_R,λ_R, and the cutoff Λ.

In order to see how these relations work in practice, let us perform calculations at one loop. The calculation of the two-point function at this order yields

$$\begin{aligned} \big[G(p,-p;m,\lambda,\Lambda)\big]^{-1} &= m^2 - p^2 + 3\lambda I(m^2,\Lambda)\,, \\ I(m^2,\Lambda) &= \int^{\Lambda}\frac{d^4k}{(2\pi)^4 i}\,\frac{1}{m^2-k^2}\,. \end{aligned} \tag{1.208}$$

Note that, at this order, $I(m^2,\Lambda)$ does not depend on p^2. Consequently, differentiating the relation

$$\Gamma_R^{(2)}(p^2;m_R,\lambda_R,\Lambda) = Z(m_R,\lambda_R,\Lambda)\big[m^2-p^2+3\lambda I(m^2,\Lambda)\big] \tag{1.209}$$

with respect to p^2 and using Eq. (1.205), we obtain

$$Z(m_R,\lambda_R,\Lambda) = 1\,. \tag{1.210}$$

Further, Eq. (1.204) at this order yields

$$m_R^2 = m^2 + 3\lambda I(m^2,\Lambda)\,, \qquad m^2 = m_R^2 - 3\lambda_R I(m_R^2,\Lambda)\,. \tag{1.211}$$

Here, we have used the fact that $m_R = m$ and $\lambda_R = \lambda$ at lowest order. Substituting the explicit expression for the self-energy integral

$$I(m^2,\Lambda) = \frac{1}{16\pi^2}\left[\Lambda^2 - m^2\ln\frac{\Lambda^2}{m^2} + O\left(\frac{1}{\Lambda^2}\right)\right], \tag{1.212}$$

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we get

$$m^2 = m_R^2 - \frac{3\lambda_R}{16\pi^2}\left[\Lambda^2 - m_R^2 \ln\frac{\Lambda^2}{m_R^2} + O\left(\frac{1}{\Lambda^2}\right)\right]. \tag{1.213}$$

Note that m_R is equal to the physical mass m_P at this order.

Next, calculating the four-point function at one loop, we get

$$\begin{aligned} T(p_1,p_2,p_3,p_4;m_R,\lambda_R,\Lambda) &= -6\lambda \\ &+(6\lambda)^2\big(J(s,m^2,\Lambda)+J(t,m^2,\Lambda)+J(u,m^2,\Lambda)\big), \end{aligned} \tag{1.214}$$

where $s=(p_1+p_2)^2$, $t=(p_1+p_3)^2$, $u=(p_1+p_4)^2$, and

$$\begin{aligned} J(s,m^2,\Lambda) &= \frac{1}{2}\int^{\Lambda}\frac{d^4k}{(2\pi)^4 i}\,\frac{1}{(m^2-k^2)(m^2-(k-p_1-p_2)^2)} \\ &= \frac{1}{32\pi^2}\int_0^1 dx\left[\ln\frac{\Lambda^2}{m^2}+1-\ln\frac{m^2-x(1-x)s}{m^2}+O\left(\frac{1}{\Lambda^2}\right)\right]. \end{aligned} \tag{1.215}$$

Using Eq. (1.207), we finally obtain

$$\lambda_R = \lambda - \frac{9\lambda^2}{16\pi^2}\left[\ln\frac{\Lambda^2}{m^2}+1\right]. \tag{1.216}$$

Fixing λ_R to a given value (independent of Λ) leads to

$$0 = \Lambda\frac{\partial}{\partial\Lambda}\lambda_R = \Lambda\frac{\partial}{\partial\Lambda}\lambda - \frac{9\lambda^2}{8\pi^2}. \tag{1.217}$$

This yields the RG equation that determines the dependence of the bare coupling λ on the cutoff Λ:

$$\Lambda\frac{\partial}{\partial\Lambda}\lambda = \beta(\lambda,m,\Lambda). \tag{1.218}$$

The arguments of the β-function on the right-hand side of this equation are the bare coupling λ, the bare mass m and the cutoff Λ. The quantities m and Λ can be further expressed through λ and the renormalized parameters m_R, λ_R. Since the latter are fixed, one could consider the explicit dependence on λ only, suppressing all other variables. Expanding now the β-function in powers of the coupling constant, we get

$$\beta(\lambda) = -\beta_0\lambda^2 - \beta_1\lambda^3 - \cdots, \tag{1.219}$$

where, in our case,

$$\beta_0 = -\frac{9}{8\pi^2} < 0. \tag{1.220}$$

Let us consider the solution of the RG equation to lowest order. Integrating the equation

$$\Lambda\frac{\partial}{\partial\Lambda}\lambda = -\beta_0\lambda^2 \tag{1.221}$$

gives

$$\frac{1}{\lambda(\Lambda)} - \frac{1}{\lambda_0} = \beta_0\ln\frac{\Lambda}{\Lambda_0}, \qquad \lambda_0 = \lambda(\Lambda)\big|_{\Lambda=\Lambda_0}. \tag{1.222}$$

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This equation can be rewritten as

$$\lambda(\Lambda) = \frac{\lambda_0}{1 + \beta_0 \lambda_0 \ln \frac{\Lambda}{\Lambda_0}} \,. \tag{1.223}$$

Recall that λ_0 is the bare coupling constant corresponding to the (fixed) renormalized constant at the value of the cutoff $\Lambda = \Lambda_0$. It has a certain numerical value that depends on λ_R, Λ_0 and m_R. Renormalizability of the theory is then equivalent to the statement that, when Λ is varied continuously from $\Lambda = \Lambda_0$ to infinity, we can always choose a value of the bare coupling $\lambda(\Lambda)$, which would correspond to the *same* values of the renormalized parameters. Let us explicitly check whether this statement is valid at one loop.

The answer to this question depends on the sign of the coefficient of β_0. As seen from Eq. (1.223), if β_0 is positive, $\lambda(\Lambda)$ decreases monotonically and approaches zero, as Λ increases from Λ_0 to infinity. (Recall that λ_0 has to be positive, otherwise the Hamiltonian of the model is not bound from below.) This means that the limit $\Lambda \to \infty$ exists and, moreover, the first-order perturbative result can be trusted at large values of Λ (asymptotic freedom). However, if β_0 is negative, as in our example (see Eq. (1.220)), the situation changes dramatically. The bare coupling constant becomes infinite at

$$\Lambda = \Lambda_0 \exp\left(-\frac{1}{\beta_0 \lambda_0}\right). \tag{1.224}$$

This phenomenon is called a *Landau pole* [25–27]. In this case, the limit $\Lambda \to \infty$ cannot be performed, unless $\lambda_R = 0$, that is, in a *trivial* theory. The latter statement, however, comes with a grain of salt. Namely, approaching the critical value of Λ, the coupling constant λ grows, and the applicability of perturbation theory is questionable. It is therefore necessary to find out whether higher-order corrections might invalidate the statement.

Below, we shall formulate the condition under which the limit $\Lambda \to \infty$ exists. To this end, let us integrate the first-order differential equation (1.218):

$$\ln \frac{\Lambda}{\Lambda_0} = \int_{\lambda_0}^{\lambda} \frac{d\lambda'}{\beta(\lambda')} \,. \tag{1.225}$$

If $\Lambda \to \infty$, the l.h.s. of this equation is positive and diverges. The integral on the r.h.s. of the equation should thus behave in the same way. There are several alternatives:

- The β-function has a zero,

  $$\beta(\lambda) = 0 \quad \text{at } \lambda = \lambda^* \,, \tag{1.226}$$

  where the quantity λ^* defines the location of a so-called *fixed point.* Let us assume, for illustrative reasons, that the zero in the β-function is of first order. (Other cases can be treated similarly.) In this case, we can perform a Taylor expansion in the vicinity of the fixed pole:

  $$\beta(\lambda) = C(\lambda - \lambda^*) + O((\lambda - \lambda^*)^2) \,, \tag{1.227}$$

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where C is the derivative of the β-function at the fixed point. We assume that $C < 0$ (ultraviolet fixed point). We also assume that $\lambda_0 > \lambda^*$ (the case $\lambda_0 < \lambda^*$ can be considered analogously). Suppose now that $\lambda(\Lambda)$ for some Λ is located in the vicinity of λ^*. Then, Eq. (1.225) can be rewritten as

$$\ln \frac{\Lambda}{\Lambda_0} = \int_{\lambda_0}^{\lambda^*+\varepsilon} \frac{d\lambda'}{\beta(\lambda')} + \int_{\lambda^*+\varepsilon}^{\lambda} \frac{d\lambda'}{C(\lambda' - \lambda^*)}. \tag{1.228}$$

Here, ε is a small quantity and, in the second integral, the function $\beta(\lambda)$ can be replaced by the first term in the Taylor expansion. Denoting the first integral by A (it does not depend on λ), we get

$$\ln \frac{\Lambda}{\Lambda_0} = A + \frac{1}{C} \ln \frac{\lambda - \lambda^*}{\varepsilon}, \tag{1.229}$$

so that

$$\lambda - \lambda^* = \varepsilon \left(\frac{\Lambda}{\Lambda_0 e^A} \right)^C. \tag{1.230}$$

If $C < 0$, $\lambda(\Lambda)$ converges toward λ^* as $\Lambda \to \infty$. For this reason, we speak of an *ultraviolet fixed point,* or an *attractor.* On the contrary, there is no attraction to this point in the ultraviolet if $C > 0$.

- The function $\beta(\lambda)$ has a definite sign. For $\Lambda \to \infty$, the l.h.s. of Eq. (1.225) diverges, and the r.h.s. should do so as well. This divergence can arise only at $\lambda \to \infty$, that is, the asymptotic behavior of $\beta(\lambda)$ is restricted by

$$\left| \frac{\beta(\lambda)}{\lambda} \right| \leq \text{constant}, \quad \text{as } \lambda \to \infty. \tag{1.231}$$

  In this case, the limit $\Lambda \to \infty$ can be also performed. There are no singularities at finite values of Λ.
- Finally, consider the situation when $\beta(\lambda)$ grows faster than $\lambda^{1+\delta}$, with $\delta > 0$, as $\lambda \to \infty$. In this case, the integral on the r.h.s. of Eq. (1.225) converges, whereas the logarithm on the l.h.s. diverges, as $\Lambda \to \infty$. Consequently, there exists a critical value of Λ, called Λ_{crit}, determined from the equation

$$\ln \frac{\Lambda_{\text{crit}}}{\Lambda_0} = \int_{\lambda_0}^{\infty} \frac{d\lambda'}{\beta(\lambda')}, \tag{1.232}$$

  where the coupling constant blows up, and thus performing the limit $\Lambda \to \infty$ is not possible. This situation resembles the previously discussed Landau pole in perturbation theory.

Thus, the question whether the limit $\Lambda \to \infty$ can be performed in the renormalized ϕ^4-theory boils down to the study of the asymptotic behavior of the β-function at large values of λ and its zeros. Clearly, non-perturbative methods should be used in order to solve this problem.

In the series of papers [28–30] Lüscher and Weisz have demonstrated that, indeed, the ϕ^4-theory is trivial, that is, the limit $\Lambda \to \infty$ *cannot* be performed at $\lambda_R \neq 0$. Schematically, the argument in these papers goes as follows. (For simplicity, we consider a

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case of a single scalar field in the phase with unbroken symmetry.) In the unrenormalized theory, one has two dimensionless quantities: $\bar{m} = m/\Lambda$ and λ. In the renormalized theory, these are $\bar{m}_R = m_R/\Lambda$ and λ_R. Further, one can express the renormalized parameters in terms of the bare ones in a wide range of bare parameters, including the physically interesting domain $m_R/\Lambda \to 0$. These relations take the form

$$\bar{m}_R = \bar{m}_R(\lambda, \bar{m}), \qquad \lambda_R = \lambda_R(\lambda, \bar{m}). \tag{1.233}$$

Now, it is possible to invert the second relation, expressing $\bar{m}$ through λ and λ_R. Substituting this into the first relation, one gets

$$\bar{m}_R = F(\lambda, \lambda_R). \tag{1.234}$$

In Refs. [28–30], the trajectories of $\bar{m}_R$ were studied for a fixed λ_R, using a combination of the high-temperature expansion of the Green's functions and a numerical solution of the RG equations. It was shown that the dependence on λ is monotonic, and the minimum is achieved when $\lambda \to \infty$. This result means that the value of $\bar{m}_R^{-1}$ is bounded *from above* at every fixed λ_R, and one may write

$$\ln \bar{m}_R^{-1} = \ln \frac{\Lambda}{m_R} \leq f(\lambda_R), \tag{1.235}$$

where $f(\lambda_R)$ can be evaluated numerically, using the methods just mentioned. In other words, for each given $\lambda_R \neq 0$, there exists an upper bound on Λ, that is, the theory is trivial. Note further, as shown in Ref. [28], the function $f(\lambda_R)$ can be well approximated by an expression of the form $f(\lambda_R) = A/\lambda_R + B \ln \lambda_R + O(1)$. If λ_R is not very large, the second-order perturbation expression works very well, yielding

$$\ln \frac{\Lambda}{m_R} \leq -\frac{1}{\beta_0 \lambda_R} - \frac{\beta_1}{\beta_0^2} \ln(-\beta_0 \lambda_R) + C(\lambda_R), \tag{1.236}$$

where $-1.7 \leq C(\lambda_R) \leq 1.3$ for $\lambda_R \leq 1.5$.

At this point, one has to discuss what the obtained result means in practice. The fact that the ultraviolet cutoff cannot be moved to infinity does not a priori invalidate the results that can be obtained from this theory at low momenta, that is, momenta much smaller than the cutoff Λ. In other words, the ϕ^4-theory perfectly makes sense as an *effective theory* if one does not insist that it should be valid at all energies. The contributions coming from high momenta, which can be characterized by the higher-dimensional operators in the Lagrangian, are small at the momenta much less than the cutoff.

This statement has important implications, if one insists that, for consistency, the mass of a scalar particle must be smaller than the ultraviolet cutoff. This results in the existence of an upper bound on the mass of the scalar particle(s), the so-called *triviality bound*, in theories, where the interactions in the scalar sector are described by the ϕ^4 Lagrangian. A prominent example is the triviality bound set on the mass of the Higgs particle in the Standard Model before the actual discovery of the Higgs. Below we shall give a simple, intuitive derivation of the bound from the one-loop running of the coupling constant.

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Consider the Lagrangian of the Higgs sector of the Standard Model that contains one complex doublet field Φ:

$$\mathcal{L}_H = \frac{1}{2}\,\partial_\mu \Phi^\dagger \partial^\mu \Phi - \frac{m^2}{2}\,\Phi^\dagger \Phi - \frac{\lambda}{4}\,(\Phi^\dagger \Phi)^2 \,. \tag{1.237}$$

For a rough estimate one can use the lowest-order perturbative results for the running of the renormalized coupling constant. To this end, by analogy with Eq. (1.216), one can define the scale-dependent renormalized constant $\lambda_R(\mu)$:

$$\lambda = \lambda_R(\mu) - \frac{1}{2}\,\beta_0 \lambda_R(\mu)^2 \left[\ln \frac{\Lambda^2}{\mu^2} + 1\right], \tag{1.238}$$

where β_0 in the theory with two complex scalar fields, described by the Lagrangian (1.237), is given by

$$\beta_0 = -\frac{3}{2\pi^2}\,, \tag{1.239}$$

and the quantity λ_R, which was defined previously, corresponds to the choice of the scale $\mu = m_R$; see Eq. (1.216). The RG running of the renormalized quartic coupling is given by

$$\mu \frac{\partial}{\partial \mu}\,\lambda_R(\mu) = -\beta_0 \lambda_R^2(\mu)\,. \tag{1.240}$$

Integrating this differential equation leads to

$$\lambda_R(\mu_0) = \frac{\lambda_R(\mu)}{1 + \frac{3}{2\pi^2}\,\lambda_R(\mu) \ln \frac{\mu}{\mu_0}}\,, \qquad \mu \geq \mu_0\,. \tag{1.241}$$

Taking into account the fact that $\lambda_R(\mu)$ is positive, the following upper bound on $\lambda_R(\mu_0)$ emerges:

$$\bar{\lambda}_R \doteq \lambda_R(\mu_0) \leq \frac{2\pi^2}{3 \ln \frac{\mu}{\mu_0}}\,. \tag{1.242}$$

Further, at the order of perturbation theory we are working, we can use tree-level results for the masses from the Standard Model:

$$M_H^2 = 2\bar{\lambda}_R v^2\,, \qquad M_W^2 = \frac{1}{4}\,g^2 v^2\,, \qquad g = \frac{e}{\sin\theta_W}\,. \tag{1.243}$$

Here, M_H and M_W are the masses of the Higgs and W-bosons, respectively, $v \simeq 246$ GeV in the vacuum expectation value of the Higgs field, e and g are the electromagnetic and the $SU(2)$ gauge couplings, respectively, and $\sin^2\theta_W \simeq 0.23$, where θ_W denotes the weak mixing (Weinberg) angle. Using now Eq. (1.243), we obtain

$$\left(\frac{M_H}{M_W}\right)^2 = \frac{8\bar{\lambda}_R}{g^2}\,. \tag{1.244}$$

Finally, substituting the value of the coupling g, the following rough estimate can be obtained:

$$\frac{M_H}{M_W} \leq \frac{4\pi}{g\sqrt{3}}\,\frac{1}{(\ln(\mu/\mu_0))^{1/2}} \simeq \frac{900\ \text{GeV}}{M_W}\,\frac{1}{(\ln(\mu/\mu_0))^{1/2}}\,. \tag{1.245}$$

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In the spontaneously broken phase, μ_0 should be chosen on the order of the Higgs mass. Further, the quantity μ defines the scale up to which the theory is consistent. (It does not make sense to move the scale μ beyond the cutoff Λ.) If μ is as small as $2\mu_0$ (the smaller values are barely consistent with the requirement $m_H \ll \Lambda$), the preceding equation yields $M_H \leq 1070$ GeV. If μ is taken up to the Planck mass $m_{\mathrm{Pl}} = 10^{19}$ GeV, the upper limit on the Higgs mass goes down to $\simeq 140$ GeV. Combining the results of lattice calculations with higher-order calculations in perturbation theory, it is possible to arrive at more refined constraints on the Higgs mass.

Very interesting further questions emerge in connection with the problem considered in the present section. How is the bound affected in the Standard Model when the interactions with the gauge bosons and fermions are taken into account? How would the results change, if the Higgs particle is composite? The answer to these intriguing questions is, however, beyond the scope of the present book. Further information on this subject can be found, for example, in Refs. [31–35].

1.13 Relevant Degrees of Freedom at Low Momenta

In this chapter we have considered theories, where both the heavy and light degrees of freedom corresponded to the fields represented in the Lagrangian, and perturbation theory was assumed to work at both high and low energies. At low energies, the heavy degrees of freedom could be neatly integrated out, and a perturbative matching could be performed. Conceptually, this is the most clean and transparent case; but in Nature the separation of the low- and high-energy modes can proceed along many different patterns. In what follows, we list a few of them.

The construction of the low-energy effective theory of QCD is perhaps the most important example, which will be considered in much detail here. The degrees of freedom of the underlying theory are quarks and gluons, none of them surviving in the low-energy limit. Formally, the (light) quarks and gluons have very small (even zero) masses. However, the strong non-perturbative interactions between them lead to confinement (no quarks and/or gluons in the asymptotic states) and to the creation of a mass gap of order of 1 GeV determined by the mass of the proton, which is the lightest *stable* particle in QCD, whose mass does not vanish in the chiral limit. It turns out that at low energies only the Goldstone bosons, which emerge as a result of the spontaneous chiral symmetry breaking in QCD, have masses lighter than the heavy scale of QCD on the order of 1 GeV, and thus represent the only relevant degrees of freedom at these energies. Perturbative matching is, of course, not possible.

Another interesting possibility emerges, for example, in nonrelativistic effective theories, or in heavy quark (heavy baryon) effective field theories, which will also be considered here in detail. In this case, the role of the heavy fields is played by the heavy components of the *same* field. These heavy components correspond to the antiparticles, whose contribution is now relegated to the effective couplings. The heavy scale of

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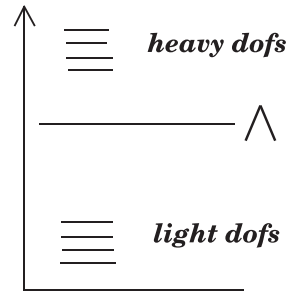

Figure 1.18 Degrees of freedom (dofs) in a typical EFT, which operates at small momenta corresponding to the light particle masses. Beyond some separation scale Λ, heavy particles appear. Their effect on the low-energy EFT is only indirect by providing the strengths of multiparticle operators in the light particle sector.

the theory is determined by the mass gap separating the particles and the antiparticles, that is, by the mass of the particle itself.

Finally, note that, in the context of the many-body problems, effective fields often describe collective excitations, a prominent example being the Landau–Ginsburg theory. The energies of the excitations should be much smaller than the natural hard momentum scale in the problem, given by the inverse of the lattice spacing in the crystal.

To summarize, there exist different scenarios in nature for how the low-energy degrees of freedom emerge from the dynamics at short distances. A universal rule to find these degrees of freedom is to examine the low-energy spectrum of the system, that is, to investigate the low-energy singularity structure of the S-matrix. This allows us to choose the appropriate variables for describing the system at low energies without any reference to the short-distance dynamics.

1.14 Construction Principles of an EFT

Here, we briefly summarize what we have learned, or, stated differently, what the principles are behind the construction of an effective field theory. These are:

- *Scale separation:* This is arguably the most basic concept underlying any EFT. The EFT is operative at small momenta/energies (at large distances), and the physics in this regime is insensitive to what happens at large momenta (small distances) as depicted in Fig. 1.18. The heavy particles can have only an indirect effect at low energies by providing the strengths of certain multiparticle operators in the light particle sector. However, the EFT can be formulated without any knowledge of these heavy degrees of freedom, fitting the appearing LECs to some data at low energies. This scale separation further implies that renormalizability in the strict sense is not applicable. In fact, it should be obvious that any quantum field theory is indeed an effective field theory.

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- *Relevant degrees of freedom:* Related to the scale separation are the active particles in the low-energy EFT, which either decouple from the heavy degrees of freedom or are generated through some symmetry breaking, as discussed in detail in Section 1.13. These relevant degrees of freedom are intimately connected to the low-energy singularity structure of the pertinent S-matrix.
- *Symmetries:* Symmetries play an important role in the construction of any EFT. First, discrete and continuous symmetries constrain the possible interactions of the particles. This is equivalent to the construction of any QFT. Second, the realization of symmetries plays an important role, as spontaneous breaking can generate the pertinent low-energy degrees of freedom. As will be discussed in detail in later chapters, this exactly happens for QCD in the confinement regime (at small momenta).
- *Power counting:* With the relevant particles and the constraints from symmetries, one can write down an infinite tower of allowed interactions in the EFT. Power counting is the tool to order all these terms according to their relevance. Symbolically, any matrix element can be perturbatively expanded in powers of energies/momenta Q over the breakdown scale Λ as

$$\mathcal{M} = \sum_{\nu} \left(\frac{Q}{\Lambda} \right)^{\nu} f_\nu(Q/\mu, g_i) \,, \tag{1.246}$$

  where μ is a renormalization scale related to the required renormalization of loop diagrams and the g_i are coupling constants, often called low-energy constants (LECs). In case of an EFT, the index ν is bounded from below, which allows for a systematic and controlled expansion. Furthermore, f_ν is a function of $O(1)$, and this property is called "naturalness." Note also that describing bound states requires some type of non-perturbative resummation.
- *Matching:* Matching is not universal to EFTs but appears very often. One example we already encountered was the matching of the EFT two-body scattering phase to the effective range expansion, which allows us to express the LECs in terms of physical parameters. Another example that will prominently appear in the next chapter is the matching of a nonrelativistic EFT to its relativistic counterpart. Yet another example of matching is the reduction of a theory at high energies down to the low-energy EFT, which might proceed in steps, and matching is performed at the boundaries between the different EFTs to guarantee a smooth transition when lowering the resolution scale. A classical example is the weak $\Delta S = 1$ Hamiltonian, where the W-bosons and the t, b and c quarks are integrated out successively; see, for example, Ref. [36].

1.15 Literature Guide

Most of the material presented in this chapter is standard and overlaps with many textbooks and lecture notes. In this context we mention here the recent books on

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effective theories by Petrov and Blechman [37] and Burgess [38], as well as a selection of lecture notes and reviews [12, 19, 39–43].

As already mentioned, the use of the effective theory approach in the problems of nonrelativistic quantum mechanics was discussed in the extremely instructive lectures by Lepage [1]. The two-body scattering with pointlike potentials and the issue of the equivalence of different regularizations and renormalization schemes as well as the relation to the Wigner bound have been extensively discussed in the literature. Apart from the papers, which were already cited [2, 3, 8], important aspects of the same problem have been considered in Refs. [7, 44].

The decoupling theorem was first proved by Appelquist and Carazzone [14]. For related work, see, for example, Refs. [45–47]. A very detailed introduction to the issue is contained in a book by Collins [6], to which the reader is referred for further references. The decoupling in the case of two scalar fields has been considered in detail in Refs. [48, 49].

The Wilson renormalization group approach was first described in Ref. [16]; see also Ref. [50]. It is discussed, in particular, in the textbook by Peskin and Schroeder [24]. The dependence of the effective couplings on the floating cutoff was first considered in Polchinski's paper [17]; see also the discussion in Weinberg's textbook [18].

The Landau pole was introduced in the seminal papers [25–27]. A thorough discussion of this issue from different points of view is given in the review article [51]. In a series of papers [28–30], Lüscher and Weisz give an extremely clear and concise discussion of triviality in the ϕ^4 theory. The triviality bound on the Higgs mass is considered in Refs. [52–56].

Various aspects of the emerging symmetries at low energy have been considered in Refs. [20–23]; see also Ref. [57].

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