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ONE-RELATOR GROUPS THAT ARE RESIDUALLY OF PRIME POWER ORDER

D. GILDENHUYS

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1. Introduction

If $\mathscr{C}$ is a class of groups, we denote by $R\mathscr{C}$ the class of groups which are residually in $\mathscr{C}$; i.e. $G \in R\mathscr{C}$ if and only if $1 \neq g \in G$ implies that there exists a normal subgroup N of G such that $g \notin N$ and $G/N \in \mathscr{C}$. A group G is residually a finite p-group if it belongs to $R\mathscr{F}_p$, where $\mathscr{F}_p$ denotes the class of finite p-groups. One also says that the groups in $R\mathscr{F}_p$ are *residually of order equal to a power of the prime p*. Given a group G with one defining relator r, one might ask for conditions on the "form" of the relator that would guarantee that G have certain residual properties. In this context, Baumslag (1971) has proved that if all the exponents of the generators appearing in r are positive, then G is residually solvable. In the same paper he also concerned himself with the residual nilpotence of one-relator groups, and found that the situation there was much more complicated. If one goes one step further and asks for conditions that will ensure that for a given prime p the one-relator group be residually a finite p-group, then very little seems to be known. Of course, if one takes r to be one of the generators:

$$G = (a, b, \cdots; a)$$

then G is freely generated by the remaining generators, and hence is in $R\mathscr{F}_p$ for all primes p (Mal'tec (1949), Lazard (1965), 3.1.4). Our main purpose in this paper is to develop methods of generating examples of one-relator groups that are residually of order equal to a given prime p.

To every group G one can in a canonical way associate a pro-p-group $\hat{G} = \varprojlim G/N$, called its pro-$p$-completion, and the canonical map $G \to \hat{G}$ is injective iff G is in $R\mathscr{F}_p$. (Here N runs through the normal subgroups in G of index a power of p.) If G has the presentation

$$G = (x_1, x_2, \cdots, x_n; r)$$

then $\hat{G}$ has the "same" presentation as a pro-p-group; i.e.

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$$\hat{G} = \hat{F}(x_1, x_2, \cdots, x_n)/(r),$$

where $\hat{F}(x_1, x_2, \cdots, x_n)$ is the free pro-p-group on the symbols $x_1, x_2, \cdots, x_n$ and (r) denotes the closed normal subgroup of $\hat{F}(x_1, x_2, \cdots, x_n)$ generated by r (Recall that $\hat{F}(x_1, x_2, \cdots, x_n)$ contains the free group $F(x_1, x_2, \cdots, x_n)$ on $x_1, \cdots, x_n$ as a dense subgroup, and is its pro-p-completion (Lazard (1965), 3.1.4)). The pro-p-completion of a one-relator group is therefore a one-relator pro-p-group, and such groups have been studied in Labute (1967),(1967a), Gildenhuys-Lim (1972), Gildenhuys-Ribes (1974), and Gildenhuys (1968), (to appear). Labute's theorem 4′ (1967) enables one to describe a large class of one relator groups B with the property that the completed group algebra $\mathbf{Z}_p[[\hat{G}]]$ is a valued ring, and hence has no zero divisors. If, in addition, G is in $R\mathscr{F}_p$, then its integral group ring $\mathbf{Z}[G]$ is embedded in $\mathbf{Z}_p[[\hat{G}]]$ and therefore has no zero divisors. We give here some examples (Proposition 2.2 and Theorem 5.1) based upon this observation, and in support of the well-known conjecture that the integral group ring of a torsion-free one-relator group is without zero divisors. The examples consist of the (discrete) one-relator groups defined by Demuškin relators and groups defined by certain types of commutators. The (t, p)-filtrations of Lazard (1965) constitute the main tool of our investigations. These filtrations share many of the properties of the derived series and the central descending series of a group, and many of Baumslag's results in (1972) carry over when the derived series is replaced by a (t, p)-filtration. Given a finitely generated group G and fixed prime p, the (t, p)-filtrations all define the same topological group structure on G, and the topology is seperated iff G is in $R\mathscr{F}_p$ (see Proposition 1.1 below).

The author thanks the referee for pointing out to him that his Theorem 1.4 may have been derived by the methods contained in Baumslag's paper: '*On the residual finiteness of generalized free products of nilpotent groups*, Trans. AMS 106 pp 193–209 (1963). Also, the absence of zero-divisors in certain group rings might have been deduced from the groups being locally indicable.

To describe the nature of our remaining results, we recall the basic breakdown of one-relator groups (Magnus, karass and Solitar (1966), section 4.4). Suppose $G = (a, b, \cdots, c; r)$, r is cyclically reduced and a occurs in r with exponent sum zero. Putting

$$b_i = a^{-i}ba^i, \cdots, c_j = a^{-j}ca^j \ (i, \cdots, j = 0, \pm 1, \cdots), \text{ we can rewrite}$$

r as a shorter word r_0 in the letters $b_{m(b)}, \cdots, b_{M(b)}, \cdots, c_{m(c)}, \cdots, c_{M(c)}$. The subgroup N_0 of G generated by these letters has the presentation

$$N_0 = (b_{m(b)}, \cdots, b_{M(b)}, \cdots, c_{m(c)}, \cdots, c_{M(c)}; r_0)$$

and we call this simpler one-relator group the *reduced one-relator group of G*. Given that N_0 is in $R\mathscr{F}_p$, what conditions do we have to impose upon the form of r to ensure that G is also in $R\mathscr{F}_p$? Theorem 4.1 gives a partial answer to this problem. Let N be the normal subgroup of G generated by $b, \cdots, c$, and let M be the closed normal subgroup of $\hat{G}$ generated by the images of these letters. One has a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \to & N & \to & G & \to & F(a) & \to & 1 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ 1 & \to & M & \to & \hat{G} & \to & \hat{F}(a) & \to & 1 \end{array}$$

The structure of the pro-p-group M can be described as an inverse limit of push-outs (colimits) of circular diagrams in the category of pro-p-groups (Gildenhuys (to appear)). The group N is a direct limit of generalized free products of isomorphic copies of N_0. The map α, and hence β, will be injective if a certain type of embedding problem for circular amajgams can be solved (Lemma 3.2). This is the basic idea behind the proof of Theorem 3.1.

2. (t, p)-Filtrations and Amalgamations of Groups

We first introduce some terminology. Let H be a subgroup of a finitely generated group G and let $\omega : G \to \boldsymbol{R} \cup \{\infty\}$ be the (t, p)-filtration on G, where t is some positive real number (Lazard (1965), 3.2.1). If the restriction $\omega | H : H \to \boldsymbol{R} \cup \{\infty\}$ of ω to H is the (t, p)-filtration on H, we will call H a *(t, p)-isometric subgroup of G*. We write

$$G_v = \{g \in G : \omega(g) \geqq v\},$$

and we say that a subgroup H of G is *(t, p)-separable in G* if $\cap_{v>0} H \cdot G_v = H$.

PROPOSITION 1.1. (a) *A finitely generated group G is in $R\mathscr{F}_p$ if and only if the identity subgroup of G is (t, p)-separable in G for some $t > 0$, if and only if the identity subgroup of G is (t, p)-separable in G for all $t > 0$.*

(b) *If G is in $R\mathscr{F}_p$ and H is a subgroup of G, then H is (t, p)-separable in G for all $t > 0$ if and only if H is (t, p)-separable in G for some $t > 0$, if and only if $G \cap \bar{H} = H$, where $\bar{H}$ denotes the closure of H in $\hat{G}$.*

PROOF. (a) Suppose $G \in R\mathscr{F}_p$ and $t > 0$. Let ω (respectively $\hat{\omega}$) denote the (t, p)-filtration on G (respectively $\hat{G}$) (Lazard (1965), 3.2.1, 3.2.8.1). The identity subgroup is (t, p)-separable in G if and only if $\omega(g) = \infty \Rightarrow g = 1$. The restriction $\hat{\omega} | G$ of $\hat{\omega}$ to G is a p-filtration with the property that $(\hat{\omega} | G)(g) \geqq t$ for all $g \in G$. Hence $\hat{\omega}(g) \geqq \omega(g)$ for all $g \in G$. Since $\hat{\omega}$ defines the topology of the separated. topological group $\hat{G}$, it follows that for all $g \in G$,

$$\omega(g) = \infty \Rightarrow \hat{\omega}(g) = \infty \Rightarrow g = 1$$

and the identity subgroup is (t, p)-separable in G. Conversely, if $\cap_{v>0} G_v = (1)$ then the canonical map

$$G \to \varprojlim G/G_\nu$$

is injective, and it only remains to verify that each quotient G/G_ν is a finite p-group Referring to Lazard (1965), 3.3.2, we let

$$0 < \lambda_1 < \lambda_2 < \cdots < \infty$$

be the set of values of the (t, p)-filtration, and we recall that the mixed Lie algebra

$$\operatorname{gr} G = \sum_{i=1}^{\infty} G_{\lambda_i}/G_{\lambda_{i+1}}$$

is generated by the images of a finite set of generators for G. It follows that each homogeneous component $G_{\lambda_i}/G_{\lambda_{i+1}}$ is finite. The first homogeneous component is G/G_{λ_2}, and if we make the induction assumption that G/G_{λ_i} is finite, then the exact sequence

$$1 \to G_{\lambda_i}/G_{\lambda_{i+1}} \to G/G_{\lambda_{i+1}} \to G/G_{\lambda_i} \to 1$$

shows that $G/G_{\lambda_{1+i}}$ is finite. This proves (a) of the Proposition.

By Lazard (1965), 3.2.8.2 the mixed Lie algebra $\operatorname{gr}\hat{G}$ of $\hat{G}$ is generated by the images of a (finite) set of (topological) generators of $\hat{G}$, so that, by the same argument as before, the closed subgroups

$$\hat{G}_\nu = \{y \in \hat{G} : \hat{\omega}(y) \geqq \nu\}$$

are of finite index in $\hat{G}$, and hence constitute a fundamental system of open neighborhoods of the identity. Since a finite set of generators for G is a set of (topological) generators for $\hat{G}$, the mixed Lie algebras $\operatorname{gr} G$ and $\operatorname{gr}\hat{G}$ are isomorphic and $G_\nu = \hat{G}_\nu \cap G$. (Lazard (1965), 1.1.8, 3.2.8). The equalities

$$\bigcap_{\nu>0} H \cdot G_\nu = \bigcap_{\nu>0} H \cdot (\hat{G}_\nu \cap G) = (\bigcap_{\nu>0} H \cdot \hat{G}_\nu) \cap G = \bar{H} \cap G$$

now show that H is (t, p)-separable if and only if $H = \bar{H} \cap G$.

We will always use the term *p-filtration* in the sense of Lazard [14]. This term should not be confused with the term *$\mathscr{F}_p$-filter*, which we will use in the sense of Gruenberg (1957), section 1; i.e. we will say that $\{A_\lambda\}_{\lambda \in \Lambda}$ is an *$\mathscr{F}_p$-filter* of a group A if:

(1) each A_λ is a normal subgroup of A;
(2) each quotient A/A_λ is in $\mathscr{F}_p$;
(3) each intersection $A_\lambda \cap A_\mu$ contains a member of the $\mathscr{F}_p$-filter.

PROPOSITION 1.2. *Let H (respectively K) be a subgroup of a group A (respectively B) and let φ: $H \to K$ be an isomorphism. Suppose that $\{A_\lambda\}_{\lambda \in \Lambda}$ and $\{B_\lambda\}_{\lambda \in \Lambda}$ are equally indexed $\mathscr{F}_p$-filters of A and B respectively such that*

(i) $H = \cap_{\lambda \in \Lambda} H \cdot A_\lambda$, $K = \cap_{\lambda \in \Lambda} K \cdot B_\lambda$;

(ii) $\varphi(H \cap A_\lambda) = K \cap B_\lambda$, *so that* φ *induces isomorphisms*;

$$\varphi_\lambda: HA_\lambda/A_\lambda \to KB_\lambda/B_\lambda, \qquad (\lambda \in \Lambda);$$

(iii) *The generalized free product*

$$P_\lambda = \{(A/A_\lambda) * (B/B_\lambda): H \circ A_\lambda/A_\lambda \overset{\varphi_\lambda}{=} K \cdot B_\lambda/B_\lambda\}$$

is residually a finite p-group, for each $\lambda \in \Lambda$.

Then

$$P = \{A * B: H \overset{\varphi}{=} K\}$$

is residually a finite p-group.

Proof. Let $\theta_\lambda: P \to P_\lambda$ be the map induced by the projections: $A \to A/A_\lambda$, $B \to B/B_\lambda$, $(\lambda \in \Lambda)$. Let S (respectively T) be a set of right coset representatives for $A \bmod H$ (respectively $B \bmod K$) containing the identity. Suppose that $1 \neq y \in P$. If $y \in H$, then y has a non-trivial image in some A/A_λ, and $\theta_\lambda(y) \neq 1$. Since P_λ is in $R\mathscr{F}_p$, there then exists a homomorphism from P into some finite p-group such that the image of y is not the identity. So we may suppose that $y \notin H$. Then y can be uniquely presented in the canonical form

$$y = hc_1c_2\cdots c_r \quad (h \in H,\ 1 \neq c_i \in S \cup T,\ r \geqq 1)$$

where c_i and c_{i+1} are not both in A and not both in B. (See Magnus, Karass and Solitar (1966), Theorem 4.4). From (i) and (3) we deduce the existence of an element λ_0 of Λ such that for all $i, j = 1, 2, \cdots, r$

(a) $c_i \in A \Rightarrow c_i \notin H \cdot A_{\lambda_0}$; $c_i \in B \Rightarrow c_i \notin K \cdot B_{\lambda_0}$;

(b) $c_i, c_j \in A,\ c_i \neq c_j \Rightarrow c_i^{-1}c_j \notin H \cdot A_{\lambda_0}$;
$c_i, c_j \in B,\ c_i \neq c_j \Rightarrow c_i^{-1} c_j \notin K \cdot B_{\lambda 0}$.

One can then find a set $\bar{S}$ (respectively $\bar{T}$) of right coset representatives for $H \cdot A_{\lambda_0}/A_{\lambda_0}$ (respectively $K \cdot B_{\lambda_0}/B_{\lambda_0}$) in A/A_{λ_0} (respectively B/B_{λ_0}) such that $\theta_{\lambda_0}(c_i) \in \bar{S} \cup \bar{T}$ for all $i = 1, \cdots, r$. It follows that

$$\theta_{\lambda_0}(y) = \theta_{\lambda_0}(h) \cdot \theta_{\lambda_0}(c_1)\,\theta_{\lambda_0}(c_2) \cdots \theta_{\lambda_0}(c_r)$$

is a canonical presentation and $\theta_{\lambda_0}(y) \neq 1$. Since $P_{\lambda_0} \in R\mathscr{F}_p$, the result follows.

Proposition 1.3. *Let* H (*respectively* K) *be a subgroup of a group* A (*respectively* B) *in* $R\mathscr{F}_p$. *Suppose that* $\{A_n\}_{n \in N}$ and $\{B_n\}_{n \in N}$ are $\mathscr{F}_p$*-filters of* A *and* B *respectively, satisfying* (i) *and* (ii) *of Proposition* 1.2, *as well as the following*:

(iii) A_n/A_{n+1} *and* B_n/B_{n+1} *are finite abelian groups of exponent* p;

(iv) *$H \cap A_n/H \cap A_{n+1}$ (respectively $K \cap B_n/K \cap B_{n+1}$) lies in the center of A/A_{n+1} (respectively B/B_{n+1}) for all $n \in N$.*

Then

$$\{A * B;\, H = K\} \in R\mathscr{F}_p.$$

PROOF. We need only verify condition (iii) of Proposition 1.2, and by Higman's theorem (1964), we need only prove that each amalgam $(A/A_n) \cup (B/B_n)$ with intersection $H/H \cap A_n = K/K \cap B_n$ is embeddable in a finite p-group. We do this by induction on $n \in N$. It is clear that the amalgam $(A/A_1) \cup (B/B_1)$ of F_p-vector spaces with intersection $H/H \cap A_1 = K/K \cap B_1$ can be embedded in an F_p-vector space M_0. More generally, the amalgam $(A_n/A_{n+1}) \cup (B_n/B_{n+1})$ of F_p-vector spaces with intersection $H \cap A_n/H \cap A_{n+1} = K \cap B_n/K \cap B_{n+1}$ can be embedded in an F_p-vector space M_n, for each $n \in N$. We now assume that the amalgam $(A/A_n) \cup (B/B_n)$ with intersection $H/H \cap A_n = K/K \cap B_n$ is embedded in a finite p-group Y_n. Let

$$\theta_n\colon H/H \cap A_{n+1} \to H/H \cap A_n \hookrightarrow Y_n$$

and consider the exact sequence

$$1 \to H \cap A_n/H \cap A_{n+1} \to H/H \cap A_{n+1} \xrightarrow{\theta_n} Y_n.$$

Using Higman's terminology and his Corollary to Lemma 1 (loc. cit.), we choose a standard embedding

$$\lambda_{n+1}\colon H/H \cap A_{n+1} \hookrightarrow (H \cap A_n/H \cap A_{n+1}) \wr Y_n$$

and extend the composite:

$$H/H \cap A_{n+1} \overset{\lambda_{n+1}}{\hookrightarrow} (H \cap A_n/H \cap A_{n+1}) \wr Y_n \hookrightarrow (A_n/A_{n+1}) \wr Y_n$$

(respectively $K/K \cap B_{n+1} \xrightarrow{\lambda_{n+1}} (K \cap B_n/K \cap B_{n+1}) \wr Y_n \hookrightarrow (B_n/B_{n+1}) \wr Y_n$) to a standard embedding

$$\mu_{n+1}\colon A/A_{n+1} \hookrightarrow (A_n/A_{n+1}) \wr Y_n$$

$$(\text{respectively} \quad \nu_{n+1}\colon B/B_{n+1} \hookrightarrow (B_n/B_{n+1}) \wr Y_n)$$

Note that we have identified $H/H \cap A_{n+1}$ with $K/K \cap B_{n+1}$ and $H \cap A_n/H \cap A_{n+1}$ with $K \cap B_n/K \cap B_{n+1}$. The two embeddings μ_{n+1} and ν_{n+1} now give rise to embeddings of A/A_{n+1} and B/B_{n+1} in $Y_{n+1} = M_n \wr Y_n$, which completes the proof.

The group P of the following theorem is our first example of a one-relator group belonging to $R\mathscr{F}_p$.

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THEOREM 1.4. *Let F and F' be free groups and suppose that $a \in F$ (respectively $b \in F'$) is not of the form z^n for $\pm 1 \neq n \in \mathbf{Z} \setminus p\mathbf{Z}$, $z \in F$ (respectively $z \in F'$). Then the generalized free product*

$$P = \{F * F' : a = b\}$$

is residually a finite p-group.

PROOF. The words a and b involve only finitely many generators. Let G be the subgroup of P generated by the remaining generators, if any; then G is a free group and is a free factor of P. Since free groups are in $R\mathscr{F}_p$, and the free product of two groups in $R\mathscr{F}_p$ is again in $R\mathscr{F}_p$ (Gruenberg (1957), Theorem 6.2), we may assume without loss in generality that F and F' are finitely generated.

Let $t, t' \in \mathbf{R}, t, t' > 1$, and let ω_t (respectively ω'_t) be the (t, p)-filtration on F (respectively F'). It follows from Lazard (1965), 3.2.6.1 that the values $\omega'_t(a)$ and $\omega_t(b)$ depend continuously on t. So, we can choose t and t' in such a way that $\omega_t(a) = \omega'_{t'}(b)$. Let $\operatorname{gr} F$ (respectively $\operatorname{gr} F'$) be the Lie algebra with coefficients in the polynomial ring $\mathbf{F}_p[\pi]$, corresponding to ω_t (respectively $\omega'_{t'}$) (see Lazard (1965), Chapter II, 1.2). Since these Lie algebras are free (Lazard (1965), 3.2.2, 3.2.5), $\operatorname{gr} a$ (respectively $\operatorname{gr} b$) generates a free Lie subalgebra $\operatorname{gr} H$ (respecpectively $\operatorname{gr} K$) in $\operatorname{gr} F$ (respectively $\operatorname{gr} F'$), where H (respectively K) denotes the cyclic subgroup of F (respectively F') generated by a (respectively b). So, the isomorphism $\varphi\colon H \to K$ that maps a onto b induces an isomorphism $\operatorname{gr} H \to \operatorname{gr} K$ of *graded* Lie algebras. It follows that if we let

$$t < \lambda_1 < \lambda_2 < \cdots < \infty$$

denote the union of the ranges of ω_t and $\omega'_{t'}$ (see Lazard (1965), 3.2.6.2), and write

$$A_n = \{a \in A : \omega_t(a) \geqq \lambda_n\},$$

$$B_n = \{b \in B : \omega'_{t'}(b) \geqq \lambda_n\},$$

then (ii) of Proposition 1.2 is satisfied. The families $\{A_n\}_{n \in \mathbf{N}}$ and $\{B_n\}_{n \in \mathbf{N}}$ are $\mathscr{F}_p$-filters. Indeed, the (t, p)-filtration on the finite p-group $A/A_n \cap A_m$ defines its discrete topology (Lazard (1965), 3.1.5, 3.2.8.1), and this filtration is the quotient filtration of the (t, p)-filtration on A (see Lazard (1965), 3.2.3); hence there exists an A_k contained in $A_n \cap A_m$. In order to show that condition (i) is satisfied we, need only prove that $\bar{H} \cap F = H$ and $\bar{K} \cap F' = K$ (see Proposition 1.1(b)). But, $\bar{H}$ is a free pro-p-group on one generator; it is isomorphic to the additive group of the ring of p-adic integers, and the *abelian* subgroup $\bar{H} \cap F$ of F must be of the form $\{z^\alpha : \alpha \in \mathbf{Z}\}$ for some $z \in F$. Since H is contained in $\bar{H} \cap F$, we must have $a = z^\gamma$ for some $\gamma \in \mathbf{Z}$, and, by hypothesis, $\gamma = \pm 1$ or p^k for some $k \in \mathbf{N}$. However, $\operatorname{gr} \bar{H} = \operatorname{gr} H$ is freely generated as an $\mathbf{F}_p[\pi]$-module by $\operatorname{gr} z$ and is also freely generated by $\operatorname{gr} a$, so that if $a = z^{p^k}$, $k \in N$, we would have $\operatorname{gr} a = \pi^k \operatorname{gr} z$, which is impossible. Thus $a = z$ and $\bar{H} \cap F = H$. Similarly $\bar{K} \cap F' = K$. The result now follows from Proposition 1.3.

3. Examples based upon Theorem 1.4.

PROPOSITION 2.1. *If a group G has the presentation*

$$(x_0, x_1, \cdots, x_m; x_0^{n_0} x_1^{n_1} \cdots x_m^{n_m})$$

where each n_i is of the form p^{k_i}, $k_i \geqq 0$, $i = 0, 1, \cdots, m$, then G is residually a finite p-group.

PROOF. The result follows from Theorem 1.4 by a simple induction argument.

Our second example in this section consists of the Demuškin relators (see Labute (1967a), (1), (2), (3), (4) on pages 106, 107). These relators appear as defining relators for Galois groups (pro-p-groups) of p-algebraic closures of finite extensions of the field Q_p of the p-adic numbers (See Serre (1962/63), (1964)). We will show that if we view these relators as defining relators for (discrete) groups, then the groups are in $R\mathscr{F}_p$, and the Demuškin groups are therefoe *Hausdorff* completions (See Bourbaki (1951), §3, no. 4, Theorem 1) of these groups. The Demuškin relators are either of the for

$$r = x_1^q (x_1, x_2)(x_3, x_4) \cdots (x_{n-1}, x_n) \tag{1}$$

where $2 \neq q = p^k$ for some prime, and n is even, or it has one of the following forms:

$$x_1^2 x_2^{2^f} (x_2, x_3) \cdots (x_{n-1}, x_n) \tag{2}$$

(where $2 \leqq f \leqq \infty$, with $2^\infty = 0$),

$$x_1^{2+2^f} (x_1, x_2)(x_3, x_4) \cdots (x_{2n-1}, x_{2n}) \qquad (2 \leqq f \leqq \infty) \tag{3}$$

$$x_1^2 (x_1, x_2) x_3^{2^f} (x_3, x_4) \cdots (x_{2n-1}, x_{2n}) \qquad (2 \leqq f \leqq \infty). \tag{4}$$

(See Labute (1967a)).

PROPOSITION 2.2. *The group $G = (x_1, x_2, \cdots, x_n; r)$ is in $R\mathscr{F}_p$ if r is of the form* (1), (2), (3), *or* (4) *and* $p = 2$. *Moreover, the integral group ring* $\boldsymbol{Z}[G]$ *has no zero divisors.*

PROOF. By Labute's Theorem 4′ (1967), there exists a q-filtration of the completed group algebra $\hat{\boldsymbol{Z}}_p[[\hat{G}]]$ with the property that the corresponding graded algebra $\operatorname{gr} \hat{\boldsymbol{Z}}_p[[\hat{G}]]$ has no zero divisors. Thus $\hat{\boldsymbol{Z}}_p[[\hat{G}]]$ is a valued ring (Lazard (1965), 2.2.1, 2.3.6) and has no zero divisors. We need only prove $G \in R\mathscr{F}_p$

since then $\boldsymbol{Z}[G] \subset \hat{\boldsymbol{Z}}_p[[\hat{G}]]$. By virtue of Theorem 1.4, we need only consider relators r of the form $r = x_1^{p^k}(x_1, x_2)$ where $k \in \boldsymbol{N}$, or of the form $r = x_1^{2+2^f}(x_1, x_2)$, $2 \leqq f \in \boldsymbol{N}$. In the first case, one has a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\pi} & F(x_2) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} & & \\ 1 & \longrightarrow & M & \longrightarrow & \hat{G} & \xrightarrow{\hat{\pi}} & \hat{F}(x_2) & \longrightarrow & 1 \end{array}$$

where N is the group with generators $z_i = x_2^{-i}x_1x_2^i$ ($i \in \boldsymbol{Z}$) and relators $z_i^{-\gamma}\, z_{i+1}$ ($i \in \boldsymbol{Z}$) with $\gamma = 1 - p_g^k$ (See Magnus, Iarass and Solitar (1966), Section 4.4). One obtains an isomorphism of N onto the additive group of the subring $\boldsymbol{Z}[\gamma^{-1}]$ of $\boldsymbol{Q}$, by mapping z_i onto γ^i. The group $F(x_2)$ is the free group on one generator and the maps μ and ν are canonical maps of groups into their pro-p-completions. The pro-p-group M is isomorphic to the additive group of the ring $\hat{\boldsymbol{Z}}_p$, and if we identify M and $\hat{F}(x_2)$ with $\hat{\boldsymbol{Z}}_p$, then we can describe $\hat{G}$ as the semi-direct product of $\hat{\boldsymbol{Z}}_p$ with itself according to the action

$$\theta\colon \boldsymbol{Z}_p \to \operatorname{Aut}(\boldsymbol{Z}_p),\ \theta(\alpha)(\beta) = \gamma^{\alpha} \circ \beta,$$

(See Gildenhuys (1972), Theorem 3.2). Note that

$$\gamma^{-1} = 1 + p^k + p^{2k} + \cdots$$

lies in the multiplicative pro-p-group of invertible elements of $\boldsymbol{Z}_p$, and if we identify N with $\boldsymbol{Z}[\gamma_{-1}^{-1}]$, then λ is just the inclusion: $\boldsymbol{Z}[\gamma_{-1}^{-1}] \subset \hat{\boldsymbol{Z}}_p$ (See Serre (1970), Chapter II, §3, Proposition 8, p. 32). It follow that μ is injective and $G \in R\mathscr{F}_p$.

If

$$r = x_1^{2+2^f}(x_1, x_2) = z_0^{1+2^f} \circ z_1, \qquad 2 \leqq f \in \boldsymbol{N}$$

we put $\gamma = -(1 + 2^f)$, and reason in the same way as before.

The above proof also shows that the cyclic subgroup H of

$$G = (x, y; x^{p^k}(x, y)), \qquad (k \in \boldsymbol{N}),$$

generated by x, is not (t, p)-separable. Indeed, $\bar{H} \cap G$ contains an isomorphic copy of $\boldsymbol{Z}[\gamma^{-1}]$, so that $\bar{H} \cap G \neq H \cong \boldsymbol{Z}$ (See Proposition 1.1 (b))).

4. Circular amalgams

In this section we keep the notation of the introduction (Baumslag's notation (1971)) for the basic breakdown of a one-relator group $G = (a, b, \cdots, c; r)$. Our purpose is to find conditions under which $N_0 \in R\mathscr{F}_p$ implies $G \in R\mathscr{F}_p$. We are assuming that r is cyclically reduced, and a occurs with exponent sum zero in r. We recall the following notation of Baumslag (1971), Section 2: N is the normal subgroup of G generated by $b, \cdots, c$; for each integer k, N_k is the subgroup of N generated by

$$b_{m(b)+k}, \cdots, b_{M(b)+k}, \cdots, c_{m(c)+k}, \cdots, c_{M(c)+k}\,;$$

H_k is the subgroup of N generated by

$$b_{m(b)+k+1}, \cdots, b_{M(b)+k}, \cdots, c_{m(c)+k+1}, \cdots, c_{M(c)+k}\,;$$

and for $i \leqq j$, we let $N_{i,j}$ denote the subgroup of N generated by $N_i, \cdots, N_j$.

Let t be some fixed positive real number. For every finitely generated group A, the range of the (t, p)-filtration ω on A is a discrete subset

$$t = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \infty$$

of $\boldsymbol{R} \cup \{\infty\}$, and we write

$$\delta_n A = \{a \in A : \omega(a) \geqq \lambda_n\}, \qquad (n \in \boldsymbol{N}).$$

The corresponding mixed Lie algebra is denoted by $\operatorname{gr} A$, and its m-th homogeneous component is denoted here by

$$\operatorname{gr}_m A = \delta_m A/\delta_{m+1} A = A_{\lambda_m}/A_{\lambda_{m+1}} \quad (m \geqq 0)$$

THEOREM 3.1. *Suppose that*

(i) *H_0 and H_j are (t, p)-isometric subgroups of $N_{1,j}$*

(ii) *H_0 and H_j are (t, p)-separable in $N_{1,j}$;*

(iii) $(\operatorname{gr} H_0) \cap (\operatorname{gr} H_j) = (0)$ *in* $\operatorname{gr} N_{1,j}$, *for all* $j \geqq$ *some* n_0

(iv) $N_0 \in R\mathscr{F}_p$

Then the original one-relator group G is in RF_p.

PROOF. Let j be some fixed power of p for which (iii) is valid, as well as:

$$j > \sup\{M(b) - m(b), \cdots, M(c) - m(c)\}.$$

Define

$$B_i = N_{ij, ij+j-1}, \qquad (i \in \boldsymbol{Z})$$

and let k be a natural number for which $p^k > 2j$. Write

$$\boldsymbol{Z}/p^k\boldsymbol{Z} = \{0, 1, \cdots, p^k - 1\}$$

and let $\pi: \boldsymbol{Z} \to \boldsymbol{Z}/p^k\boldsymbol{Z}$ denote the canonical projection. Using functional notation:

$$r_i = r_i(b_{m(b)+i}, \cdots, b_{M(b)+i}, \cdots, c_{m(c)+i}, \cdots, c_{M(c)+i})$$

we now define for each $i \in \boldsymbol{Z}/p^k\boldsymbol{Z}$:

$$r_i' = r_i(b_{\pi(m(b)+i)}, \cdots, b_{\pi(M(b)+i)}, \cdots, c_{\pi(m(c)+i)}, \cdots, c_{\pi(M(c)+i)})$$

and we note that in the presentation for B_i:

$$B_i = (b_{m(b)+ij}, \cdots, b_{M(b)+ij+j-1}, \cdots, c_{m(c)+ij}, \cdots, c_{M(c)+ij+j-1};$$
$$r_{ij}, \cdots, r_{ij+j-1})$$

we may replace the relators $r_{ij}, \cdots, r_{ij+j-1}$ by $r'_{ij}, \cdots, r'_{ij+j-1}$, and view the indices of the generators as elements of the cyclic group $\boldsymbol{Z}/p^k\boldsymbol{Z}$. For all $i \in \boldsymbol{Z}/p^k\boldsymbol{Z}$ one has

$$B_i \cap B_{i+1} = H_{ij+j-1}.$$

In view of condition (i), we may also write

$$(B_i/\delta_m B_i) \cap (B_{i+1}/\delta_m B_{i+1}) = H_{ij+j-1}/\delta_m H_{ij+j-1}$$

for all $m \in \boldsymbol{N}$. We will say that a finite p-group Y_m *contains an image of the circular amalgam* $\{B_i/\delta_m B_i\}_{i \in \boldsymbol{Z}/p^k\boldsymbol{Z}}$ if there are homomorphisms

$$\alpha_i \colon B_i/\delta_m B_i \to Y_m,$$

(not necessarily injective), such that for all $i \in \boldsymbol{Z}/p^k\boldsymbol{Z}$ the restriction of α_i to $H_{ij+j-1}/\delta_m H_{ij+j-1}$ agrees with the restriction of α_{i+1} to $H_{ij+j-1}/\delta_m H_{ij+j-1}$. A sequence

$$B_s/\delta_m B_s,\ B_{s+1}/\delta_m B_{s+1}, \cdots, B_{s+t}/\delta_m B_{s+t}$$

contained in the above circular amalgam is said to be *properly mapped into* Y_m if there are maps α_i satisfying the above condition for $i, i+1 \in \{s, s+1, \cdots, s+t\}$. If these maps are all injective, we say that the sequence is *properly embedded in* Y_m.

LEMMA 3.2. *Assume the hypotheses of Theorem* 3.1; *let*

$$n = \sup\{M(b) - m(b), \cdots, M(c) - m(c)\}$$

and suppose that: $v, k, m \in \boldsymbol{N}$, $j = p^v$, $w = k - v > 0$, $j > n$, $p^k > n + j + 1$, $j \geqq n_0$, $t = m - 1$, $p^k > 3t + n + 1$. *One can then find a finite p-group* Y_m *containing an image of the circular amalgam* $\{B_i/\delta_m B_i\}_{i \in \mathbf{Z}/p^w\mathbf{Z}}$, *such taht the sequence*

$$B_t/\delta_m B_t,\ B_{t+1}/\delta_m B_{t+1}, \cdots, B_{p^w-2t-1}/\delta_m B_{p^w-2t-1}$$

is (*properly*) *embedded* in Y_m.

PROOF. The proof goes by induction on m. For each $m \in \boldsymbol{N}$ and $i \in \boldsymbol{Z}/p^w\boldsymbol{Z}$ we choose in the $\boldsymbol{F}_p$-vector space $\mathrm{gr}_m B_i$ a complementary subspace $C_{m,i}$ for the subspace $\mathrm{gr}_m H_{ij-1} + \mathrm{gr}_m H_{ij+j-1}$. We recall that by the Freiheitsatz, H_{ij-1} and H_{ij+j-1} are free groups, so that for each $i \in \boldsymbol{Z}/p^w\boldsymbol{Z}$ there is a natural embedding $\alpha_{m,i}$ of $\mathrm{gr}_m H_{ij-1} + \mathrm{gr}_m H_{ij+j-1}$ into the m-th homogeneous component

$$\mathrm{gr}_m F(b_0, \cdots, b_{p^k-1}, \cdots, c_0, \cdots, c_{p^k-1}),$$

of the mixed Lie algebra of the free group on the indicated letters, with $\alpha_{m,i}$ respecting the indices $(\mathrm{mod}\, p^k\boldsymbol{Z})$ of the generators, so that $\alpha_{m,i}$ and $\alpha_{m,i+1}$ agree on

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$\operatorname{gr}_m H_{ij+j-1}$. (We use here the hypothesis (iii))). Hence there is a natural embedding of $\operatorname{gr}_m B_i$ into

$$U_m = (\bigoplus_{i \in \mathbf{Z}/p^w\mathbf{Z}} C_{m,i}) \oplus \operatorname{gr}_m F(b_0, \cdots, b_{p^k-1}, \cdots, c_0, \cdots, c_{p^k-1})$$

for each $i \in \mathbf{Z}/p^w\mathbf{Z}$. So for $m = 1$, we have embedded the circular amalgam $\{B_i/\delta_1 B_i\}_{i \in \mathbf{Z}/p^w\mathbf{Z}}$ into $Y_1 = U_1$.

Suppose now that $p^k \geqq 3m + n + 1$. Our induction hypothesis is that the circular amalgam $\{B_i/\delta_m B_i\}_{i \in \mathbf{Z}/p^w\mathbf{Z}}$ is mapped into a finite p-group Y_m by the maps $\lambda_i\colon B_i/\delta_m B_i \to Y_m$ say, and λ_i is injective for

$$i \in \{t, t+1, \cdots, p^w - 2t - 1\}, \qquad (t = m - 1).$$

Let $\bar{U}_m$ be the direct sum of the vector space

$$C_m = \bigoplus_{i \in \mathbf{Z}/p^w\mathbf{Z}} C_{m,i}$$

and the m-th homogeneous component of the mixed Lie algebra of the free group on the letters

$$b_{M(b)+mj-j}, \cdots, b_{m(b)+p^k-2mj+2j-1}, \cdots,$$

$$c_{M(c)+mj-j}, \cdots, c_{m(c)+p^k-2mj+2j-1}.$$

Let $\pi_m\colon U_m \to \bar{U}_m$ be the identity map on C_m and let it coincide on

$$\operatorname{gr}_m F(b_0, \cdots, b_{p^k-1}, \cdots, c_0, \cdots, c_{p^k-1})$$

with the restriction of the projection q of the free mixed Lie algebra (Lazard (1965), 3.2.5)

$$\operatorname{gr} F(b_0, \cdots, b_{p^k-1}, \cdots, c_0, \cdots, c_{p^k-1})$$

onto the free mixed Lie algebra

$$\operatorname{gr} F(b_{M(b)+mj-j}, \cdots, b_{m(b)+p^k-2mj+2j-1}, \cdots,$$

$$c_{M(c)+mj-j}, \cdots, c_{m(c)+p^k-2mj+2j-1})$$

such that q maps each of the generators

$$\operatorname{gr} b_{M(b)+mj-j}, \cdots, \operatorname{gr} b_{m(b)+p^k-2mj+2j-1}, \cdots,$$

$$\operatorname{gr} c_{M(c)+mj-j}, \cdots, \operatorname{gr} c_{m(c)+p^k-2mj+2j-1}$$

onto itself, and each of the remaining generators onto 0.

In what follows, we will use the terminology and the Corollary to Lemma 1 of Higman (1964). For every

$$i \in \{t, t+1, \cdots, p^w - 2m + 1\}$$

the map

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$$\theta_i: B_i/\delta_{m+1}B_i \to B_i/\delta_m B_i \xrightarrow{\lambda_i} Y_m$$

gives rise to a commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{gr}_m H_{ij-1} \arrow[r] \arrow[d, hook] & H_{ij-1}/\delta_{m+1}H_{ij-1} \arrow[r] \arrow[d, hook] & Y_m \arrow[d, dotted, no head] \\
1 \arrow[r] & \mathrm{gr}_m B_i \arrow[r] & B_i/\delta_{m+1}B_i \arrow[r] & Y_m \\
1 \arrow[r] & \mathrm{gr}\, H_{ij+j-1} \arrow[r] \arrow[u, hook] & H_{ij+j-1}/\delta_{m+1}H_{ij+j-1} \arrow[r] \arrow[u, hook] & Y_m \arrow[u, equal]
\end{tikzcd}$$

Let

$$H_{tj+j-1}/\delta_{m+1}H_{tj+j-1} \hookrightarrow (\mathrm{gr}_m H_{tj+j-1}) \wr Y_m$$

be a standard embedding of $H_{tj+j-1}/\delta_{m+1}H_{tj+j-1}$ in the weath product of $\mathrm{gr}_m H_{tj+j-1}$ and Y_m. Since $\mathrm{gr}_m H_{tj+j-1}$ is in the center of $B_t/\delta_{m+1}B_t$ and in the center of $B_{t+1}/\delta_{m+1}B_{t+1}$, we can find standard embeddings

$$\varepsilon_t: B_t/\delta_{m+1}B_t \hookrightarrow (\mathrm{gr}_m B_t) \wr Y_m$$

$$\varepsilon_{t+1}: B_{t+1}/\delta_{m+1}B_{t+1} \hookrightarrow (\mathrm{gr}_m B_{t+1}) \wr Y_m$$

such that the following diagram commutes:

$$\begin{tikzcd}
 & B_t/\delta_{m+1}B_t \arrow[r, phantom, "\varepsilon_t"] & (\mathrm{gr}_m B_t) \wr Y_m \arrow[dr] & \\
H_{tj+j-1}/\delta_{m+1}H_{tj+j-1} \arrow[ur, hook] \arrow[rr, hook] \arrow[dr, hook] & & (\mathrm{gr}_m H_{tj+j-1}) \wr Y_m \arrow[u, hook] \arrow[d, hook] \arrow[r] & U_m \wr Y_m \\
 & B_{t+1}/\delta_{m+1}B_{t+1} \arrow[r, hook, "\varepsilon_{t+1}"] & (\mathrm{gr}_m B_{t+1}) \wr Y_m \arrow[ur] &
\end{tikzcd}$$

Moreover, the image of $\varepsilon_{t+1}(\mathrm{gr}_m H_{(t+1)j+j-1})$ in $U_m \wr Y_m$ is contained in the center of $U_m \wr Y_m$, since ε_{t+1} maps the subgroup $\mathrm{gr}_m B_{t+1}$ of $B_{t+1}/\delta_{m+1}B_{t+1}$ into the diagonal of $(\mathrm{gr}_m B_{t+1})^{Y_m}$. (By the "diagonal" we mean the subgroup of constant maps). Consider now the commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & U_m^{Y_m} \arrow[r] & U_m \wr Y_m \arrow[r] & Y_m \\
1 \arrow[r] & \mathrm{gr}_m H_{(t+1)j+j-1} \arrow[r] \arrow[u, hook] \arrow[d, hook] & H_{(t+1)j+j-1}/\delta_{m+1}H_{(t+1)j+j-1} \arrow[r] \arrow[u, hook, "\eta"'] \arrow[d, hook] & Y_m \arrow[u, equal] \arrow[d, equal] \\
1 \arrow[r] & \mathrm{gr}_m B_{t+2} \arrow[r] & B_{t+2}/\delta_{m+1}B_{t+2} \arrow[r] & Y_m
\end{tikzcd}$$

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where η is obtained by restricting ε_{t+1} to $H_{(t+1)j+j-1}/\delta_{m+1}H_{(t+1)j+j-1}$. Let

$$\delta\colon H_{(t+1)j+j-1}/\delta_{m+1}H_{(t+1)j+j-1} \to (\mathrm{gr}_m H_{(t+1)j+j-1}) \wr Y_m$$

be a standard embedding. By Higman's Corollary to Lemma 1 (loc. cit.), the composition of δ and the map:

$$(\mathrm{gr}_m H_{(t+1)j+j-1}) \wr Y_m \hookrightarrow (U_m^{Y_m}) \wr Y_m$$

can be extended to a standard embedding:

$$U_m \wr Y_m \hookrightarrow (U_m^{Y_m}) \wr Y_m,$$

and the composition of δ and the map:

$$(\mathrm{gr}_m H_{(t+1)j+j-1}) \wr Y_m \to (\mathrm{gr}_m B_{t+2}) \wr Y_m$$

can be extended to a standard embedding:

$$B_{t+2}/\delta_{m+1}B_{t+2} \xhookrightarrow{\varepsilon_{t+2}} (\mathrm{gr}_m B_{t+2}) \wr Y_m \to U_m \wr Y_m \hookrightarrow (U_m^{Y_m}) \wr Y_m.$$

The image of $\mathrm{gr}_m B_{t+2}$, and hence of $\mathrm{gr}_m H_{(t+2)j+j-1}$ is in the center of $(U_m^{Y_m}) \wr Y_m$. In the above commutative diagram one can replace the top row by

$$1 \to (U_m^{Y_m})^{Y_m} \to (U_m^{Y_m}) \wr Y_m \to Y_m$$

and t by $t+1$. We can then again apply Higman's Corollary of Lemma 1, to obtain a commutative diagram:

$$\begin{tikzcd}
 & B_t/\delta_{m+1}B_t \arrow[dr, hook] & & & \\
H_{tj+j-1}/\delta_{m+1}H_{tj+j-1} \arrow[ur, hook] \arrow[dr, hook] & & U_m \wr Y_m \arrow[dr, hook] & & \\
 & B_{t+1}/\delta_{m+1}B_{t+1} \arrow[ur, hook] & & (U_m^{Y_m}) \wr Y_m \arrow[ddr, hook] & \\
H_{(t+1)j+j-1}/\delta_{m+1}H_{(t+1)j+j-1} \arrow[ur] \arrow[dr, hook] & & & & \\
 & B_{t+2}/\delta_{m+1}B_{t+2} \arrow[uurr, hook] & & & (U_m^{Y_m})^{Y_m} \wr Y_m \\
H_{(t+2)j+j-1}/\delta_{m+1}H_{(t+2)j+j-1} \arrow[ur, hook] \arrow[dr, hook] & & & & = U_m^{Y_m^2} \wr Y_m \\
 & B_{t+3}/\delta_{m+1}B_{t+3} \arrow[uurrr, hook] & & &
\end{tikzcd}$$

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We proceed in this manner upto the index $p^w - 2m + 1$ of B to obtain maps

$$\varphi_i \colon B_i/\delta_{m+1}B_i \to U_m^Z \wr Y_m,$$

where $Z = Y_m^{p^w - 3m + 1}$, that define a proper embedding of the sequence

$$B_t/\delta_{m+1}B_t,\ B_{t+1}/\delta_{m+1}B_{i+1}, \cdots, B_{p^w-2m+1}/\delta_{m+1}B_{p^w-2m+1}$$

into $U_m^Z \wr Y_m$. For

$$i \in \{t, t+1, \cdots, p^w - 2m + 1\}$$

define

$$\lambda_i' \colon B_i/\delta_{m+1}B_i \xhookrightarrow{\varphi_i} U_m^Z \wr Y_m \to \bar{U}_m^Z \wr Y_m.$$

Note that λ_i' maps $\mathrm{gr}_m B_i$ into the diagonal of the subgroup $\bar{U}_m^{Z \times Y_m} = (\bar{U}_m^Z)^{Y_m}$ of $\bar{U}_m^Z \wr Y_m$. For

$$i \in \{-2m+2, -2m+3, \cdots, m-2\}$$

define

$$\lambda_i' \colon B_i/\delta_{m+1}B_i \to B_i/\delta_m B_i \xrightarrow{\lambda_i} Y_m \to \bar{U}_m^Z \wr Y_m,$$

where the last map sends $y \in Y_m$ to the pair $(y, 0)$.

Claim. *The circular amalgam $\{B_i/\delta_{m+1}B_i\}_{i \in Z/p^w Z}$ is mapped into $Y_{m+1} = \bar{U}_m^Z \wr Y_m$ by the maps λ_i', and the sequence*

$$B_m/\delta_{m+1}B_m,\ B_{m+1}/\delta_{m+1}B_m, \cdots, B_{p^w-2m-1}/\delta_{m+1}B_{p^w-2m-1}$$

is (properly) embedded in Y_{m+1}.

Proof. Clearly the sequence

$$B_t/\delta_{m+1}B_t,\ B_{t+1}/\delta_{m+1}B_{t+1}, \cdots, B_{p^w-2m+1}/\delta_{m+1}B_{p^w-2m+1}$$

is properly mapped into Y_{m+1}; and the sequence

$$B_{-2m+2}/\delta_{m+1}B_{-2m+2},\ B_{-2m+3}/\delta_{m+1}B_{-2m+3}, \cdots, B_{m-2}/\delta_{m+1}B_{m-2}$$

is properly mapped into Y_m, hence into Y_{m+1}, by the induction hypothesis. We must show that the two sequences are properly linked together at the edges. So suppose that

$$z \in H_{p^k-2mj+2j-1}/\delta_{m+1}H_{p^k-2mj+2j-1}.$$

Then

$$\lambda'_{p^w-2m+2}(z) = (\lambda_{p^w-2m+2}(\bar{z}), 0) \in Y_{m+1},$$

where $\bar{z}$ is the image of z in $B_{p^w-2m+2}/\delta_m B_{p^w-2m+2}$. On the other hand, $\lambda'_{p^w-2m+1}(z)$ is of the form $(\lambda_{p^w-2m+1}(\bar{z}), \pi^Z \circ \psi)$, where $\pi^Z \colon U_m^Z \to \bar{U}_m^Z$ is induced

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by $\pi: U_m \to \bar{U}_m$; ψ is a mapping: $Y_m \to U_m^{\mathbf{Z}}$, and $\tilde{z}$ is the image of z in $B_{p^w-2m+1}/\delta_m B_{p^w-2m+1}$. By the induction hypothesis,

$$\lambda_{p^w-2m+1}(\tilde{z}) = \lambda_{p^w-2m+2}(\bar{z}).$$

The map λ'_{p^w-2m+1} factors through a map

$$B_{p^w-2m+1}/\delta_{m+1}B_{p^w-2m+1} \hookrightarrow (\operatorname{gr}_m B_{p^w-2m+1}) \wr Y_m \hookrightarrow U_m \wr Y_m$$

that sends $\operatorname{gr}_m H_{p^k-2mj+2j-1}$ into the diagonal of $U_m^{Y_m}$. Hence the mapping ψ factors as follows:

$$Y_m \to \operatorname{gr}_m H_{p^k-2mj+2j-1} \to U_m \xrightarrow{\Delta} U_m^{\mathbf{Z}},$$

where the last map is the diagonal map. The images in U_m of the elements of $\operatorname{gr}_m H_{p^k-2mj+2j-1}$ are homogeneous mixed Lie polynomials in the symbols

$$\operatorname{gr} b_{m(b)+p^k-2mj+2j}, \cdots, \operatorname{gr} b_{M(b)+p^k-2mj+2j-1},$$

$$\cdots, \operatorname{gr} c_{m(c)+p^k-2mj+2j}, \cdots, \operatorname{gr} c_{M(c)+p^k-2mj+2j-1}.$$

But, π maps these polynomials to 0, and we conclude that $\pi^{\mathbf{Z}} \circ \psi = 0$. Thus

$$\lambda'_{p^w-2m+1}(z) = \lambda'_{p^w-2m+2}(z).$$

Finally, if $z \in H_{mj-j-1}/\delta_{m+1}H_{mj-j-1}$, then

$$\lambda'_{m-1}(z) = (\lambda_{m-1}(\bar{z}), 0),$$

where $\bar{z}$ is the image of z in $B_t/\delta_m B_t$. On the other hand, $\lambda'_{m-2}(z)$ is of the form $(\lambda_{m-2}(\tilde{z}), \pi^{\mathbf{Z}} \circ \psi)$ where $\psi: Y_m \to U_m^{\mathbf{Z}}$ factors as follows:

$$Y_m \to \operatorname{gr}_m H_{mj-j-1} \to U_m \xrightarrow{\Delta} U_m^{\mathbf{Z}}.$$

The images in U_m of the elements of $\operatorname{gr}_m H_{mj-j-1}$ are homogeneous mixed Lie polynomials in the symbols

$$\operatorname{gr} b_{m(b)+mj-j}, \cdots, \operatorname{gr} b_{M(b)+mj-j-1},$$

$$\cdots, \operatorname{gr} c_{m(c)+mj-j}, \cdots, \operatorname{gr} c_{M(c)+mj-j-1}.$$

Since π maps these elements to 0, it follows that $\pi^{\mathbf{Z}} \circ \psi = 0$. By the induction hypothesis, $\lambda_{m-1}(\tilde{z}) = \lambda_{m-2}(\tilde{z})$; hence

$$\lambda'_{m-1}(z) = \lambda'_{m-2}(z).$$

To prove the second statement of the Claim, we recall that for

$$i \in \{t+1, t+2, \cdots, p^w - 2m - 1\},$$

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λ_i' is the composite map:

$$\lambda_i'\colon B_i/\delta_{m+1}B_i \xrightarrow{\varphi_i} U_m^{\mathbf{Z}} \wr Y_m \to \bar{U}_m^{\mathbf{Z}} \wr Y_m = Y_{m+1},$$

and φ_i is injective. If $g \in B_i/\delta_{m+1}B_i$ and $\lambda_i'(g) = 1$, then λ_i maps the image of g in $B_i/\delta_m B_i$ to the identity. Hence g belongs to the kernel $\operatorname{gr}_m B_i$ of λ_i. The restriction of φ_i to $\operatorname{gr}_m B_i$ factors as follows:

$$\operatorname{gr}_m B_i \to U_m \xrightarrow{\Delta} U_m^{\mathbf{Z}\times Y_m} = (U_m^{\mathbf{Z}})^{Y_m} \to (U_m^{\mathbf{Z}}) \wr Y_m.$$

The images of the elements of $\operatorname{gr}_m B_i$ in U_m are uniquely expressible in the form $c + d$, where $c \in C_i$ and d is a homogeneous, mixed Lie polynomial in the symbols

$$\operatorname{gr} b_{m(b)+ij}, \cdots, \operatorname{gr} b_{M(b)+ij+j-1},$$
$$\cdots, \operatorname{gr} c_{m(c)+ij}, \cdots, \operatorname{gr} c_{M(c)+ij+j-1}.$$

Since for $i \in \{t+1, t+2, \cdots, p^w - 2m - 1\}$ the indices

$$m(b) + ij, \cdots, M(b) + ij + j - 1$$
$$\text{belong to } \{M(b) + mj - j, \cdots, m(b) + p^k - 2mj + 2j - 1\},$$

etc., we have $\pi(d) = d$; i.e. the restriction of π to the image of $\operatorname{gr}_m B_i$ in U_m is injective. It follows that λ_i' is injective for all

$$i \in \{m, m+1, \cdots, p^w - 2m - 1\}.$$

This proves the Claim and the Lemma.

We can now complete the proof of the Theorem. As explained in the introduction, we need only show that the map α of the following commutative diagram with exact rows is injective:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, "\alpha"] & G \arrow[r] \arrow[d, "\beta"] & F(a) \arrow[r] \arrow[d, "\gamma"] & 1 \\
1 \arrow[r] & M \arrow[r] & \hat{G} \arrow[r] & \hat{F}(a) \arrow[r] & 1
\end{tikzcd}$$

The bottom row is an image of the exact sequence

$$1 \to \hat{F}(\hat{\mathbf{Z}}_1 \dot{\cup} \cdots \dot{\cup} \hat{\mathbf{Z}}_p) \to \hat{F}(a, b, \cdots, c) \to \hat{F}(a) \to 1$$

of free pro-p-groups, where

$$\hat{F}(\hat{\mathbf{Z}}_p \dot{\cup} \cdots \dot{\cup} \hat{\mathbf{Z}}_p) = \lim_{\overleftarrow{k}} \hat{F}((\mathbf{Z}/p^k\mathbf{Z}) \dot{\cup} \cdots \dot{\cup} (\mathbf{Z}/p^k\mathbf{Z}))$$

is the free pro-p-group generated by the coproduct (disjoint union), in the category of topological spaces, of as many copies of the underlying topological space of

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$\hat{Z}_p$ as there are generators $b, \cdots, c$. (See Gildenhuys, Lim (1972), Corollary 2.2 and Proposition 1.7). It follows that M is the inverse limit of the pro-p-groups

$$D_k = \hat{F}(b_0, b_1, \cdots, b_{p^k-1}, \cdots, c_0, c_1, c_{p^k-1})/(r'_0, r'_1, \cdots, r'_{p^k-1}),$$
$$(p^k > 2j > 2n).$$

Each D_k is the colimit in the category of pro-p-groups of the circular diagram consisting of the pro-p-completions

$$\hat{B}_i = \hat{F}(b_{m(b)+ij}, \cdots, b_{M(b)+ij+j-1},$$
$$\cdots, c_{m(c)+ij}, \cdots, c_{M(c)+ij+j-1})/(r'_{ij}, \cdots, r'_{ij+j-1})$$

and the inclusions

$$\hat{H}_{ij+j-1} \to \hat{B}_i, \hat{H}_{ij+j-1} \to \hat{B}_{i+1}, \qquad (i \in Z/p^w Z),$$

where $j = p^v$, k, w and n are as in Lemma 4.2. (We use the hypothesis (i) of the theorem.) Let

$$\gamma_i: \hat{B}_i \overset{\cong}{\to} \hat{B}_{i+m-1} \qquad (i \in Z/p^w Z),$$

be the isomorphism that sends the sequence of (topological) generators

$$(b_{m(b)+ij}, \cdots, b_{M(b)+ij+j-1}, \cdots, c_{m(c)+ij}, \cdots, c_{M(c)+ij+j-1})$$

for $\hat{B}_i$ onto the sequence

$$(b_{m(b)+ij+mj-j}, \cdots, b_{M(b)+ij+mj-1}, \cdots, c_{m(c)+ij+mj-j},$$
$$\cdots, c_{M(c)+ij+mj-1})$$

of generators for $\hat{B}_{i+m-1}$. The group Y_m of the Lemma also depends on k, and we write $Y_{m,k} = Y_m$. Consider the maps

$$\rho_{i,m,k}: \hat{B}_i \overset{\gamma_i}{\to} \hat{B}_{i+m-1} = \varprojlim_n B_{i+m-1}/\delta_n B_{i+m-1} \to B_{i+m-1}/\delta_m B_{i+m-1} \to Y_{m,k},$$

(see Lemma 3.2), and let $L_{i,m}$ denote the kernel of the canonical map:

$$\hat{B}_i \to B_i/\delta_m B_i.$$

Note that $\rho_{0,m,k}$ has kernel $L_{0,m}$. The maps $\rho_{i,m,k}$ $(i \in Z/p^w Z)$ induce a map $\sigma_k: D_k \to Y_{m,k}$ out of the colimit D_k of the circular diagram described above. Since $\rho_{0,m,k}$ factors through this map:

$$\rho_{0,m,k}: \hat{B}_0 \overset{\tau_k}{\to} D_k \overset{\sigma_k}{\to} Y_{m,k},$$

the canonical map τ_k has kernel K_k contained in $L_{0,m}$. Note that τ_k preserves the indices of the generators. We now choose an ascending sequence $\{k_m\}_{m \in N}$ of positive integers, such that $p^{k_m} > 3(m-1) + n + 1$ for all m. The maps

$$\tau_k : B_0 \to D_k$$

now induce an embedding

$$\xi\colon \hat{B}_0 = \varprojlim_m \hat{B}_0/K_{k_m} \to \varprojlim D_{k_m} = \varprojlim D_i = M.$$

From Proposition 1.3 and the equality

$$B_0 = N_{0,j-1} = \{N_{0,j-2} * N_{j-1}\,;\, H_{j-2}\},$$

one deduces by a simple induction argument that $B_0 \in RF_p$, so that the maps

$$\zeta_{0,j-1}\colon N_{0,j-1} = B_0 \to \hat{B}_0 \xrightarrow{\xi} M$$

are all injective. By shifting the indices, we conclude that the natural maps

$$\zeta_{-i,h}\colon N_{-i,j} \to M$$

are injective for all $i, h \in N$, with $j = h+i+1 = p^v$ sufficiently large. Now, N is the direct limit of these groups $N_{-i,h}$ and $\alpha\colon N \to M$ is induced by the maps $\zeta_{-i,h}$. So α, and hence $\beta\colon G \to \hat{G}$ is injective, and we are done.

5. One-relator groups whose reduced one-relator groups are residually of order equal to a power of p.

THEOREM 4.1. *Suppose that*

(i) *the reduced one-relator group N_0 is residually a finite p-group;*

(ii) *r_0 belongs to $V \cap W$, where V (resp. W) is the normal subgroup of*

$$F(b_{m(b)}, \cdots, b_{M(b)}, \cdots, c_{m(c)}, \cdots, c_{M(c)})$$

generated by the first elements $b_{m(b)}, \cdots, c_{m(c)}$ (resp. last elements $b_{M(b)}, \cdots, c_{M(c)}$) of each sequence

$$(b_{m(b)}, \cdots, b_{M(b)}), \cdots, (c_{m(c)}, \cdots, c_{M(c)}).$$

Then H_0 and H_1 are (t,p)-isometric subgroups of N_1, and if L_S denotes the (free) subgroup of N_1 generated by a proper subset S of the given set of generators for N_1, then $\operatorname{gr} L_S$ is embedded in $\operatorname{gr} N_1$. Suppose furthermore that

(iii) $\operatorname{gr} L_S \cap \operatorname{gr} L_T = \operatorname{gr} L_{S\cap T}$ *(with $\operatorname{gr} L_{S\cap T} = (0)$ if $S \cap T = \phi$).*

Then $G \in R\mathscr{F}_p$.

PROOF. Let $j \geqq 1$. By (ii) the inclusion maps $H_0 \to N_{1,j}$ and $H_j \to N_{1,j}$ have left inverses, say α_j and β_j respectively. If $t \in H_0 \cap \delta_i N_{1,j}$ then $t = \alpha_j(t) \in \delta_i H_0$. Thus $H_0 \cap \delta_i N_{1,j} = \delta_i H_0$, and similarly $H_j \cap \delta_i N_{1,j} = \delta_i H_j$. Taking $j = 1$ we see that the first statement of our theorem is verified. We will now assume that

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the hypotheses (i), (ii) and (iii) are satisfied and proceed to verify condition (ii) of Theorem 3.1. One has a commutative diagram:

$$\begin{array}{ccc} N_1 & \xrightarrow{\alpha_1} & H_0 \\ \downarrow & & \downarrow \\ \hat{N}_1 & \xrightarrow[\hat{\alpha}_1]{} & \hat{H}_0 \end{array}$$

and

$$x \in \bigcap_{i \in N} H_0 \cdot \delta_i N_1 \Rightarrow x \in \bar{H}_0 = \hat{H}_0 \Rightarrow \hat{\alpha}_1(x) = x \Rightarrow \alpha_1(x) = x.$$

Thus

$$\bigcap_{i \in N} H_0 \cdot \delta_i N_1 = H_0$$

and, similarly

$$\bigcap_{i \in N} H_1 \cdot \delta_i N_1 = H_1 .$$

Since

$$N_{1,j+1} = \{N_{1,j} * N_{j+1} ; H_j\}$$

we can argue by induction on j, and deduce the result from Proposition 1.3. (The separability condition of Proposition 1.3 is verified by replacing the above diagram by

$$\begin{array}{ccc} N_{1,j} & \xrightarrow{\alpha_j} & H_0 \\ \downarrow & & \downarrow \\ \hat{N}_{1,j} & \xrightarrow{\hat{\alpha}_j} & \hat{H}_0 \end{array}$$

and using the same argument).

The third hypothesis of our theorem guarantees that for j sufficiently large, condition (iii) of Theorem 3.1 is satisfied, and the result now follows

Unfortunately, properties (i), (ii), (iii) are in general not inherited by the reduced one-relator group of a given one-relator group. Nevertheless, this theorem can be used in conjunction with Labute's Theorem 4′ to generate many examples of one-relator groups in $R\mathscr{F}_p$, whose integral group rings have no zero divisors. The simplest examples are probably of the type $(b, a), (b, (b, a)), (b, (b, (b, a))), \cdots$. Denoting the j-th term of this sequence by r_j, we see that $G_j = (a, b; r_j)$ is the reduced one-relator group of G_{j+1}. One easily proves by induction that G_j is in $R\mathscr{F}_p$ for every $j \in N$ and prime p. Moreover, $Z[G_j]$ is without zero-divisors. This result is only a very special case of Theorem 5.1 below.

6. One-relator groups defined by commutators

THEOREM 5.1.

Let r be the commutator (u, v) in the free group $F = F(x_1, \cdots, x_n, y_1, \cdots, y_m)$, where $v = v(y_1, , y_m)$ and u is in the normal subgroup of F generated by $x_1, \cdots, x_n$.

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Suppose $u = u_1^t$, $v = v_1^s$, $t, s \in \boldsymbol{Z}$, where u_1 and v_1 are not proper powers in F. Then the one-relator group

$$G = (x_1, \cdots, x_n, y_1, \cdots, y_m ; r)$$

is residually a finite p-group iff t and s are of the form $t = \pm p^h$, $s = \pm p^k$, with $k, h \geqq 0$.

PROOF. Let $Y = F(y_1, \cdots, y_m)$ and let $\hat{Y}$ (respectivelly $\hat{G}$) be the pro-p-completion of Y (respectivelly G). One has a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \to & N & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & Y & \to & 1 \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow & & \downarrow & & \\ 1 & \to & M & \to & \hat{G} & \xrightarrow{\hat{\beta}} & \hat{Y} & \to & 1 \end{array}$$

where $\beta(x_i) = 1$, $\beta(y_j) = y_j$, $i = 1, \cdots, n, j = 1, \cdots, m$. Clearly G is residually a finite p-group iff γ is injective. Our proof of the theorem will depend upon an explicit description of N as a tree product of free groups and of M as an inverse limit of generalized free products (pushouts) of free pro-p-groups.

In order to obtain presentations for N and M, we first note that the following sequences are exact:

$$1 \to F(Y \times \{x_1, \cdots, x_n\}) \xrightarrow{\delta} F(x_1, \cdots, x_n, y_1, \cdots, y_m) \xrightarrow{\varepsilon} Y \to 1$$

$$1 \to \hat{F}(\hat{Y} \times \{x_1, \cdots, x_n\}) \xrightarrow{\varphi} \hat{F}(x_1, \cdots, x_n, y_1, \cdots, y_m) \xrightarrow{\lambda} \hat{Y} \to 1$$

where $\hat{F}(\hat{Y} \times \{x_1, \cdots, x_n\})$ is the free pro-p-group generated by the *topological space* $\hat{Y} \times \{x_1, \cdots, x_n\}$ (see [6], Corollary 2.2), $\varepsilon(x_i) = 1$,

$$\varepsilon(y_j) = y_j, \lambda(x_i) = 1, \lambda(y_j) = y_j, i = 1, \cdots, n, j = 1, \cdots, m;$$

$$\delta(w, x_i) = w^{-1} x_i w \text{ for } w \in Y, i = 1, \cdots, n,$$

$$\varphi(t, x_i) = t^{-1} x_i t \text{ for every generating pair } (t, x_i), t \in \hat{Y}, i = 1, \cdots, n.$$

We can write u as a word in the pairs $(w_1, x_{i_1}), \cdots, (w_s, x_{i_s})$:

$$u = u((w_1, x_{i_1}), \cdots, (w_s, x_{i_s})), w_i \in Y, i_j \in \{1, \cdots, n\}, j = 1, \cdots, s,$$

and we will now write

$$\tilde{u}(w) = u((ww_1, x_{i_1}), \cdots, (ww_s, x_{i_s})).$$

Then δ maps $\tilde{u}(w)^{-1} \tilde{u}(vw)$ to

$$w^{-1} u^{-1} w w^{-1} v^{-1} u v w = w^{-1}(u, v) w = w^{-1} r w.$$

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So we see that N is the group generated by the pairs

$$(w, x_i)\ (w \in Y,\ i = 1, \cdots, n),$$

with defining relations

$$\tilde{u}(w) = \tilde{u}(vw)\ (w \in Y).$$

Let V be the cyclic subgroup of Y generated by v. Then V acts upon the set Y by left multiplication. We decompose Y into a disjoint union of orbits and we choose a set of representatives

$$\{w_\gamma : \gamma \in \Gamma\},$$

one from each orbit. For each $h \in \mathbf{Z}$, we let $A(w_\gamma, h)$ denote the free group on the n pairs

$$(v^h w_\gamma, x_1), \cdots, (v^h w_\gamma, x_n)$$

and for $h' > h$, we consider the generalized free product

$$P(w_\gamma, h, h') = \{A(w_\gamma, h) * A(w_\gamma, h+1) * \cdots * A(w_\gamma, h');$$
$$\tilde{u}(v^h w_\gamma) = \tilde{u}(v^{h+1} w_\gamma) = \cdots = \tilde{u}(v^{h'} w_\gamma)\}.$$

Let $C(w_\gamma) = \varinjlim_{h \in \mathbf{N}} P(w_\gamma, -h, h)$, with respect to the obvious (injective) maps. Then N is easily seen to be the free product $\ast_{\gamma \in \Gamma} C(w_\gamma)$. It can also be viewed as a graph-product of free groups on n generators, where a cyclic subgroup is amalgamated along each connected path (orbit) of the graph.

In order to describe the structure of M, we first note that for each normal subgroup U of Y, of index a power of p, one has an obvious map

$$\pi_U : \hat{F}(\hat{Y} \times \{x_1, \cdots, x_n\}) \to \hat{F}((Y/U) \times \{x_1, \cdots, x_n\})$$

and, by Gildenhugs and Lim (1972) Prop. 1.7,

$$\hat{F}(\hat{Y} \times \{x_1, \cdots, x_n\}) = \varprojlim_U \hat{F}((Y/U) \times \{x_1, \cdots, x_n\}).$$

Let R denote the closed normal subgroup of $\hat{F}(x_1, \cdots, x_n, y_1, \cdots, y_m)$ generated by r. Then $S = \varphi^{-1}(R)$ is the closed normal subgroup of $\hat{F}(\hat{Y} \times \{x_1, \cdots, x_n\})$ generated by $\tilde{u}(w)^{-1}\tilde{u}(vw)$, $(w \in Y)$. Define

$$E_U = \hat{F}((Y/U) \times \{x_1, \cdots, x_n\})/\pi_U(S)$$

Then $M = \varprojlim E_U$, with respect to the obvious maps. Let $\bar{v}$ denote the image of v in Y/U, and let $\bar{V}$ be the cyclic subgroup of Y/U generated by $\bar{v}$. Then $\bar{V}$ acts on the set Y/U by left multiplication. There exists a finite subset Λ of Γ, such that the images $\bar{w}_\lambda$ $(\lambda \in \Lambda)$ of w_λ in Y/U form a complete system of representatives for the orbits. We can now describe E_U as the pro-p-group on the set

$$(Y/U) \times \{x_1, \cdots, x_n\},$$

of generators, with defining relations

$$\tilde{u}(\bar{w}_\lambda) = \tilde{u}(v^{-h}\bar{w}_\lambda), \ (\lambda \in \Lambda, \ h = 0, 1, \cdots, c-1),$$

where c is the order of v in Y/U. I.e. E_U is just the pro-p-completion of the free product

$$\mathop{*}_{\lambda \in \Lambda} P(w_\lambda, 0, c-1)$$

Suppose now that $u = u_1^{p^h}$, $v = v_1^{p^s}$, and u_1, v_1 are not proper powers in Y. It follows from a trivial generalization of theorem 1.4, that $P(w_\lambda, 0, c-1)$, and hence $\mathop{*}_{\lambda \in \Lambda} P(w_\lambda, 0, c-1)$, is residually a finite p-group (Gildenhuys (1968), Theorem 4.1). Thus $\mathop{*}_{\lambda \in \Lambda} P(w_\lambda, 0, c-1)$ is naturally embedded in E_U. The structures of N and M have now been completely described, and it remains to show that $N \underset{\gamma}{\rightarrow} M$ is injective. So let

$$1 \neq z = z_1 z_2 \cdots z_l \in N = \mathop{*}_{\gamma \in \Gamma} C(w_\gamma)$$

be in reduced form, with $z_i \in C(w_{\gamma_i})$, $i = 1, 2, \cdots, l$. Each z_i belongs to $P(w_{\gamma_i}, h_i, h_i')$ say ($h_i' > h_i \in \mathbf{Z}$). Since we can always replace the orbit representative w_{γ_i} by $v^h w_{\gamma_i}$, we may assume without loss in generality that $h_i = 0$.

CLAIM 1. *Y contains a normal subgroup U of index a power of p, such that* (i) *the images $\bar{w}_{\gamma_i}$ and $\bar{w}_{\gamma_j}$ are in distinct orbits in Y/U whenever w_{γ_i} and w_{γ_j} are distinct in Y, ($i = 1, 2, \cdots, l$), and*

(ii) *the order c of $\bar{v}$ in Y/U is larger than all the integers $h_1', \cdots, h_l'$.*

PROOF. The family Φ of normal subgroups of index a power of p in Y is closed under finite intersections. So we need only to prove that if $a, b \in Y$ are such that for all $U \in \Phi$ there exists $n_U \in \mathbf{Z}$ with

$$ab^{-1} = v^{n_U} \bmod U,$$

then there exists an integer k such that $ab^{-1} = v^k$ in Y. Since the ring $\hat{\mathbf{Z}}_p$ of p-adic integers is compact, there exists a p-adic integer α and a chain of normal subgroups

$$U_1 \supset U_2 \supset \cdots$$

in Φ such that

$$\lim_{i \to \infty} n_{U_i} = \alpha \text{ in } \hat{Z}_p,$$

and $\cap_{i=1}^{\infty} U_i = (1)$. One easily sees then that $ab^{-1} = v^\alpha$ in $\hat{Y}$. However

$$\{v^\beta \in \hat{Y} : \beta \in \hat{\mathbf{Z}}_p\} \cap Y$$

is an abelian subgroup of the free group Y, hence is the cyclic subgroup of Y generated by some element v_2. The hypothesis on v now implies that $v = v_2^{p^t}$ for some $t \geqq 0$. There also exists an integer f such that $ab^{-1} = v_2^f = v_2^{p^t} \cdot \alpha$. It follows that α lies in $\boldsymbol{Z}$ and this completes the proof of Claim 1.

We now choose U as in the claim, and we let Λ be a finite subset of Γ, containing $\gamma_1, \cdots, \gamma_l$, and with the property that

$$\{\bar{w}_\lambda \in Y/U : \lambda \in \Lambda\}$$

is a set of orbit representatives. We may view z as an element of

$$\mathop{*}_{\lambda \in \Lambda} P(w_\lambda, 0, c-1) \subset N$$

As pointed out before, this group is embedded in its pro-p-completion E_U. Hence the image of z under the map

$$\mathop{*}_{\lambda \in \Lambda} P(w_\lambda, 0, c-1) \subset N \to M \xrightarrow{\gamma} E_U$$

is non trivial. It follows that γ is injective.

To prove the converse of the theorem, we need the following

Lemma. *Let a, b be two elements of a pro-p-group K, and let T be the closed normal subgroup of K generated by the commutators $(f, (a, b)), f \in K$. Then*

(i) $(a^\alpha, b) \equiv (a, b)^\alpha \bmod T$ *for all* $\alpha \in \hat{\boldsymbol{Z}}_p$.

(ii) *if* $\alpha \notin p\hat{\boldsymbol{Z}}_p$ *and* $(a^\alpha, b) = 1$ *in* K, *then* $(a, b) = 1$.

Proof. (i) Since T is closed, we may assume without loss in generality that $\alpha \in \boldsymbol{Z}$, and since

$$(a^{-n}, b) = (b, a^n)((b, a^n), a^{-n}),$$

we may assume without loss in generality that $n \in \boldsymbol{N}$. But one has

$$(a^{n+1}, b) = (a, b) \cdot ((a, b), a^n) \cdot (a^n, b)$$

and the result follows immediately by induction on n.

(ii) Suppose $(a, b) \neq 1$. Let K_m be the central descending series of the pro-p-group K, and suppose that m is the smallest integer such that $(a, b) \notin K_m$. Then $T \subset K_m$. Let $\beta = \alpha^{-1}$ in $\hat{\boldsymbol{Z}}_p$. Then

$$(a, b) = (a^{\alpha\beta}, b) \equiv (a^\alpha, b)^\beta = 1 \bmod K_m, \text{ the desired}$$
contradiction.

Suppose now that $u = {u_1}^k$, $1 \neq k \in \boldsymbol{Z} - p\boldsymbol{Z}$. It follows from the Lemma that the image of (u_1, v) in $\hat{G}$ is the identity. So we need only show that the image of (u_1, v) in G is not the identity. Clearly the image e of (u_1, v) in G lies in N, and from the description given of N as a generalized free product, we see that e can be identified with the element

$$\tilde{u}_1(1)^{-1}\tilde{u}_1(v) = u_1((w_1, x_{i_1}), \cdots, (w_s, x_{i_s}))^{-1} \cdot u_1((vw_1, x_i), \cdots, (vw_s, x_{i_s}))$$

of the generalized free product

$$P(1,0,1) = \{A(1,0) * A(1,1); \tilde{u}(1) = \tilde{u}(v)\}$$

contained in N, where it has been assumed that 1 is the chosen representative of the orbit $\{v^n \in Y : n \in \mathbf{Z}\}$. Clearly $e \neq 1$ in $P(1,0,1)$, and the proof is complete.

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Department of Mathematics
McGill University,
Montreal 110
Canada