KERNEL FUNCTORS FOR WHICH THE ASSOCIATED IDEMPOTENT KERNEL FUNCTOR IS STABLE

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1. Preliminaries. Let R be a ring with unity and let $\mathfrak{M}_R$ denote the category of unital right R-modules. A *preradical* γ of $\mathfrak{M}_R$ is a functor $\gamma : \mathfrak{M}_R \to \mathfrak{M}_R$ such that

(i) $\gamma(M) \subseteq M$ for each R-module M,
(ii) for $f: M \to N$, $\gamma(f)$ is the restriction of f to $\gamma(M)$.

γ is a *radical* if (iii) $\gamma(M/\gamma(M)) = 0$ for all R-modules M. γ is *left exact* or γ is a *kernel functor* in the sense of Goldman [2] if (iii)$'$ for a submodule N of an R-module M, $\gamma(N) = \gamma(M) \cap N$. A *left exact radical* is nothing but an idempotent kernel functor as defined in [2].

Let σ be an idempotent kernel functor. An R-module M is said to be *σ-torsion* (*σ-torsion-free*) if $\sigma(M) = M$ ($\sigma(M) = 0$). If we denote the classes of σ-torsion and σ-torsion-free modules by $\mathscr{T}_\sigma$ and $\mathscr{F}_\sigma$ respectively, then the pair $(\mathscr{T}_\sigma, \mathscr{F}_\sigma)$ is a hereditary torsion theory for $\mathfrak{M}_R$. More precisely: a *torsion theory* for $\mathfrak{M}_R$ is a pair $(\mathscr{T}, \mathscr{F})$ of classes of R-modules such that

$$\mathscr{F} = \{N_R \mid \mathrm{Hom}_R[K, N] = 0 \text{ for all } K \in \mathscr{T}\},$$
$$\mathscr{T} = \{M_R \mid \mathrm{Hom}_R[M, L] = 0 \text{ for all } L \in \mathscr{F}\}.$$

$\mathscr{T}$ is closed under homomorphic images, direct sums and extensions. $\mathscr{F}$ is closed under submodules, direct products and extensions.

The torsion theory $(\mathscr{T}, \mathscr{F})$ is said to be *hereditary* if $\mathscr{T}$ (or equivalently $\mathscr{F}$) is closed under submodules (injective envelopes).

We have a one-to-one correspondence between idempotent kernel functors on $\mathfrak{M}_R$ and hereditary torsion theories for $\mathfrak{M}_R$. The correspondence is given by

$$\sigma \to \mathscr{T}_\sigma = \{M_R \mid \sigma(M) = M\}$$

with the inverse correspondence $\mathscr{T} \to \sigma_{\mathscr{T}}$, where, for an R-module M, $\sigma_{\mathscr{T}}(M) = \Sigma\{N \mid N$ is a submodule of M and $N \in \mathscr{T}\}$. For details, we refer the reader to Goldman [2], Lambek [3] and Stenstrom [7].

If γ_1 and γ_2 are preradicals, $\gamma_1 \leqq \gamma_2$ if $\gamma_1(M) \subseteq \gamma_2(M)$ for all R-modules M.

For the proof of the following proposition, we refer to Stenstrom [7, Proposition 1.1] or Goldman [2, Proposition 1.1, Theorem 1.6].

PROPOSITION 1.1. *With each preradical γ, one can associate a radical to be denoted by $\bar{\gamma}$, such that*

(i) $\gamma \leqq \bar{\gamma}$,
(ii) *$\bar{\gamma}$ is a radical,*
(iii) *if μ is a radical and $\gamma \leqq \mu$, then $\bar{\gamma} \leqq \mu$.*

Moreover, if γ is a kernel functor, so is $\bar{\gamma}$. That is, $\bar{\gamma}$ defines an idempotent kernel functor.

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$\bar{\gamma}$ is obtained by transfinite induction as follows: let M be an R-module. For a non-limit ordinal β, define γ_β by $\gamma_\beta(M)/\gamma_{\beta-1}(M) = \gamma(M/\gamma_{\beta-1}(M))$ and for a limit ordinal β, define γ_β by $\gamma_\beta(M) = \sum_{\alpha<\beta} \gamma_\alpha(M)$. This yields an ascending sequence of preradicals. $\bar{\gamma}$ is now given by $\bar{\gamma}(M) = \sum_\beta \gamma_\beta(M)$. Equivalently, we can define $\bar{\gamma}(M) = \cap \{N \mid N \subseteq M \text{ and } \gamma(M/N) = 0\}$. We note that $\gamma(M) = 0$ implies that $\bar{\gamma}(M) = 0$.

2. Main result and applications. Let $\mathscr{E}$ be a class of R-modules. By the *hereditary torsion class generated by* $\mathscr{E}$ is meant the smallest class $\mathscr{T}_{\mathscr{E}}$ containing $\mathscr{E}$ such that $\mathscr{T}_{\mathscr{E}}$ is a hereditary torsion class for some hereditary torsion theory.

LEMMA 2.1. *Let γ be a kernel functor, $\mathscr{E}_\gamma = \{M \mid \gamma(M) = M\}$ and $\mathscr{T}_{\mathscr{E}_\gamma}$ the hereditary torsion class generated by $\mathscr{E}_\gamma$. Then $\mathscr{T}_{\mathscr{E}_\gamma} = \mathscr{T}_{\bar{\gamma}}$, where $\mathscr{T}_{\bar{\gamma}}$ is the class of torsion modules corresponding to the idempotent kernel functor $\bar{\gamma}$.*

Proof. $\bar{\gamma}$ is the smallest idempotent kernel functor larger than γ, by Proposition 1.1. Since there is a one-to-one correspondence between idempotent kernel functors on $\mathfrak{M}_R$ and hereditary torsion theories for $\mathfrak{M}_R$, $\mathscr{T}_{\bar{\gamma}}$ must correspond to the smallest hereditary torsion class containing $\mathscr{E}_\gamma$. Thus $\mathscr{T}_{\mathscr{E}_\gamma} = \mathscr{T}_{\bar{\gamma}}$.

LEMMA 2.2. *Let γ be a kernel functor. Then for each R-module M, $\gamma(M)$ is an essential submodule of $\bar{\gamma}(M)$.*

Proof. Let $N \subseteq \bar{\gamma}(M)$ be such that $N \cap \gamma(M) = 0$. We show that $N = 0$. Since γ is a kernel functor, $\gamma(N) = N \cap \gamma(M) = 0$. This implies that $\bar{\gamma}(N) = 0$. But $N \subseteq \bar{\gamma}(M)$ and hence $\bar{\gamma}(N) = N$. Thus $N = 0$ and the lemma follows.

DEFINITION 2.3. A hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ is said to be *stable* if $\mathscr{T}$ is closed under essential extensions. We shall call an idempotent kernel functor σ *stable* if the corresponding hereditary torsion theory is stable. (See Stenstrom [7, §4] and Gabriel [**1**].)

THEOREM 2.4. *Let γ be a kernel functor such that $\bar{\gamma}$ is stable. Then the following statements are equivalent.*

(i) *$\gamma(R)$ is an essential right ideal of R.*
(ii) *$\gamma(M)$ is an essential submodule of M for all R-modules M.*
(iii) *If $M \neq 0$, then $\gamma(M) \neq 0$.*
(iv) *$\bar{\gamma}(R) = R$.*
(v) *$\bar{\gamma}(M) = M$ for all M; that is $\bar{\gamma}$ is the identity functor on $\mathfrak{M}_R$.*
(vi) *Each hereditary torsion theory for $\mathfrak{M}_R$ is generated by a class of γ-torsion modules.* (Here an R-module M is γ-torsion if $\gamma(M) = M$.)

Proof. $(i) \Rightarrow (iv)$. $\bar{\gamma}(\gamma(R)) = \gamma(R)$. Since $\bar{\gamma}$ is stable, we have $\bar{\gamma}(R) = R$.

$(iv) \Rightarrow (v)$. Since $\mathscr{T}_{\bar{\gamma}}$ is closed under homomorphic images and direct sums, $\bar{\gamma}(R) = R$ implies that $\bar{\gamma}(M) = M$ for all M.

$(v) \Rightarrow (iv)$. Trivial.

$(v) \Rightarrow (ii)$. By Lemma 2.2, $\gamma(M)$ is an essential submodule of $\bar{\gamma}(M)$. Thus since $\bar{\gamma}(M) = M$, the implication follows.

$(ii) \Rightarrow (iii)$. Trivial.

$(iii) \Rightarrow (i)$. Suppose not. Then there exists a non-zero right ideal I such that $\gamma(R) \cap I = 0$. Now $\gamma(I) \subseteq \gamma(R)$. This implies that $\gamma(I) = 0$, a contradiction.

$(v) \Rightarrow (vi)$. The class $\mathscr{E} = \{M \mid \gamma(M) = M\}$ is closed under submodules and factor modules. By Lemma 2.1, the hereditary torsion class generated by $\mathscr{E}$ is all of $\mathfrak{M}_R$. Now let $\mathscr{T}$ be a hereditary torsion class. Then by Stenstrom [7, Exercise 3, p. 11] $\mathscr{T}$ is generated by $\mathscr{T} \cap \mathscr{E}$. That is $\mathscr{T}$ is generated by a class of γ-torsion modules.

$(vi) \Rightarrow (v)$. Take $\mathscr{T} = \mathfrak{M}_R$. Then $\mathfrak{M}_R$ is generated by a class of γ-torsion modules. By Lemma 2.1, $\bar{\gamma}$ is the identity functor on $\mathfrak{M}_R$. Thus $\bar{\gamma}(M) = M$ for all R-modules M.

Let R be a ring and M an R-module. The singular submodule of M, to be denoted by $Z_R(M)$, is the set of all elements of M which are annihilated by essential right ideals of R. $Z_R(\ )$ defines a kernel functor on $\mathfrak{M}_R$. The idempotent kernel functor corresponding to $Z_R(\ )$ is called the *Goldie torsion functor*. It will be denoted by $\mathscr{G}$.

We note that the Goldie torsion class is generated by the class of modules of the form A/B, where A is an essential extension of B. Moreover, the Goldie torsion functor is stable.

As a special case of Theorem 2.4 we have the following result.

PROPOSITION 2.5. *Let R be a ring. Then the following statements are equivalent.*

(i) $Z_R(R_R)$ *is an essential right ideal of R.*
(ii) $Z_R(M)$ *is an essential submodule for each R-module M.*
(iii) $Z_R(M) \neq 0$ *for every non-zero R-module M.*
(iv) $\mathscr{G}(R) = R$.
(v) $\mathscr{G}(M) = M$ *for each R-module M.*
(vi) *Each hereditary torsion theory for* $\mathfrak{M}_R$ *is generated by a class of singular modules.*

REMARK. Using different methods, Ming [4] has also established the equivalence of (i), (ii) and (iii).

PROPOSITION 2.6. *Let R be a commutative noetherian ring and let* γ *be a kernel functor. Then the following statements are equivalent.*

(i) $\gamma(R)$ *is an essential ideal of R.*
(ii) *For each R-module M,* $\gamma(M)$ *is an essential submodule of M.*
(iii) $\gamma(M) \neq 0$ *for a non-zero module M.*
(iv) *Each hereditary torsion theory for* $\mathfrak{M}_R$ *is generated by a class of* γ*-torsion modules.*

Proof. By a result of Gabriel [1], every hereditary torsion class for a commutative noetherian ring is stable. The result follows from Theorem 2.4.

As an application we have

PROPOSITION 2.7. *Let R be a commutative noetherian ring. Then the following statements are equivalent.*

(i) *Socle (R) is an essential ideal of R.*
(ii) *R is an artinian ring.*

(Here, for an R-module M, Socle (M) is the sum of all simple submodules of M.)

Proof. $(i) \Rightarrow (ii)$. Socle ( ) defines a kernel functor on $\mathfrak{M}_R$. From the last theorem, Socle $(M) \neq 0$ for each non-zero module M. Define an ascending sequence of ideals as follows: $I_0 = \text{Socle}(R)$ and $I_{n+1} \supseteq I_n$ with $I_{n+1}/I_n = \text{Socle}(R/I_n)$. Either $R = I_m$ for some m or we get a strictly ascending sequence $I_0 \subsetneqq I_1 \subsetneqq \ldots$, since Socle $(R/I_n) \neq 0$. Since R is noetherian, this sequence terminates, say at m. Thus $R = I_m$ for some integer m. Now I_{n+1}/I_n has finite length for each n. Hence R itself is of finite length. Thus R is artinian.
$(ii) \Rightarrow (i)$. Trivial.

REMARK. The above sharpens a result of Nita in **[6]** where, using different methods, the above equivalence is proved assuming further that R is an S-ring in the sense of Morita **[5]**.

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