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On ℓ-independence for the étale cohomology of rigid spaces over local fields

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Abstract

We investigate the action of the Weil group on the compactly supported ℓ-adic étale cohomology groups of rigid spaces over a local field. We prove that the alternating sum of the traces of the action is an integer and is independent of ℓ when either the rigid space is smooth or the characteristic of the base field is equal to 0. We modify the argument of T. Saito to prove a result on ℓ-independence for nearby cycle cohomology, which leads to our ℓ-independence result for smooth rigid spaces. In the general case, we use the finiteness theorem of Huber, which requires the restriction on the characteristic of the base field.

1. Introduction

Let K be a complete discrete valuation field with finite residue field $\mathbb{F}_q$ and $\overline{K}$ a separable closure of K. We denote by Fr_q the geometric Frobenius element (the inverse of the qth power map) in $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. The Weil group W_K of K is defined as the inverse image of the subgroup $\langle \mathrm{Fr}_q \rangle \subset \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ by the canonical map $\mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. For $\sigma \in W_K$, let $n(\sigma)$ be the integer such that the image of σ in $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is $\mathrm{Fr}_q^{n(\sigma)}$. Put $W_K^+ = \{\sigma \in W_K \mid n(\sigma) \geqslant 0\}$.

Let X be a separated rigid space over K. We consider the action of W_K on the compactly supported ℓ-adic cohomology group $H_c^i(\mathsf{X} \otimes_K \overline{K}, \mathbb{Q}_\ell)$, where ℓ is a prime number that does not divide q. This cohomology group is defined by using the étale site of X (cf. [Hub96, Hub98b]). Our main theorem is the following.

Theorem 1.1 (Theorems 7.1.6 and 7.2.3). *Let X be a quasi-compact separated rigid space over K. Assume one of the following conditions:*

(i) *the rigid space X is smooth over K;*

(ii) *the characteristic of K is equal to 0.*

Then for every $\sigma \in W_K^+$, the number

$$\sum_{i=0}^{2\dim \mathsf{X}} (-1)^i \operatorname{Tr}(\sigma_*; H_c^i(\mathsf{X} \otimes_K \overline{K}, \mathbb{Q}_\ell))$$

is an integer that is independent of ℓ.

Note that $H_c^i(\mathsf{X} \otimes_K \overline{K}, \mathbb{Q}_\ell)$ is known to be a finite-dimensional $\mathbb{Q}_\ell$-vector space when one of the above conditions is satisfied [Hub96, Propositions 6.1.1 and 6.2.1; Hub98a, Corollary 2.3; Hub98b, Theorem 3.1]. In the previous paper, under the same assumption, the author proved that every eigenvalue of the action of $\sigma \in W_K^+$ on $H_c^i(\mathsf{X} \otimes_K \overline{K}, \mathbb{Q}_\ell)$ is a Weil number [Mie06, Theorems 4.2 and 5.5].

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For a scheme over K, the property in Theorem 1.1 was proven by Ochiai [Och99, Theorem 2.4]. However, it seems difficult to prove Theorem 1.1 by the same method as in [Och99], since the induction on the dimension does not work well. In this paper, we modify the method in [SaT03], which treats the composite action of an element of W_K and a correspondence.

We sketch the outline of the paper. In § 2, we derive ℓ-independence of the alternating sum of the traces of the action of a correspondence from Fujiwara's trace formula [Fuj97]. This result seems well known, but we include its proof for completeness. In § 3, by using localized Chern characters, we prove a lemma which is a refined version of [SaT03, Lemma 2.17]. This lemma is needed in § 5. In § 4, we introduce partially supported cohomology and investigate its several functorial properties. In terms of partially supported cohomology, we can describe the action of a correspondence on the compactly supported cohomology of a scheme which is not necessarily proper. The required properties of nearby cycles and their cohomology are also included in this section. In § 5, we introduce a spectral sequence converging to nearby cycle cohomology, which is a generalization of the weight spectral sequence studied in [RZ82] and [SaT03]. By the same method as in [SaT03, §§ 2.3 and 2.4], we can prove the compatibility of the spectral sequence with the action of a correspondence. In § 6, we prove ℓ-independence for nearby cycle cohomology by using the result in § 5 and de Jong's alteration [deJ96]. The method is almost the same as that in [SaT03, § 3]. Several applications to algebraic geometry (not to rigid geometry) are also included (Theorems 6.2.2 and 6.3.8). Finally in § 7 we give a proof of our main result. When X is smooth over K, we can reduce our theorem to the case where X is the Raynaud generic fiber of the completion of a scheme over $\mathcal{O}_K$ with smooth generic fiber (though the reduction does not seem so immediate in comparison with [Mie06]). In this case we can use the result in § 6. Finally, assuming that the characteristic of K is 0, we prove our theorem for a general X by induction on $\dim \mathsf{X}$. In this process, we need the finiteness theorem of Huber [Hub98a].

Notation. Let K be a field. For a scheme X (or a rigid space) over K and an extension L of K, we denote the base change $X \times_{\operatorname{Spec} K} \operatorname{Spec} L$ by X_L. For a scheme X of finite type over K, we denote the group of k-cycles on X by $Z_k(X)$ and the kth Chow group (the group of k-cycles modulo rational equivalences) by $\mathrm{CH}_k(X)$. Let X be a scheme of finite type over K and Y be a closed subscheme of X. Put $d = \dim X$. We denote by $\mathrm{cl}_Y^X\colon \mathrm{CH}_{d-k}(Y) \longrightarrow H_Y^{2k}(X, \mathbb{Q}_\ell(k))$ the cycle map defined in [Del77, cycle], where ℓ is a prime number distinct from the characteristic of K.

Convention on correspondences. Let K be a field and ℓ a prime number distinct from the characteristic of K. Put $\Lambda = \mathbb{Q}_\ell$. For schemes X and Y separated of finite type over K, a correspondence between X and Y is a morphism $\gamma\colon \Gamma \longrightarrow X \times Y$, where Γ is a scheme separated of finite type over K. A morphism $f\colon X \longrightarrow X$ can be regarded as the correspondence $f \times \mathrm{id}\colon X \longrightarrow X \times X$. Note that this convention is different from that in [SaT03], while it is the same as that in [Ill77] and [Fuj97]. We sometimes assume that γ is a closed immersion.

Let $\gamma\colon \Gamma \longrightarrow X \times Y$ be a correspondence such that Y is smooth and purely d-dimensional. Put $c = \dim \Gamma$ and $\gamma_i = \mathrm{pr}_i \circ \gamma$. When γ_1 is proper, Γ induces a homomorphism between cohomology groups

$$\Gamma^*\colon H_c^q(X, \Lambda) \xrightarrow{\mathrm{pr}_1^*} H_c^q(\Gamma, \Lambda) \xrightarrow{\mathrm{pr}_{2*}} H_c^{q+2d-2c}(Y, \Lambda(d-c)).$$

More generally, for $\alpha \in Z_k(\Gamma)$, we can define a homomorphism

$$\alpha^*\colon H_c^q(X, \Lambda) \longrightarrow H_c^{q+2d-2k}(Y, \Lambda(d-k)).$$

It is easy to see that the map α^* depends only on the rational equivalence class of α. Therefore for an element α of the Chow group $\mathrm{CH}_k(\Gamma)$, we can define the map

$$\alpha^*\colon H_c^q(X, \Lambda) \longrightarrow H_c^{q+2d-2k}(Y, \Lambda(d-k)).$$

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2. On ℓ-independence for schemes over finite fields

2.1 The ℓ-independence

2.1.1 In this section we give a result on ℓ-independence for schemes over finite fields. Though the result seems well known for specialists, we include its proof for completeness.

Theorem 2.1.2. *Let X be a separated smooth purely d-dimensional scheme of finite type over $\mathbb{F}_q$ and $\gamma\colon \Gamma \longrightarrow X \times X$ a correspondence such that Γ is purely d-dimensional. We denote the characteristic of $\mathbb{F}_q$ by p. Assume that $\gamma_1\colon \Gamma \longrightarrow X$ is proper. Then the number*

$$\mathrm{Tr}(\Gamma^*; H_c^*(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) = \sum_{i=0}^{2d} (-1)^i \,\mathrm{Tr}(\Gamma^*; H_c^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell))$$

lies in $\mathbb{Z}[1/p]$ and is independent of ℓ.

Proof. Let $\gamma^{(n)}\colon \Gamma^{(n)} \longrightarrow X \times X$ be the correspondence satisfying $\gamma_1^{(n)} = \mathrm{Fr}_X^n \circ \gamma_1$ and $\gamma_2^{(n)} = \gamma_2$, where Fr_X is the qth power Frobenius morphism. Take a compactification $\overline{\gamma}\colon \overline{\Gamma} \longrightarrow \overline{X} \times \overline{X}$ of $\gamma\colon \Gamma \longrightarrow X \times X$ and define $\overline{\gamma}^{(n)}\colon \overline{\Gamma}^{(n)} \longrightarrow \overline{X} \times \overline{X}$ in the same way. We may assume that $D = \overline{X} \setminus X$ is a Cartier divisor of $\overline{X}$. Then for sufficiently large n, any connected component of $\overline{\Gamma}^{(n)} \cap \Delta_{\overline{X}}$ which meets D is (set-theoretically) contained in D (here we identify $\Delta_{\overline{X}}$ and $\overline{X}$). This easily follows from Fujiwara's result on contractility [Fuj97, Propositions 5.3.5 and 5.4.1]. See also [Var05, Theorem 2.1.3 and Lemma 2.2.3].

By this fact and Fujiwara's trace formula [Fuj97, Propositions 5.3.4 and 5.4.1], there exists an integer N such that for every $n \geqslant N$ and ℓ the equality

$$\mathrm{Tr}(\Gamma^{(n)*}; H_c^*(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) = (\Gamma^{(n)}, \Delta_X)_{X\times X}$$

holds. The right-hand side denotes the intersection number (note that $\Gamma^{(n)} \cap \Delta_X$ is proper over $\mathbb{F}_q$ for sufficiently large n by the argument above), which is an integer and is independent of ℓ. Since $\Gamma^{(n)*} = \Gamma^* \circ (\mathrm{Fr}_X^*)^n$, the number $\mathrm{Tr}(\Gamma^* \circ (\mathrm{Fr}_X^*)^n; H_c^*(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell))$ is an integer that is independent of ℓ for $n \geqslant N$.

Let $\alpha_{\ell,i,1}, \ldots, \alpha_{\ell,i,m_i}$ and $\lambda_{\ell,i,1}, \ldots, \lambda_{\ell,i,m_i}$ be eigenvalues of Γ^* and Fr_X^* on $H_c^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ respectively. By [DK73, Exposé XXI, Corollaire 5.5.3], $\lambda_{\ell,i,k}$ and $q^d\lambda_{\ell,i,k}^{-1}$ are algebraic integers. Since Γ^* and Fr_X^* commute with each other, the trace of $\Gamma^* \circ (\mathrm{Fr}_X^*)^n$ on $H_c^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ is equal to $\sum_{k=1}^{m_i} \alpha_{\ell,i,k}\lambda_{\ell,i,k}^n$ with $\lambda_{\ell,i,1}, \ldots, \lambda_{\ell,i,m_i}$ permuted suitably. Therefore the theorem follows from the subsequent lemma. $\square$

Lemma 2.1.3. *Let p be a prime number, K a field of characteristic 0 and $\alpha_1, \ldots, \alpha_m, \lambda_1, \ldots, \lambda_m$ elements of K such that $\lambda_k \neq 0$ for every k and $\lambda_i \neq \lambda_j$ for $i \neq j$. Put $b_n = \sum_{k=1}^m \alpha_k \lambda_k^n$. Assume the following conditions.*

(i) *There exists an integer d_0 such that λ_k and $p^{d_0}\lambda_k^{-1}$ are integral over $\mathbb{Z}$ for every k.*

(ii) *There exists an integer N such that $b_n \in \mathbb{Z}$ for every $n \geqslant N$.*

Then α_k is algebraic over $\mathbb{Q}$ and $b_n \in \mathbb{Z}[1/p]$ for every non-negative integer n.

Proof. We may assume that K is algebraically closed. Denote the algebraic closure of $\mathbb{Q}$ in K by $\overline{\mathbb{Q}}$ and the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$ by $\overline{\mathbb{Z}}$. By the first condition above, $\lambda_k \in \overline{\mathbb{Z}}[1/p]^\times$ for every k.

Consider the matrices $L, V_n \in GL_m(K)$ defined by

$$L = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix}, \quad V_n = \begin{pmatrix} \lambda_1^n & \lambda_2^n & \cdots & \lambda_m^n \\ \lambda_1^{n+1} & \lambda_2^{n+1} & \cdots & \lambda_m^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n+m-1} & \lambda_2^{n+m-1} & \cdots & \lambda_m^{n+m-1} \end{pmatrix}.$$

It is easy to see that $V_r L^s = V_{r+s}$ for every integers r, s. Define $\mathbf{a}, \mathbf{b}_n \in K^m$ by

$$\mathbf{a} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}, \quad \mathbf{b}_n = \begin{pmatrix} b_n \\ \vdots \\ b_{n+m-1} \end{pmatrix}.$$

Note that $\mathbf{b}_n = V_n \mathbf{a}$. By the second assumption, $b_n \in \mathbb{Z}^m$ for $n \geqslant N$. Thus all the entries of $\mathbf{a} = V_N^{-1}\mathbf{b}_N$ are in $\overline{\mathbb{Q}}$, which is the first part of the lemma.

Put $M = V_N^{-1}(\mathbb{Z}^m)$. Then $L^n\mathbf{a} \in M$ for $n \geqslant 0$. Let M_0 be the $\mathbb{Z}$-submodule of M generated by $\{L^n\mathbf{a}\}_{n\geqslant 0}$, which is evidently a free $\mathbb{Z}$-module of finite rank. Consider the minimal polynomial $\mu(T) = T^d + a_1T^{d-1} + \cdots + a_d \in \mathbb{Q}[T]$ of $L \in \mathrm{End}(M_0 \otimes \mathbb{Q})$. Since it divides the characteristic polynomial of $L \in \mathrm{End}(M_0)$, it lies in $\mathbb{Z}[T]$. Moreover, in $\overline{\mathbb{Q}}[T]$, it divides the characteristic polynomial $\prod_{i=1}^m (T - \lambda_i) \in \overline{\mathbb{Z}}[1/p][T]$ of the matrix L, therefore in $\overline{\mathbb{Z}}[1/p][T]$ since $\mu(T)$ is monic. Hence in $\overline{\mathbb{Z}}[1/p]$, a_d divides $\lambda_1 \cdots \lambda_m \in \overline{\mathbb{Z}}[1/p]^\times$. This implies that $a_d \in \mathbb{Z} \cap \overline{\mathbb{Z}}[1/p]^\times$ and we may conclude that $a_d = \pm p^k$ for some integer k.

Now we will prove $b_n \in \mathbb{Z}[1/p]^m$ by descending induction on n. By the assumption, it holds for $n \geqslant N$. Let $n \geqslant d$ be an integer. We have $L^n\mathbf{a} + a_1L^{n-1}\mathbf{a} + \cdots + a_dL^{n-d}\mathbf{a} = 0$. Multiplying by V_0, we have $\mathbf{b}_n + a_1\mathbf{b}_{n-1} + \cdots + a_d\mathbf{b}_{n-d} = 0$. Therefore the assumption $\mathbf{b}_n, \mathbf{b}_{n-1}, \dots, \mathbf{b}_{n-d+1} \in \mathbb{Z}[1/p]^m$ implies

$$\mathbf{b}_{n-d} = -\frac{1}{a_d}(\mathbf{b}_n + a_1\mathbf{b}_{n-1} + \cdots + a_{d-1}\mathbf{b}_{n-d+1}) \in \mathbb{Z}[1/p]^m.$$

This completes the proof. □

Remark 2.1.4. In [BE05], Bloch and Esnault gave another proof of Theorem 2.1.2 by using the theory of relative motivic cohomology defined by Levine. They also prove the integrality of the alternating sum of the trace in Theorem 2.1.2. They only consider the case where X has a good compactification, but we can easily reduce the general case to their case by de Jong's alteration (cf. (6.1.6)).

3. Complements on cycle classes

3.1 Localized Chern characters

3.1.1 Here we briefly recall localized Chern characters. Let S be a noetherian regular scheme. By an arithmetic S-scheme, we mean a separated scheme of finite type over S. Let ℓ be a prime number which is invertible in S and denote $\mathbb{Q}_\ell$ by Λ.

3.1.2 Let X be a purely d-dimensional arithmetic S-scheme and $i\colon Y \longrightarrow X$ a closed subscheme of X. Let $\mathcal{E}_\bullet$ be a bounded complex of locally free $\mathcal{O}_X$-module which is exact over $X \setminus Y$. With such $\mathcal{E}_\bullet$, we associate $\mathrm{ch}_Y^X(\mathcal{E}_\bullet) \in \mathrm{CH}_{d-\bullet}(Y)_\mathbb{Q}$, called the *localized Chern character* [Ful98, § 18.1]. We denote the degree-k part of $\mathrm{ch}_Y^X(\mathcal{E}_\bullet)$ by $\mathrm{ch}_{k,Y}^X(\mathcal{E}_\bullet) \in \mathrm{CH}_{d-k}(Y)_\mathbb{Q}$. Note that in [Ful98, § 18.1], $\mathrm{ch}_Y^X(\mathcal{E}_\bullet)$ is defined as an element of $\mathrm{CH}(Y \to X)_\mathbb{Q}$. In the notation there, $\mathrm{ch}_Y^X(\mathcal{E}_\bullet) \in \mathrm{CH}_{d-\bullet}(Y)_\mathbb{Q}$ here should be denoted by $\mathrm{ch}_Y^X(\mathcal{E}_\bullet) \cap [X]$.

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3.1.3 We need the following property of ch^X_Y: assume that $S = \operatorname{Spec} K$ where K is a field, X is smooth over S and Y is irreducible. Let $\mathcal{E}_\bullet \longrightarrow i_*\mathcal{O}_Y$ be a resolution of $i_*\mathcal{O}_Y$ consisting of locally free $\mathcal{O}_X$-modules (such a resolution always exists since X is regular). Put $d' = \dim Y$. Then $\mathrm{ch}^X_{d-d',Y}(\mathcal{E}_\bullet) = [Y] \in \mathrm{CH}_{d'}(Y)_{\mathbb{Q}}$. This is a corollary of the Riemann–Roch theorem [Ful98, Theorem 18.3 (3), (5)].

3.1.4 Let the notation be the same as in paragraph 3.1.2. We can associate the cohomology class $\mathrm{ch}^X_{\ell,k,Y}(\mathcal{E}_\bullet) \in H^{2k}_Y(X, \Lambda(k))$ for each k, which is also called the *localized Chern character* (cf. [Ive76]).

3.1.5 We list some properties of $\mathrm{ch}^X_{\ell,k,Y}$ needed later.

(i) The localized Chern character $\mathrm{ch}^X_{\ell,k,Y}(\mathcal{E}_\bullet)$ is compatible with any pull-back.

(ii) Assume that $S = \operatorname{Spec} K$ where K is a field and X is smooth over S. Then we have

$$\mathrm{cl}^X_Y(\mathrm{ch}^X_{k,Y}(\mathcal{E}_\bullet)) = \mathrm{ch}^X_{\ell,k,Y}(\mathcal{E}_\bullet) \text{ (cf. [Ful98, Example 19.2.6]).}$$

3.2 A lemma on cycle classes

3.2.1 Let $S = \operatorname{Spec} A$ be a henselian trait and ℓ a prime number that is invertible in S. We denote the generic (respectively special) point of S by η (respectively s). For an S-scheme X, we denote its generic (respectively special) fiber by X_η (respectively X_s).

An arithmetic S-scheme X is said to be *strictly semistable* if it is Zariski locally on X, étale over $\operatorname{Spec} A[T_0, \dots, T_n]/(T_0 \cdots T_r - \pi)$ for a uniformizer π of A and integers n, r with $0 \leqslant r \leqslant n$. Let $D_1, \dots, D_m$ be irreducible components of X_s. We put $D_I = \bigcap_{i \in I} D_i$ for $I \subset \{1, \dots, m\}$ and $D^{(p)} = \coprod_{I \subset \{1,\dots,m\}, \#I = p+1} D_I$ for a non-negative integer p. We write $a_i \colon D_i \hookrightarrow X$ and $a^{(p)} \colon D^{(p)} \longrightarrow X$ for the canonical morphisms.

Lemma 3.2.2. *Let X be a strictly semistable S-scheme of purely relative dimension d and Y a closed subscheme of X with $(d-k)$-dimensional generic fiber. Assume that Y is flat over S. Then there exists a cohomology class $\xi_\ell \in H^{2k}_Y(X, \mathbb{Q}_\ell(k))$ for each prime number ℓ which is invertible in S satisfying the following conditions:*

(i) $\xi_\ell|_{X_\eta} = \mathrm{cl}^{X_\eta}_{Y_\eta}(Y_\eta) \in H^{2k}_{Y_\eta}(X_\eta, \mathbb{Q}_\ell(k))$;

(ii) $\xi_\ell|_{D^{(p)}} = \mathrm{cl}^{D^{(p)}}_{D^{(p)} \cap Y}(a^{(p)!}[Y]) \in H^{2k}_{D^{(p)} \cap Y}(D^{(p)}, \mathbb{Q}_\ell(k))$.

Here we are abusing notation since $D^{(p)}$ is not a subscheme of X.

Proof. Take a resolution $\mathcal{E}_\bullet \longrightarrow i_*\mathcal{O}_Y$ of $i_*\mathcal{O}_Y$ by locally free $\mathcal{O}_X$-modules, where i denotes the canonical closed immersion $Y \hookrightarrow X$. Put $\xi_\ell = \mathrm{ch}^X_{\ell,k,Y}(\mathcal{E}_\bullet)$. Then it satisfies the first condition above by Paragraphs 3.1.3 and 3.1.5.

We will prove that the second condition holds. Since the cycle map for a scheme over a field is compatible with the refined Gysin map, we may assume $p = 0$. In other words, we should prove $\xi_\ell|_{D_i} = \mathrm{cl}^{D_i}_{D_i \cap Y}(a_i^![Y])$. Since Y is flat, $D_i \cap Y \hookrightarrow Y$ is a Cartier divisor. Thus $a_i^![Y] = [D_i \cap Y]$ in $\mathrm{CH}_{d-k}(D_i \cap Y)$. Moreover Y and D_i are Tor-independent over X and $\mathcal{E}_\bullet|_{D_i}$ is a resolution of $\mathcal{O}_{D_i \cap Y}$ by locally free $\mathcal{O}_{D_i}$-modules. Therefore by Paragraphs 3.1.3 and 3.1.5, we have

$$\xi_\ell|_{D_i} = \mathrm{ch}^{D_i}_{\ell,k,D_i\cap Y}(\mathcal{E}_\bullet|_{D_i}) = \mathrm{cl}^{D_i}_{D_i \cap Y}(\mathrm{ch}^{D_i}_{k,D_i \cap Y}(\mathcal{E}_\bullet|_{D_i})) = \mathrm{cl}^{D_i}_{D_i\cap Y}(D_i \cap Y) = \mathrm{cl}^{D_i}_{D_i \cap Y}(a_i^![Y]).$$

This completes the proof. ∎

Remark 3.2.3. We can prove that the class ξ_ℓ constructed above coincides with the refined cycle class of Y defined by using the absolute purity theorem of Gabber (cf. [Fuj02]). In particular, we have the canonical element $\xi'_\ell \in H^{2k}_Y(X, \mathbb{Z}_\ell(k))$ whose image in $H^{2k}_Y(X, \mathbb{Q}_\ell(k))$ is equal to ξ_ℓ.

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Remark 3.2.4. In the same way, we can remove the denominator $k!$ in [SaT03, Lemma 2.17].

4. Partially supported cohomology and nearby cycle cohomology

4.1 Partially supported cohomology

4.1.1 Let K be a separably closed field and ℓ a prime number that does not divide the characteristic of K. Put $\Lambda = \mathbb{Q}_\ell$.

4.1.2 Consider a triple (X, U_1, U_2) of schemes over K such that

$$U_1 \text{ is an open subscheme of } X \text{ and } U_2 \text{ is an open subscheme of } U_1. \tag{$\star$}$$

We call such a triple a $\star$-*triple*. The scheme X is often assumed to be proper over K. We denote the canonical open immersions $U_1 \hookrightarrow X$ and $U_2 \hookrightarrow U_1$ by j_1 and j_{12} respectively. Put $j_2 = j_1 \circ j_{12}$. A morphism $f\colon (X, U_1, U_2) \longrightarrow (Y, V_1, V_2)$ of $\star$-triples means a triple of morphisms $f\colon X \longrightarrow Y$, $f_1\colon U_1 \longrightarrow V_1$, and $f_2\colon U_2 \longrightarrow V_2$, which makes the following diagram commutative.

$$\begin{tikzcd}
U_2 \arrow[r] \arrow[d, "f_2"] & U_1 \arrow[r] \arrow[d, "f_1"] & X \arrow[d, "f"] \\
V_2 \arrow[r] & V_1 \arrow[r] & Y
\end{tikzcd}$$

DEFINITION 4.1.3. Let (X, U_1, U_2) be a $\star$-triple and $\mathcal{F} \in \operatorname{obj} D_c^b(U_2, \Lambda)$. We define the *partially supported cohomology* $H^q_{!*}(X, U_1, U_2; \mathcal{F})$ as $H^q(X, j_{1!}Rj_{12*}\mathcal{F})$ and $H^q_{*!}(X, U_1, U_2; \mathcal{F})$ as $H^q(X, Rj_{1*}j_{12!}\mathcal{F})$. Note that if X is proper, $H^q_{!*}(X, U, U; \mathcal{F}) = H^q_c(U, \mathcal{F})$. Needless to say, $H^q_{*!}(X, U_1, U_2; \mathcal{F}) = H^q(U_1, j_{12!}\mathcal{F})$ is independent of X.

4.1.4 Let $f\colon (X, U_1, U_2) \longrightarrow (Y, V_1, V_2)$ be a morphism of $\star$-triples and $k_1\colon V_1 \hookrightarrow X$, $k_{12}\colon V_2 \hookrightarrow V_1$ the canonical open immersions. Put $k_2 = k_1 \circ k_{12}$. Consider the diagram below.

$$\begin{tikzcd}
U_2 \arrow[r, "j_{12}"] \arrow[d, "f_2"] & U_1 \arrow[r, "j_1"] \arrow[d, "f_1"] & X \arrow[d, "f"] \\
V_2 \arrow[r, "k_{12}"] & V_1 \arrow[r, "k_1"] & Y
\end{tikzcd}$$

Assume that one of the following conditions is fulfilled.

(i) The right rectangle is cartesian.

(ii) The morphism f_1 is proper.

(iii) The morphism k_1 is proper.

Then we have the pull-back homomorphism

$$f^*\colon H^q_{!*}(Y, V_1, V_2; \mathcal{F}) \longrightarrow H^q_{!*}(X, U_1, U_2; f_2^*\mathcal{F})$$

induced by the composite

$$k_{1!}Rk_{12*}\mathcal{F} \xrightarrow{\text{adj}} Rf_*f^*k_{1!}Rk_{12*}\mathcal{F} \xrightarrow{\text{b.c.}} Rf_*j_{1!}f_1^*Rk_{12*}\mathcal{F} \xrightarrow{\text{b.c.}} Rf_*j_{1!}Rj_{12*}f_2^*\mathcal{F},$$

where b.c. denotes the base change map. Moreover if f is proper (for example X and Y are proper over K), we have the push-forward homomorphism

$$f_*\colon H^q_{*!}(X, U_1, U_2; Rf_2^!\mathcal{F}) \longrightarrow H^q_{*!}(Y, V_1, V_2; \mathcal{F})$$

induced by the composite

$$Rf_!Rj_{1*}j_{12!}Rf_2^!\mathcal{F} \xrightarrow{\text{b.c.}} Rf_!Rj_{1*}Rf_1^!k_{12!}\mathcal{F} \xrightarrow{\text{b.c.}} Rf_!Rf^!Rk_{1*}k_{12!}\mathcal{F} \xrightarrow{\text{adj}} Rk_{1*}k_{12!}\mathcal{F}.$$

4.1.5 Assume that one of the following conditions is fulfilled.

(i) The left rectangle is cartesian.

(ii) The morphism f_2 is proper.

(iii) The morphism k_{12} is proper.

Then we have $f^*\colon H^q_{*!}(Y, V_1, V_2; \mathcal{F}) \longrightarrow H^q_{*!}(X, U_1, U_2; f_2^*\mathcal{F})$ defined similarly. Moreover if f is proper, we have $f_*\colon H^q_{!*}(X, U_1, U_2; Rf_2^!\mathcal{F}) \longrightarrow H^q_{!*}(Y, V_1, V_2; \mathcal{F})$.

4.1.6 Next we define a cup product. Let (X, U_1, U_2) be a $\star$-triple such that X is proper and $\mathcal{F}, \mathcal{G} \in \operatorname{obj} D^b_{ctf}(U_2, \Lambda)$. By Lemma 4.1.7 below, we can define a cup product

$$H^p_{!*}(X, U_1, U_2; \mathcal{F}) \otimes H^q_{*!}(X, U_1, U_2; \mathcal{G}) \xrightarrow{\cup} H^{p+q}_c(U_2, \mathcal{F} \overset{\mathbb{L}}{\otimes} \mathcal{G})$$

as the composite

$$\begin{aligned} H^p_{!*}(X, U_1, U_2; \mathcal{F}) \otimes H^q_{*!}(X, U_1, U_2; \mathcal{G}) &= H^p(X, j_{1!}Rj_{12*}\mathcal{F}) \otimes H^q(X, Rj_{1*}j_{12!}\mathcal{G}) \\ &\longrightarrow H^{p+q}(X, j_{1!}Rj_{12*}\mathcal{F} \overset{\mathbb{L}}{\otimes} Rj_{1*}j_{12!}\mathcal{G}) \\ &\cong H^{p+q}(X, j_{2!}(\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathcal{G})) \\ &= H^{p+q}_c(U_2, \mathcal{F} \overset{\mathbb{L}}{\otimes} \mathcal{G}). \end{aligned}$$

LEMMA 4.1.7. *Let the notation be the same as above. We have the isomorphism*

$$j_{1!}Rj_{12*}\mathcal{F} \overset{\mathbb{L}}{\otimes} Rj_{1*}j_{12!}\mathcal{G} \cong j_{2!}(\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathcal{G}).$$

Proof. Denote the canonical closed immersion $X \setminus U_2 \longrightarrow X$ by i. Since $j_2^*(j_{1!}Rj_{12*}\mathcal{F} \overset{\mathbb{L}}{\otimes} Rj_{1*}j_{12!}\mathcal{G}) = \mathcal{F} \overset{\mathbb{L}}{\otimes} \mathcal{G}$, then

$$j_{2!}(\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathcal{G}) \longrightarrow j_{1!}Rj_{12*}\mathcal{F} \overset{\mathbb{L}}{\otimes} Rj_{1*}j_{12!}\mathcal{G} \longrightarrow i^*(j_{1!}Rj_{12*}\mathcal{F} \overset{\mathbb{L}}{\otimes} Rj_{1*}j_{12!}\mathcal{G}) \xrightarrow{+1}$$

is a distinguished triangle. Moreover we have $i^*(j_{1!}Rj_{12*}\mathcal{F} \overset{\mathbb{L}}{\otimes} Rj_{1*}j_{12!}\mathcal{G}) = 0$ since $i^*Rj_{1*}j_{12!}\mathcal{G} = 0$. Thus $j_{1!}Rj_{12*}\mathcal{F} \overset{\mathbb{L}}{\otimes} Rj_{1*}j_{12!}\mathcal{G} \cong j_{2!}(\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathcal{G})$. □

4.1.8 Let X, Y be proper schemes over K and $U \subset X$, $V \subset Y$ open subschemes. For $\mathcal{F} \in \operatorname{obj} D^b_{ctf}(U, \Lambda)$ and $\mathcal{G} \in \operatorname{obj} D^b_{ctf}(V, \Lambda)$, we have the following Künneth formula:

$$\begin{aligned} H^q_{!*}(X \times Y, U \times Y, U \times V; \mathcal{F} \overset{\mathbb{L}}{\boxtimes} \mathcal{G}) &= H^q_{*!}(X \times Y, X \times V, U \times V; \mathcal{F} \overset{\mathbb{L}}{\boxtimes} \mathcal{G}) \\ &= \bigoplus_{i+j=q} H^i_c(U, \mathcal{F}) \otimes H^j(V, \mathcal{G}). \end{aligned}$$

Proof. Denote the canonical open immersions $U \longrightarrow X$ and $V \longrightarrow Y$ by j and k respectively. By the Künneth formula [AGV73, Exposé XVII, Théorème 5.4.3; Del77, Finitude, Théorème 1.9], we have

$$(j \times 1)_!R(1 \times k)_*(\mathcal{F} \overset{\mathbb{L}}{\boxtimes} \mathcal{G}) = R(1 \times k)_*(j \times 1)_!(\mathcal{F} \overset{\mathbb{L}}{\boxtimes} \mathcal{G}) = j_!\mathcal{F} \overset{\mathbb{L}}{\boxtimes} Rk_*\mathcal{G}.$$

This completes the proof. □

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4.1.9 For simplicity, we will write $H^q_{!*}(X,U_1,U_2)$ and $H^q_{*!}(X,U_1,U_2)$ for $H^q_{!*}(X,U_1,U_2;\Lambda)$ and $H^q_{*!}(X,U_1,U_2;\Lambda)$ respectively. Let $f\colon (X,U_1,U_2)\longrightarrow (Y,V_1,V_2)$ be a morphism of $\star$-triples. If the condition in Paragraph 4.1.4 is satisfied, we have

$$f^*\colon H^q_{!*}(Y,V_1,V_2)\longrightarrow H^q_{!*}(X,U_1,U_2;f_2^*\Lambda)=H^q_{!*}(X,U_1,U_2).$$

Assume further that f is proper, V_2 is smooth and U_2, V_2 are equidimensional. Then we have

$$f_*\colon H^{q+2d}_{*!}(X,U_1,U_2)(d)\longrightarrow H^q_{*!}(X,U_1,U_2;Rf_2^!\Lambda)\longrightarrow H^q_{*!}(Y,V_1,V_2)$$

where $d=\dim U_2-\dim V_2$.

It is easy to see that f^* and f_* are dual to each other and the following projection formula holds.

Proposition 4.1.10. *Assume that X and Y are proper over K. For every $x\in H^p_{!*}(Y,V_1,V_2)$ and $y\in H^q_{*!}(X,U_1,U_2)$, the equality $f_{2*}(f^*(x)\cup y)=x\cup f_*(y)$ holds in $H_c^{p+q-2d}(V_2,\Lambda(-d))$.*

4.1.11 Next assume that the condition in Paragraph 4.1.5 is satisfied for a morphism $f\colon (X,U_1,U_2)\longrightarrow (Y,V_1,V_2)$ of $\star$-triples. Then we have

$$f^*\colon H^q_{*!}(Y,V_1,V_2)\longrightarrow H^q_{*!}(X,U_1,U_2;f_2^*\Lambda)=H^q_{*!}(X,U_1,U_2).$$

Assume further that f is proper, V_2 is smooth and U_2, V_2 are equidimensional. Then we have

$$f_*\colon H^{q+2d}_{!*}(X,U_1,U_2)(d)\longrightarrow H^q_{!*}(X,U_1,U_2;Rf_2^!\Lambda)\longrightarrow H^q_{!*}(Y,V_1,V_2)$$

where $d=\dim U_2-\dim V_2$.

It is easy to see that f^* and f_* are dual to each other and the following projection formula holds.

Proposition 4.1.12. *Assume that X and Y are proper over K. For every $x\in H^p_{*!}(Y,V_1,V_2)$ and $y\in H^q_{!*}(X,U_1,U_2)$, the equality $f_{2*}(f^*(x)\cup y)=x\cup f_*(y)$ holds in $H_c^{p+q-2d}(V_2,\Lambda(-d))$.*

4.1.13 Let (X,U_1,U_2) be a $\star$-triple such that U_2 is smooth and equidimensional. Let Y be a closed subscheme of X which is purely of codimension c. Assume that $Y\cap U_1=Y\cap U_2$ and put $V=Y\cap U_1$. Then the diagram

$$\begin{tikzcd} V \arrow[r, equal] \arrow[d, "i_2"] & V \arrow[r] \arrow[d, "i_1"] & Y \arrow[d, "i"] \\ U_2 \arrow[r, "j_{12}"] & U_1 \arrow[r, "j_1"] & X \end{tikzcd}$$

is cartesian and we have the base change map $Ri_2^!\Lambda=\mathrm{id}_!\,Ri_2^!\Lambda\longrightarrow Ri_1^!j_{12!}\Lambda$. By this, we have the morphisms $Rj_{1*}i_{1*}Ri_2^!\Lambda\longrightarrow Rj_{1*}j_{12!}\Lambda$ and

$$H^{2c}_V(U_2,\Lambda(c))\longrightarrow H^{2c}_{*!}(X,U_1,U_2)(c).$$

Lemma 4.1.14. *The image of $\mathrm{cl}_V^{U_2}(V)\in H^{2c}_V(U_2,\Lambda(c))$ under the map above is equal to the image of $1\in H^0(V,\Lambda)=H^0_{*!}(Y,V,V)$ under the map $i_*\colon H^0_{*!}(Y,V,V)\longrightarrow H^{2c}_{*!}(X,U_1,U_2)(c)$.*

Proof. By the definition of i_*, we have the following commutative diagram:

$$\begin{tikzcd} H^0(V,\Lambda) \arrow[r, equal] \arrow[d, "i_{2*}"] & H^0_{*!}(Y,V,V) \arrow[d, "i_*"] \\ H^{2c}_V(U_2,\Lambda(c)) \arrow[r] & H^{2c}_{*!}(X,U_1,U_2)(c) \end{tikzcd}$$

where the map i_{2*} is induced by the canonical map $\Lambda\longrightarrow Ri_2^!\Lambda(c)[2c]$. By [Del77, cycle, Théorème 2.3.8(i)], we have $i_{2*}(1)=\mathrm{cl}_V^{U_2}(V)$. This completes the proof. □

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4.1.15 Let X, Y be schemes proper over K and $j\colon U \hookrightarrow X$, $j'\colon V \hookrightarrow Y$ dense open subschemes of X, Y respectively. Assume that U, V are equidimensional and V is smooth. Put $c = \dim U$ and $d = \dim V$. Let $\Gamma \subset U \times V$ be a purely d-dimensional closed subscheme such that $\Gamma \hookrightarrow U \times V \xrightarrow{\mathrm{pr}_1} U$ is proper and $\overline{\Gamma}$ the closure of Γ in $X \times Y$. Then $(X \times Y, U \times Y, U \times V)$ and $\overline{\Gamma}$ satisfy the condition in Paragraph 4.1.13. By Lemma 4.1.14, we can describe the action Γ^* of the correspondence Γ by means of partially supported cohomology, as follows.

Proposition 4.1.16. *Let the notation be the same as in Paragraph 4.1.15. Then $\Gamma^*\colon H^q_c(U, \Lambda) \longrightarrow H^q_c(V, \Lambda)$ coincides with the composite*

$$H^q_c(U,\Lambda) = H^q_{!*}(X, U, U) \xrightarrow{\mathrm{pr}_1^*} H^q_{!*}(X \times Y, U \times Y, U \times V) \xrightarrow{\cup \mathrm{cl}(\Gamma)} H^{q+2c}_c(U \times V)(c) \xrightarrow{\mathrm{pr}_{2*}} H^q_c(V, \Lambda).$$

Here $\mathrm{cl}(\Gamma)$ denotes the image of $\mathrm{cl}^{U\times V}_\Gamma(\Gamma) \in H^{2c}_\Gamma(U \times V, \Lambda(c))$ in $H^{2c}_{*!}(X \times Y, U \times Y, U \times V)(c)$.

Proof. This follows immediately from Proposition 4.1.10 and Lemma 4.1.14. □

4.1.17 Let $f\colon X' \longrightarrow X$ be a proper morphism of equidimensional schemes over K and $Z \subset X$ a closed subscheme of X which is purely k-dimensional. Put $Z' = Z \times_X X'$ and $d = \dim X' - \dim X$. Assume that X is smooth over K. Then the map $\mathrm{id} \times f\colon X' \longrightarrow X' \times X$ is a regular immersion. By applying the construction in [Ful98, ch. 6] to the cartesian diagram we have the element $(\mathrm{id} \times f)^![Z] \in \mathrm{CH}_{k+d}(X')$.

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & X' \times Z \arrow[d] \\ X' \arrow[r] & X' \times X \end{tikzcd}$$

We denote it by $f^![Z]$. It is well known that this construction is compatible with cycle class, i.e. $f^*(\mathrm{cl}^X_Z(Z)) = \mathrm{cl}^{X'}_{Z'}(f^![Z])$.

Lemma 4.1.18. *Let X, Y, X' and Y' be equidimensional schemes over K and $f\colon X' \longrightarrow X$, $g\colon Y' \longrightarrow Y$ be proper surjective generically finite morphisms over K. Put $c = \dim X = \dim X'$ and $d = \dim Y = \dim Y'$. Let $\Gamma \subset X \times Y$ be a purely d-dimensional subscheme such that the composite $\Gamma \hookrightarrow X \times Y \xrightarrow{\mathrm{pr}_1} X$ is proper. Assume that X and Y are smooth over K. Then the following diagram is commutative.*

$$\begin{tikzcd} H^i_c(X', \Lambda) \arrow[r, "((f\times g)^![\Gamma])^*"] \arrow[d, "f_*"] & H^i_c(Y', \Lambda) \\ H^i_c(X, \Lambda) \arrow[r, "\Gamma^*"] & H^i_c(Y, \Lambda) \arrow[u, "g^*"] \end{tikzcd}$$

Proof. First note that $((f \times g)^![\Gamma])^*$ makes sense since $(f \times g)^![\Gamma]$ is supported on $(f \times g)^{-1}(\Gamma)$, which is proper over X'. Take a compactification $\overline{f}\colon \overline{X}' \longrightarrow \overline{X}$ of $f\colon X' \longrightarrow X$ and $\overline{g}\colon \overline{Y}' \longrightarrow \overline{Y}$ of $g\colon Y' \longrightarrow Y$. Since f and g are proper, we have $\overline{f}^{-1}(X) = X'$ and $\overline{g}^{-1}(Y) = Y'$. Put $\xi = \mathrm{cl}(\Gamma) \in H^{2c}_{*!}(\overline{X} \times \overline{Y}, X \times \overline{Y}, X \times Y)(c)$. Then $\mathrm{cl}((f \times g)^![\Gamma]) = (\overline{f} \times \overline{g})^*\xi$.

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Consider the following diagram.

$$\begin{tikzcd}
H^i_c(X',\Lambda) \arrow[r, "\mathrm{pr}_1^*"] \arrow[d, "f_*"] & H^i_{!*}(\overline{X}'\times\overline{Y}', X'\times\overline{Y}', X'\times Y') \arrow[r, "\cup(\overline{f}\times\overline{g})^*\xi"] \arrow[d, "(\overline{f}\times\mathrm{id})_*"] & H^{i+2c}_c(X'\times Y',\Lambda(c)) \arrow[r, "\mathrm{pr}_{2*}"] \arrow[d, "(f\times\mathrm{id})_*"] & H^i_c(Y',\Lambda) \arrow[d, equal] \\
H^i_c(X,\Lambda) \arrow[r, "\mathrm{pr}_1^*"] \arrow[d, equal] & H^i_{!*}(\overline{X}\times\overline{Y}', X\times\overline{Y}', X\times Y') \arrow[r, "\cup(\mathrm{id}\times\overline{g})^*\xi"] & H^{i+2c}_c(X\times Y',\Lambda(c)) \arrow[r, "\mathrm{pr}_{2*}"] & H^i_c(Y',\Lambda) \\
H^i_c(X,\Lambda) \arrow[r, "\mathrm{pr}_1^*"] & H^i_{!*}(\overline{X}\times\overline{Y}, X\times\overline{Y}, X\times Y) \arrow[r, "\cup\xi"] \arrow[u, "(\mathrm{id}\times\overline{g})^*"] & H^{i+2c}_c(X\times Y,\Lambda(c)) \arrow[r, "\mathrm{pr}_{2*}"] \arrow[u, "(\mathrm{id}\times g)^*"] & H^i_c(Y,\Lambda) \arrow[u, "g^*"]
\end{tikzcd}$$

By Proposition 4.1.16, the composite of the upper horizontal arrows is equal to $((f\times g)^![\Gamma])^*$ and that of the lower horizontal arrows is equal to Γ^*. The lower left rectangle, the lower middle rectangle, and the upper right rectangle in the diagram above are clearly commutative. The upper left rectangle and the lower right rectangle are commutative by the Künneth formula. The upper middle rectangle is commutative by the projection formula. This completes the proof. $\square$

4.2 Nearby cycle cohomology

4.2.1 Let $S=\operatorname{Spec} A$ be a strict henselian trait and denote its generic (respectively special) point by η (respectively s). Let K be a quotient field of A and $\overline{K}$ a separable closure of K. For an S-scheme X, we denote its special fiber, generic fiber, and geometric generic fiber by X_s, X_η, and $X_{\overline{\eta}}$ respectively. Denote the integral closure of A in $\overline{K}$ by $\overline{A}$ and put $\overline{S}=\operatorname{Spec}\overline{A}$. For an S-scheme $f\colon X\longrightarrow S$, we write $\overline{f}\colon\overline{X}\longrightarrow\overline{S}$ for the base change of f from S to $\overline{S}$. Then we have the cartesian diagrams below.

$$\begin{tikzcd}
X_s \arrow[r, "i"] \arrow[d, "f_s"] & X \arrow[d, "f"] & X_\eta \arrow[l, "j"'] \arrow[d, "f_\eta"] \\
s \arrow[r] & S & \eta \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_s \arrow[r, "\overline{i}"] \arrow[d, "f_s"] & \overline{X} \arrow[d, "\overline{f}"] & X_{\overline{\eta}} \arrow[l, "\overline{j}"'] \arrow[d, "f_{\overline{\eta}}"] \\
s \arrow[r] & \overline{S} & \overline{\eta} \arrow[l]
\end{tikzcd}$$

Let ℓ be a prime which is invertible on S and denote $\Lambda=\mathbb{Q}_\ell$. For $\mathcal{F}\in\operatorname{obj} D^b_c(X_\eta,\Lambda)$, we define $R\psi_X\mathcal{F}=\overline{i}^*R\overline{j}_*\varphi^*\mathcal{F}$, where $\varphi\colon X_{\overline{\eta}}\longrightarrow X_\eta$ is the canonical morphism. If no confusion occurs, we omit the subscript X of $R\psi_X$.

4.2.2 First we recall some functorialities of nearby cycles. Let $f\colon X\longrightarrow Y$ be a morphism between S-schemes. We define $f^*\colon R\psi_Y\Lambda\longrightarrow Rf_{s*}R\psi_X\Lambda$ as the composite of

$$R\psi_Y\Lambda\longrightarrow Rf_{s*}f_s^*R\psi_Y\Lambda\xrightarrow{\text{b.c.}}Rf_{s*}R\psi_X f_{\overline{\eta}}^*\Lambda=Rf_{s*}R\psi_X\Lambda.$$

Assume further that X_η, Y_η are equidimensional and Y_η is smooth. Put $d=\dim X_\eta-\dim Y_\eta$. Then we define $f_*\colon Rf_{s!}R\psi_Y\Lambda(d)[2d]\longrightarrow R\psi_X\Lambda$ as the composite of

$$Rf_{s!}R\psi_X\Lambda(d)[2d]\xrightarrow{\text{b.c.}}R\psi_Y Rf_{\overline{\eta}!}\Lambda(d)[2d]\xrightarrow{\mathrm{Tr}}R\psi_Y Rf_{\overline{\eta}!}Rf^!_{\overline{\eta}}\Lambda\xrightarrow{\text{adj}}R\psi_Y\Lambda.$$

Here the map Tr is induced by $\mathrm{Tr}\colon\Lambda(d)[2d]\longrightarrow Rf^!_{\overline{\eta}}\Lambda$ defined as follows. Denote the structure morphisms $X_{\overline{\eta}}\longrightarrow\overline{\eta}$, $Y_{\overline{\eta}}\longrightarrow\overline{\eta}$ by $\varphi_{X_{\overline{\eta}}}$, $\varphi_{Y_{\overline{\eta}}}$. Since Y_η is smooth and equidimensional, we have $R\varphi^!_{Y_{\overline{\eta}}}\Lambda=\Lambda(\dim Y_\eta)[2\dim Y_\eta]$. Therefore the trace map relative to $\varphi_{X_{\overline{\eta}}}$ induces

$$\Lambda(\dim X_\eta)[2\dim X_\eta]\longrightarrow R\varphi^!_{X_{\overline{\eta}}}\Lambda=Rf^!_{\overline{\eta}}R\varphi^!_{Y_{\overline{\eta}}}\Lambda=Rf^!_{\overline{\eta}}\Lambda(\dim Y_\eta)[2\dim Y_\eta],$$

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which again induces $\Lambda(d)[2d] \longrightarrow Rf^!_{\overline{\eta}}\Lambda$. Note that if $f_{\overline{\eta}}$ is flat, the composite $Rf_{\overline{\eta}!}\Lambda(d)[2d] \xrightarrow{\mathrm{Tr}} Rf_{\overline{\eta}!}Rf^!_{\overline{\eta}}\Lambda \xrightarrow{\mathrm{adj}} \Lambda$ coincides with the usual trace map $\mathrm{Tr}_{f_{\overline{\eta}}}$ relative to $f_{\overline{\eta}}$.

Lemma 4.2.3. *Let X and Y be arithmetic S-schemes and $f\colon X \longrightarrow Y$ a proper surjective generically finite S-morphism. Denote the degree of f by n. Assume that X_η is equidimensional, Y_η is smooth and connected. Note that by these conditions $\dim X_\eta$ and $\dim Y_\eta$ are equal. Then the composite*

$$R\psi_Y\Lambda \xrightarrow{f^*} Rf_{s*}R\psi_X\Lambda \xrightarrow{f_*} R\psi_Y\Lambda$$

is the multiplication by n.

Proof. Since Y_η is smooth and connected, we have the homomorphism

$$\Lambda \xrightarrow{\mathrm{adj}} Rf_{\overline{\eta}*}f^*_{\overline{\eta}}\Lambda = Rf_{\overline{\eta}*}\Lambda \xrightarrow{\mathrm{Tr}} Rf_{\overline{\eta}*}Rf^!_{\overline{\eta}}\Lambda \xrightarrow{\mathrm{adj}} \Lambda \qquad (*)$$

between constant sheaves on $Y_{\overline{\eta}}$ (here Tr is the map defined in Paragraph 4.2.2). By the assumption and the generic flatness [Gro64, Théorème 6.9.1], there exists a dense open $U \subset Y$ such that $f_{\overline{\eta}}$ is finite flat over U. Therefore over U the map $(*)$ coincides with the composite

$$\Lambda \xrightarrow{\mathrm{adj}} Rf_{\overline{\eta}*}\Lambda \xrightarrow{\mathrm{Tr}_{f_{\overline{\eta}}}} \Lambda,$$

which is known to be the multiplication by n. Therefore the map $(*)$ is also the multiplication by n. Thus we have only to prove that the given map is equal to the composite

$$R\psi_Y\Lambda \xrightarrow{R\psi_Y(\mathrm{adj})} R\psi_Y Rf_{\overline{\eta}*}\Lambda \xrightarrow{R\psi_Y(\mathrm{adj}\circ\mathrm{Tr})} R\psi_Y\Lambda.$$

Now we recall some basic facts on the base change map. For a commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & X' \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$

the following hold.

(a) For $\mathcal{F} \in \operatorname{obj} D^b_c(Y,\Lambda)$, the composite

$$Rg_*\mathcal{F} \xrightarrow{\mathrm{adj}} Rf_*f^*Rg_*\mathcal{F} \xrightarrow{Rf_*(\mathrm{b.c.})} Rf_*Rg'_*f'^*\mathcal{F} = Rg_*Rf'_*f'^*\mathcal{F}$$

is equal to $Rg_*(\mathrm{adj})$.

(b) For $\mathcal{F} \in \operatorname{obj} D^b_c(X,\Lambda)$, the composite

$$g^*\mathcal{F} \xrightarrow{g^*(\mathrm{adj})} g^*Rf_*f^*\mathcal{F} \xrightarrow{\mathrm{b.c.}} Rf'_*g'^*f^*\mathcal{F} = Rf'_*f'^*g^*\mathcal{F}$$

is equal to adj.

Fact (a) is nothing but the definition of the base change map. Fact (b) is also easy.

Consider the following cartesian diagram.

$$\begin{tikzcd} X_s \arrow[r, "\overline{i}'"] \arrow[d, "f_s"] & \overline{X} \arrow[d, "\overline{f}"] & X_{\overline{\eta}} \arrow[l, "\overline{j}'"'] \arrow[d, "f_{\overline{\eta}}"] \\ Y_s \arrow[r, "\overline{i}"] & \overline{Y} & Y_{\overline{\eta}} \arrow[l, "\overline{j}"'] \end{tikzcd}$$

By (b), the composite

$$\overline{i}^*R\overline{j}_*\Lambda \xrightarrow{\mathrm{adj}} Rf_{s*}f_s^*\overline{i}^*R\overline{j}_*\Lambda = Rf_{s*}\overline{i}'^*\overline{f}^*R\overline{j}_*\Lambda \xleftarrow[\cong]{\mathrm{b.c.}} \overline{i}^*R\overline{f}_*\overline{f}^*R\overline{j}_*\Lambda$$

is equal to $\overline{i}^*(\mathrm{adj})$. Together with (a), we may infer that the composite

$$\begin{aligned}\overline{i}^* R\overline{j}_*\Lambda &\xrightarrow{\mathrm{adj}} Rf_{s*}f_s^*\overline{i}^* R\overline{j}_*\Lambda = Rf_{s*}\overline{i}'^*\overline{f}^* R\overline{j}_*\Lambda \xleftarrow[\cong]{\mathrm{b.c.}} \overline{i}^* R\overline{f}_*\overline{f}^* R\overline{j}_*\Lambda \xrightarrow{\mathrm{b.c.}} \overline{i}^* R\overline{f}_* R\overline{j}'_* f_{\overline{\eta}}^*\Lambda \\ &= \overline{i}^* R\overline{j}_* Rf_{\overline{\eta}*} f_{\overline{\eta}}^*\Lambda\end{aligned}$$

is equal to $\overline{i}^* R\overline{j}_*(\mathrm{adj})$.

Now consider the following diagram.

$$\begin{tikzcd}
R\psi_Y\Lambda \arrow[r, "\mathrm{adj}"] \arrow[rr, bend left, "f^*"] \arrow[drr, "R\psi_Y(\mathrm{adj})"'] & Rf_{s*}Rf_s^*R\psi_Y\Lambda \arrow[r, "\mathrm{b.c.}"] & Rf_{s*}R\psi_Y\Lambda \arrow[d, "\mathrm{b.c.}"] \arrow[dr, "f_*"] & \\
 & & R\psi_Y Rf_{\overline{\eta}*}\Lambda \arrow[r, "R\psi_Y(\mathrm{adj}\circ\mathrm{Tr})"'] & R\psi_Y\Lambda
\end{tikzcd}$$

We have just proved that the left triangle is commutative. The right triangle is commutative by the definition of f_*. Our claim immediately follows from this. $\square$

4.2.4 Let X and Y be arithmetic S-schemes with equidimensional smooth generic fibers. Put $c = \dim X_\eta$ and $d = \dim Y_\eta$. Let $\Gamma \subset X \times_S Y$ be a closed subscheme with purely d-dimensional generic fiber such that $\Gamma \hookrightarrow X \times_S Y \xrightarrow{\mathrm{pr}_1} X$ is proper. Denote the closed immersion $\Gamma \hookrightarrow X \times Y$ by γ and put $\gamma_i = \mathrm{pr}_i \circ \gamma$. Then we have the maps

$$H_c^q(X_s, R\psi_X\Lambda) \xrightarrow{\gamma_1^*} H_c^q(\Gamma_s, R\psi_\Gamma\Lambda), \quad H_c^q(\Gamma_s, R\psi_\Gamma\Lambda) \xrightarrow{\gamma_{2*}} H_c^q(Y_s, R\psi_Y\Lambda).$$

We define $\Gamma^*\colon H_c^q(X_s, R\psi_X\Lambda) \longrightarrow H_c^q(Y_s, R\psi_Y\Lambda)$ as the composite of the maps above.

4.2.5 As in § 4.1, we can consider a $\star$-triple (X, U_1, U_2) over S. We will always assume that X, U_1, U_2 are arithmetic S-schemes. For such a $\star$-triple, we consider the partially supported nearby cycle cohomology $H^q_{!*}(X_s, U_{1s}, U_{2s}; R\psi_{U_2}\Lambda)$ and $H^q_{*!}(X_s, U_{1s}, U_{2s}; R\psi_{U_2}\Lambda)$. We have the same functorialities as in Paragraphs 4.1.9 and 4.1.11, the projection formulas, and the Künneth formula (cf. [Ill94, Théorèmes 4.2 and 4.7]).

4.2.6 Let (X, U_1, U_2) be a $\star$-triple over S such that $U_{2\eta}$ is smooth and equidimensional. Let Y be a closed subscheme of X such that $Y_\eta \subset X_\eta$ is purely of codimension c. Assume that $Y \cap U_1 = Y \cap U_2$ and put $V = Y \cap U_1$. Then as in Paragraph 4.1.13, we have the canonical map

$$H^{2c}_{V_\eta}(U_{2\eta}, \Lambda(c)) \longrightarrow H^{2c}_{V_{\overline{\eta}}}(U_{2\overline{\eta}}, \Lambda(c)) \longrightarrow H^{2c}_{V_s}(U_{2s}, R\psi_{U_2}\Lambda(c)) \longrightarrow H^{2c}_{*!}(X_s, U_{1s}, U_{2s}; R\psi_{U_2}\Lambda(c)).$$

Lemma 4.2.7. *The image of* $\mathrm{cl}^{U_{2\eta}}_{V_\eta}(V_\eta)$ *by the map above coincides with the image of* $1 \in H^0(V_s, R\psi_V\Lambda) = H^0_{*!}(Y_s, V_s, V_s; R\psi_V\Lambda)$ *under the push-forward map*

$$H^0_{*!}(Y_s, V_s, V_s; R\psi_V\Lambda) \longrightarrow H^{2c}_{*!}(X_s, U_{1s}, U_{2s}; R\psi_{U_2}\Lambda(c)).$$

We denote it by $\mathrm{cl}(V_\eta)$.

Proof. Consider the diagram below.

$$\begin{tikzcd}
H^0(V_\eta, \Lambda) \arrow[r] \arrow[d] & H^0(V_{\overline{\eta}}, \Lambda) \arrow[r] \arrow[d] & H^0(V_s, R\psi_V\Lambda) \arrow[r, equal] \arrow[d] & H^0_{*!}(Y_s, V_s, V_s; R\psi_V\Lambda) \arrow[d] \\
H^{2c}_{V_\eta}(U_{2\eta}, \Lambda(c)) \arrow[r] & H^{2c}_{V_{\overline{\eta}}}(U_{2\overline{\eta}}, \Lambda(c)) \arrow[r] & H^{2c}_{V_s}(U_{2s}, R\psi_{U_2}\Lambda(c)) \arrow[r] & H^{2c}_{*!}(X_s, U_{1s}, U_{2s}; R\psi_V\Lambda(c))
\end{tikzcd}$$

The two left rectangles are clearly commutative. As in the proof of Lemma 4.1.14, we can see that the right one is commutative. Since the image of $1 \in H^0(V_\eta, \Lambda)$ under the map $H^0(V_\eta, \Lambda) \longrightarrow H^{2c}_{V_\eta}(U_{2\eta}, \Lambda(c))$ is $\mathrm{cl}^{U_{2\eta}}_{V_\eta}(V_\eta)$, the lemma follows. $\square$

Proposition 4.2.8. *Let X, Y be proper arithmetic S-schemes and $U \subset X$, $V \subset Y$ open subschemes which are arithmetic S-schemes. Assume that U_η and V_η are equidimensional and smooth, and put $c = \dim U_\eta$, $d = \dim V_\eta$. Let $\Gamma \subset U \times_S V$ be a closed subscheme with purely d-dimensional generic fiber such that the composite $\Gamma \hookrightarrow U \times_S V \xrightarrow{\mathrm{pr}_1} U$ is proper. Then $\Gamma^*\colon H^q_c(U_s, R\psi_U\Lambda) \longrightarrow H^q_c(V_s, R\psi_V\Lambda)$ coincides with the composite*

$$H^q_c(U_s, R\psi_U\Lambda) = H^q_{!*}(X_s, U_s, U_s; R\psi_U\Lambda) \xrightarrow{\mathrm{pr}_1^*} H^q_{!*}(X_s \times Y_s, U_s \times Y_s, U_s \times V_s; R\psi_{U\times V}\Lambda)$$
$$\xrightarrow{\cup\,\mathrm{cl}(\Gamma_\eta)} H^{q+2c}_c(U_s \times V_s, R\psi_{U\times V}\Lambda(c)) \xrightarrow{\mathrm{pr}_{2*}} H^q_c(V_s, R\psi_V\Lambda).$$

In particular, Γ^* depends only on Γ_η (as long as $\Gamma \hookrightarrow U \times_S V \xrightarrow{\mathrm{pr}_1} U$ is proper).

Proof. This follows immediately from Lemma 4.2.7 and the projection formula. $\square$

Lemma 4.2.9. *Let X, Y, X' and Y' be arithmetic S-schemes with smooth equidimensional generic fibers and $f\colon X' \longrightarrow X$, $g\colon Y' \longrightarrow Y$ be proper surjective generically finite S-morphisms. Put $c = \dim X_\eta = \dim X'_\eta$ and $d = \dim Y_\eta = \dim Y'_\eta$. Let $\Gamma \subset X \times_S Y$ be a closed subscheme with purely d-dimensional generic fiber such that the composite $\Gamma \hookrightarrow X \times_S Y \xrightarrow{\mathrm{pr}_1} X$ is proper. As in Paragraph 4.1.17, we have $(f_\eta \times g_\eta)^![\Gamma_\eta] \in \mathrm{CH}_d((f_\eta \times g_\eta)^{-1}(\Gamma_\eta))$. Take $\Gamma' \in Z_d((f \times g)^{-1}(\Gamma))$ whose image in $\mathrm{CH}_d((f_\eta \times g_\eta)^{-1}(\Gamma_\eta))$ is equal to $(f_\eta \times g_\eta)^![\Gamma_\eta]$. Such Γ' is not unique, but Γ'^* is independent of the choice of Γ' by the previous proposition. Then the following diagram is commutative.*

$$\begin{tikzcd} H^i_c(X'_s, R\psi_{X'}\Lambda) \arrow[r, "\Gamma'^*"] \arrow[d, "f_*"] & H^i_c(Y'_s, R\psi_{Y'}\Lambda) \\ H^i_c(X_s, R\psi_X\Lambda) \arrow[r, "\Gamma^*"] & H^i_c(Y_s, R\psi_Y\Lambda) \arrow[u, "g^*"'] \end{tikzcd}$$

Proof. As in the proof of Lemma 4.1.18, we derive the commutativity from the projection formula, the Künneth formula, and Proposition 4.2.8. $\square$

Lemma 4.2.10. *Let (X, U_1, U_2) be a $\star$-triple over S where $U_{2\eta}$ is smooth and $i\colon Y \hookrightarrow X$ a closed subscheme such that $Y_\eta \hookrightarrow X_\eta$ is purely of codimension c. Assume that $Y \cap U_1 = Y \cap U_2$ and put $V = Y \cap U_1$. Let $\xi \in H^{2c}_V(U_2, \Lambda(c))$ be an element satisfying $\xi|_{U_{2\eta}} = \mathrm{cl}^{U_{2\eta}}_{V_\eta}(V_\eta)$. Then $\mathrm{cl}(V_\eta) \in H^{2c}_{*!}(X_s, U_{1s}, U_{2s}; R\psi_{U_2}\Lambda(c))$ coincides with the image of ξ under the map*

$$H^{2c}_V(U_2, \Lambda(c)) \longrightarrow H^{2c}_{V_s}(U_{2s}, \Lambda(c)) \longrightarrow H^{2c}_{*!}(X_s, U_{1s}, U_{2s})(c) \longrightarrow H^{2c}_{*!}(X_s, U_{1s}, U_{2s}; R\psi_{U_2}\Lambda(c)).$$

Proof. This follows from the commutative diagram below.

$$\begin{tikzcd} H^{2c}_V(U_2, \Lambda(c)) \arrow[r] \arrow[d] & H^{2c}_{V_s}(U_{2s}, \Lambda(c)) \arrow[r] \arrow[dd] & H^{2c}_{*!}(X_s, U_{1s}, U_{2s}; \Lambda(c)) \arrow[dd] \\ H^{2c}_{V_\eta}(U_{2\eta}, \Lambda(c)) \arrow[d] & & \\ H^{2c}_{V_{\overline{\eta}}}(U_{2\overline{\eta}}, \Lambda(c)) \arrow[r] & H^{2c}_{V_s}(U_{2s}, R\psi_{U_2}\Lambda(c)) \arrow[r] & H^{2c}_{*!}(X_s, U_{1s}, U_{2s}; R\psi_{U_2}\Lambda(c)) \end{tikzcd}$$
$\square$

Corollary 4.2.11. *Let the notation be the same as in Proposition 4.2.8. Let $\xi \in H^{2c}_\Gamma(U \times_S V, \Lambda(c))$ be an element satisfying $\xi|_{U_\eta \times V_\eta} = \mathrm{cl}(\Gamma_\eta)$. We denote by ξ' the image of ξ under the*

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map $H^{2c}_{\Gamma}(U \times V, \Lambda(c)) \longrightarrow H^{2c}_{\Gamma_s}(U_s \times V_s, \Lambda(c)) \longrightarrow H^{2c}_{*!}(X_s \times Y_s, U_s \times Y_s, U_s \times V_s)(c)$. *Then* $\Gamma^*\colon H^q_c(U_s, R\psi_U\Lambda) \longrightarrow H^q_c(V_s, R\psi_V\Lambda)$ *coincides with the composite*

$$H^q_c(U_s, R\psi_U\Lambda) = H^q_{!*}(X_s, U_s, U_s; R\psi_U\Lambda) \xrightarrow{\mathrm{pr}_1^*} H^q_{!*}(X_s \times Y_s, U_s \times Y_s, U_s \times V_s; R\psi_{U\times V}\Lambda)$$
$$\xrightarrow{\cup\xi'} H^{q+2c}_c(U_s \times V_s, R\psi_{U\times V}\Lambda(c)) \xrightarrow{\mathrm{pr}_{2*}} H^q_c(V_s, R\psi_V\Lambda).$$

Proof. This is clear from Proposition 4.2.8 and Lemma 4.2.10. □

5. An analogue of the weight spectral sequence and its functorialities

5.1 An analogue of the weight spectral sequence

5.1.1 Let $S = \operatorname{Spec} A$ be a strict henselian trait as in § 4.2. Let (X, U_1, U_2) be a $\star$-triple over S such that U_2 is strictly semistable (cf. Paragraph 3.2.1) over S. We say that such a $\star$-triple itself is strictly semistable. We denote the irreducible components of U_{2s} by $D''_1, \dots, D''_m$. We write D_i (respectively D'_i) for the closure of D''_i in X (respectively U_1). They form $\star$-triples (D_i, D'_i, D''_i). We have the following maps between $\star$-triples.

$$\begin{tikzcd} D''_i \arrow[r, "k'_i"] \arrow[d, "a''_i"] & D'_i \arrow[r, "k_i"] \arrow[d, "a'_i"] & D_i \arrow[d, "a_i"] \\ U_{2s} \arrow[r, "j_{12}"] & U_{1s} \arrow[r, "j_1"] & X_s \end{tikzcd}$$

For a subset $I \subset \{1, \dots, m\}$, put $D_I = \bigcap_{i\in I} D_i$, $D'_I = \bigcap_{i\in I} D'_i$, and $D''_I = \bigcap_{i\in I} D''_i$. For every I, D''_I is smooth over s. We write a_I, a'_I, a''_I, k_I and k'_I for the maps induced by a_i, a'_i, a''_i, k_i and k'_i respectively. For an integer p, put $D^{(p)} = \coprod_{I\subset\{1,\dots,m\},\#I=p+1} D_I$, $D'^{(p)} = \coprod_{I\subset\{1,\dots,m\},\#I=p+1} D'_I$, and $D''^{(p)} = \coprod_{I\subset\{1,\dots,m\},\#I=p+1} D''_I$. If X is purely of relative dimension n over S, they are purely of relative dimension $n-p$ over s. We write $a^{(p)}$, $a'^{(p)}$, $a''^{(p)}$, $k^{(p)}$ and $k'^{(p)}$ for the maps induced by a_I, a'_I, a''_I, k_I and k'_I respectively. We have the following maps between $\star$-triples.

$$\begin{tikzcd} D''_I \arrow[r, "k'_I"] \arrow[d, "a''_I"] & D'_I \arrow[r, "k_I"] \arrow[d, "a'_I"] & D_I \arrow[d, "a_I"] \\ U_{2s} \arrow[r, "j_{12}"] & U_{1s} \arrow[r, "j_1"] & X_s \end{tikzcd} \qquad \begin{tikzcd} D''^{(p)} \arrow[r, "k'^{(p)}"] \arrow[d, "a''^{(p)}"] & D'^{(p)} \arrow[r, "k^{(p)}"] \arrow[d, "a'^{(p)}"] & D^{(p)} \arrow[d, "a^{(p)}"] \\ U_{2s} \arrow[r, "j_{12}"] & U_{1s} \arrow[r, "j_1"] & X_s \end{tikzcd}$$

5.1.2 By [SaT03, § 2.1], we have the monodromy filtration $M_\bullet$ on $R\psi_{U_2}\Lambda$. This is a filtration in the category of perverse sheaves on X_s. The filtration $M_\bullet$ defines a quasi-filtered object $(R\psi_{U_2}\Lambda, (M_sR\psi_{U_2}\Lambda/M_rR\psi_{U_2}\Lambda)_{s\geqslant r})$ of the category $D^b_c(U_{2s}, \Lambda)$ (see [SaM88, § 5.2.17]). Since the functors Rj_{12*} and $j_{1!}$ preserve distinguished triangles, we have a quasi-filtered object

$$(j_{1!}Rj_{12*}R\psi_{U_2}\Lambda, (j_{1!}Rj_{12*}(M_sR\psi_{U_2}\Lambda/M_rR\psi_{U_2}\Lambda))_{s\geqslant r})$$

of the category $D^b_c(X_s, \Lambda)$.

Theorem 5.1.3. *Let the notation be the same as above. The above quasi-filtered object induces the spectral sequence*

$$E_1^{p,q} = \bigoplus_{i\geqslant\max(0,-p)} H^{q-2i}_{!*}(D^{(p+2i)}, D'^{(p+2i)}, D''^{(p+2i)})(-i) \Longrightarrow H^{p+q}_{!*}(X_s, U_{1s}, U_{2s}; R\psi_{U_2}\Lambda).$$

Proof. By [SaM88, Lemme 5.2.18], we have the spectral sequence

$$E_1^{p,q} = H^{p+q}(X_s, j_{1!}Rj_{12*}\operatorname{Gr}^M_{-p} R\psi_{U_2}\Lambda) \Longrightarrow H^{p+q}_{!*}(X_s, U_{1s}, U_{2s}; R\psi_{U_2}\Lambda).$$

On the other hand, by [SaT03, Proposition 2.7] we have the canonical isomorphism

$$j_{1!}Rj_{12*}\Big(\bigoplus_{p-q=r} a''^{(p+q)}_{*}\Lambda(-p)[-p-q]\Big) \xrightarrow{\cong} j_{1!}Rj_{12*}\operatorname{Gr}^M_r R\psi_{U_2}\Lambda.$$

Since $j_{1!}Rj_{12*}a''_{i*}\Lambda = j_{1!}a'_{i*}Rk'_{i*}\Lambda = j_{1!}a'_{i!}Rk'_{i*}\Lambda = a_{i!}k_{i!}Rk'_{i*}\Lambda = a_{i*}k_{i!}Rk'_{i*}\Lambda$, we have the canonical isomorphism

$$\begin{aligned} H^{p+q}(X_s, j_{1!}Rj_{12*}\operatorname{Gr}^M_{-p} R\psi_{U_2}\Lambda) &\cong H^{p+q}\Big(X_s,\, j_{1!}Rj_{12*}\Big(\bigoplus_{i\geqslant \max(0,-p)} a''^{(p+2i)}_{*}\Lambda(-i)[-p-2i]\Big)\Big)\\ &= \bigoplus_{i\geqslant \max(0,-p)} H^{q-2i}(X_s, a^{(p+2i)}_{*}k^{(p+2i)}_{!}Rk'^{(p+2i)}_{*}\Lambda(-i))\\ &= \bigoplus_{i\geqslant \max(0,-p)} H^{q-2i}_{!*}(D^{(p+2i)}, D'^{(p+2i)}, D''^{(p+2i)})(-i). \end{aligned}$$

This completes the proof. □

5.1.4 If $X = U_1 = U_2$ and X is proper over S, the spectral sequence above coincides with the weight spectral sequence in [RZ82] up to sign (see [SaT03, p. 613]).

5.2 Functoriality: pull-back

5.2.1 Let (X, U_1, U_2) and (Y, V_1, V_2) be strictly semistable $\star$-triples and f a morphism between them. Assume that the diagram is cartesian.

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & X \arrow[d] \\ V_1 \arrow[r] & Y \end{tikzcd}$$

Then we have the pull-back map

$$f^*\colon H^q_{!*}(Y, V_1, V_2; R\psi_{V_2}\Lambda) \longrightarrow H^q_{!*}(X, U_1, U_2; R\psi_{U_2}\Lambda).$$

5.2.2 Let $E''_1, \dots, E''_{m'}$ be the irreducible components of V_{2s} and E_i (respectively E'_i) a closure of E_i in Y (respectively V_1). As in Paragraph 5.1.1, we have the following diagram.

$$\begin{tikzcd} E''_i \arrow[r, "l'_i"] \arrow[d, "b''_i"] & E'_i \arrow[r, "l_i"] \arrow[d, "b'_i"] & E_i \arrow[d, "b_i"] \\ V_{2s} \arrow[r, "j'_{12}"] & V_{1s} \arrow[r, "j'_1"] & Y_s \end{tikzcd}$$

We also define E_I, E'_I, E''_I, $E^{(p)}$, $E'^{(p)}$ and $E''^{(p)}$ as in Paragraph 5.1.1.

Since U_2 and V_2 are strictly semistable, we have $f_2^*(\sum_{i=1}^{m'} E''_i) = \sum_{i=1}^{m} D''_i$ as Cartier divisors on U_2. Therefore there exists a unique map $\varphi\colon \{1, \dots, m\} \longrightarrow \{1, \dots, m'\}$ satisfying $f_2(D''_i) \subset E''_{\varphi(i)}$ for every $i \in \{1, \dots, m\}$. Renumbering the D''_i if necessary, we may assume that φ is increasing. Then we have $f(D_i) \subset E_{\varphi(i)}$ and $f_1(D'_i) \subset E'_{\varphi(i)}$ for every i. Moreover the right rectangle of the

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following commutative diagram is cartesian.

$$\begin{tikzcd} D''_i \arrow[r, "k'"] \arrow[d, "f_2"] & D'_i \arrow[r, "k"] \arrow[d, "f_1"] & D_i \arrow[d, "f"] \\ E''_{\varphi(i)} \arrow[r, "l'"] & E'_{\varphi(i)} \arrow[r, "l"] & E_{\varphi(i)} \end{tikzcd}$$

5.2.3 For a non-negative integer p, we put $\mathcal{I}_{f,p} = \{I \subset \{1,\dots,m\} \mid \#I = \#\varphi(I) = p+1\}$ and $D''^{(p)}_f = \coprod_{I\in\mathcal{I}_{f,p}} D''_I$. We define $D^{(p)}_f$ and $D'^{(p)}_f$ similarly. For $I \in \mathcal{I}_{f,p}$, we have a morphism of $\star$-triples $f_{\varphi(I)I}\colon (D_I, D'_I, D''_I) \longrightarrow (E_{\varphi(I)}, E'_{\varphi(I)}, E''_{\varphi(I)})$, which is a restriction of f. Put $f^{(p)} = \coprod f_{\varphi(I)I}\colon (D^{(p)}_f, D'^{(p)}_f, D''^{(p)}_f) \longrightarrow (E^{(p)}, E'^{(p)}, E''^{(p)})$. Since the right rectangle of the commutative diagram

$$\begin{tikzcd} D''^{(p)}_f \arrow[r, "k'^{(p)}"] \arrow[d, "f_2^{(p)}"] & D'^{(p)}_f \arrow[r, "k^{(p)}"] \arrow[d, "f_1^{(p)}"] & D^{(p)}_f \arrow[d, "f^{(p)}"] \\ E''^{(p)} \arrow[r, "l'^{(p)}"] & E'^{(p)} \arrow[r, "l^{(p)}"] & E^{(p)} \end{tikzcd}$$

is cartesian, we have the pull-back map

$$f^{(p)*} = \sum_{I\in\mathcal{I}_{f,p}} f^*_{\varphi(I)I}\colon H^q_{!*}(E^{(p)}, E'^{(p)}, E''^{(p)}) \longrightarrow H^q_{!*}(D^{(p)}_f, D'^{(p)}_f, D''^{(p)}_f) \hookrightarrow H^q_{!*}(D^{(p)}, D'^{(p)}, D''^{(p)}).$$

Proposition 5.2.4. *We have a map of spectral sequences as follows.*

$$\begin{tikzcd} E'^{p,q}_1 = \bigoplus_{i\geqslant \max(0,-p)} H^{q-2i}_{!*}(E^{(p+2i)}, E'^{(p+2i)}, E''^{(p+2i)})(-i) \arrow[r, Rightarrow] \arrow[d, "\oplus f^{(p+2i)*}"] & H^{p+q}_{!*}(Y_s, V_{1s}, V_{2s}; R\psi_{V_2}\Lambda) \arrow[d, "f^*"] \\ E^{p,q}_1 = \bigoplus_{i\geqslant \max(0,-p)} H^{q-2i}_{!*}(D^{(p+2i)}, D'^{(p+2i)}, D''^{(p+2i)})(-i) \arrow[r, Rightarrow] & H^{p+q}_{!*}(X_s, U_{1s}, U_{2s}; R\psi_{U_2}\Lambda) \end{tikzcd}$$

Proof. We have a morphism of quasi-filtered objects

$$\begin{aligned} &(j'_{1!}Rj'_{12*}R\psi_{V_2}\Lambda, (j'_{1!}Rj'_{12*}(M_sR\psi_{V_2}\Lambda/M_rR\psi_{V_2}\Lambda))_{s\geqslant r}) \\ &\quad\longrightarrow (Rf_{s*}f_s^*j'_{1!}Rj'_{12*}R\psi_{V_2}\Lambda, (Rf_{s*}f_s^*j'_{1!}Rj'_{12*}(M_sR\psi_{V_2}\Lambda/M_rR\psi_{V_2}\Lambda))_{s\geqslant r}) \\ &\quad\longrightarrow (Rf_{s*}j_{1!}Rj_{12*}f^*_{2s}R\psi_{V_2}\Lambda, (Rf_{s*}j_{1!}Rj_{12*}f^*_{2s}(M_sR\psi_{V_2}\Lambda/M_rR\psi_{V_2}\Lambda))_{s\geqslant r}). \end{aligned}$$

By [SaT03, Proposition 2.11(1)], we have a morphism of quasi-filtered objects

$$(f^*_{2s}R\psi_{V_2}\Lambda, f^*_{2s}(M_sR\psi_{V_2}\Lambda/M_rR\psi_{V_2}\Lambda)_{s\geqslant r}) \longrightarrow (R\psi_{U_2}\Lambda, (M_sR\psi_{U_2}\Lambda/M_rR\psi_{U_2}\Lambda)_{s\geqslant r}),$$

which induces

$$\begin{aligned} &(Rf_{s*}j_{1!}Rj_{12*}f^*_{2s}R\psi_{V_2}\Lambda, Rf_{s*}j_{1!}Rj_{12*}f^*_{2s}(M_sR\psi_{V_2}\Lambda/M_rR\psi_{V_2}\Lambda)_{s\geqslant r}) \\ &\quad\longrightarrow (Rf_{s*}j_{1!}Rj_{12*}R\psi_{U_2}\Lambda, Rf_{s*}j_{1!}Rj_{12*}(M_sR\psi_{U_2}\Lambda/M_rR\psi_{U_2}\Lambda)_{s\geqslant r}). \end{aligned}$$

Therefore we have a morphism of quasi-filtered objects

$$\begin{aligned} &(j'_{1!}Rj'_{12*}R\psi_{V_2}\Lambda, (j'_{1!}Rj'_{12*}(M_sR\psi_{V_2}\Lambda/M_rR\psi_{V_2}\Lambda))_{s\geqslant r}) \\ &\quad\longrightarrow (Rf_{s*}j_{1!}Rj_{12*}R\psi_{U_2}\Lambda, Rf_{s*}j_{1!}Rj_{12*}(M_sR\psi_{U_2}\Lambda/M_rR\psi_{U_2}\Lambda)_{s\geqslant r}). \end{aligned}$$

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The associated morphism of spectral sequences is as follows.

$$\begin{tikzcd} E_1'^{p,q} = H^{p+q}(Y_s, j'_{1!}Rj'_{12*}\operatorname{Gr}^M_{-p} R\psi_{V_2}\Lambda) \arrow[r, Rightarrow] \arrow[d] & H^{p+q}_{!*}(Y_s, V_{1s}, V_{2s}; R\psi_{V_2}\Lambda) \arrow[d, "f^*"] \\ E_1^{p,q} = H^{p+q}(X_s, j_{1!}Rj_{12*}\operatorname{Gr}^M_{-p} R\psi_{U_2}\Lambda) \arrow[r, Rightarrow] & H^{p+q}_{!*}(X_s, U_{1s}, U_{2s}; R\psi_{U_2}\Lambda) \end{tikzcd}$$

On the other hand, by [SaT03, Proposition 2.11(2)], we have the following commutative diagram for every r.

$$\begin{tikzcd} \bigoplus_{p-q=r} j'_{1!}Rj'_{12*}b''^{(p+q)}_*\Lambda(-p)[-p-q] \arrow[r, "\cong"] \arrow[d] & j'_{1!}Rj'_{12*}\operatorname{Gr}^M_r R\psi_{V_2}\Lambda \arrow[d] \\ \bigoplus_{p-q=r} Rf_{s*}j_{1!}Rj_{12*}f^*_{2s}b''^{(p+q)}_*\Lambda(-p)[-p-q] \arrow[r, "\cong"] \arrow[d] & Rf_{s*}j_{1!}Rj_{12*}f^*_{2s}\operatorname{Gr}^M_r R\psi_{V_2}\Lambda \arrow[d] \\ \bigoplus_{p-q=r} Rf_{s*}j_{1!}Rj_{12*}a''^{(p+q)}_*\Lambda(-p)[-p-q] \arrow[r, "\cong"] & Rf_{s*}j_{1!}Rj_{12*}\operatorname{Gr}^M_r R\psi_{U_2}\Lambda \end{tikzcd}$$

where the horizontal arrows are the canonical isomorphisms in [SaT03, Proposition 2.7]. We know that $j'_{1!}Rj'_{12*}b''^{(p+q)}_*\Lambda = b^{(p+q)}_* l_! Rl'_*\Lambda$ and $Rf_{s*}j_{1!}Rj_{12*}a''^{(p+q)}_*\Lambda = Rf_{s*}a^{(p+q)}_* k_! Rk'_*\Lambda$. Thus we have the map $b^{(p+q)}_* l_! Rl'_*\Lambda \longrightarrow Rf_{s*}a^{(p+q)}_* k_! Rk'_*\Lambda$ induced by the composite of

$$j'_{1!}Rj'_{12*}b''^{(p+q)}_*\Lambda \longrightarrow Rf_{s*}j_{1!}Rj_{12*}f^*_{2s}b''^{(p+q)}_*\Lambda \longrightarrow Rf_{s*}j_{1!}Rj_{12*}a''^{(p+q)}_*\Lambda,$$

appearing in the above diagram. We can easily see that (by taking $R\Gamma(Y_s, *)$) this map induces

$$f^{(p+q)*}\colon H^k_{*!}(E^{(p+q)}, E'^{(p+q)}, E''^{p+q}) \longrightarrow H^k_{*!}(D^{(p+q)}, D'^{(p+q)}, D''^{(p+q)}).$$

The proposition immediately follows from this. □

5.3 Functoriality: cup product

PROPOSITION 5.3.1. *Let (X, U_1, U_2) be a strictly semistable $\star$-triple over S and $\xi \in H^m_{*!}(X_s, U_{1s}, U_{2s})(l)$. Then the cup product with ξ induces a map of spectral sequences as follows.*

$$\begin{tikzcd} E_1^{p,q} = \bigoplus_{i\geqslant \max(0,-p)} H^{q-2i}_{!*}(D^{(p+2i)}, D'^{(p+2i)}, D''^{(p+2i)})(-i) \arrow[r, Rightarrow] \arrow[d, "\cup\xi|_{D^{(p+2i)}}"] & H^{p+q}_{!*}(X_s, U_{1s}, U_{2s}; R\psi_{U_2}\Lambda) \arrow[d, "\cup\xi"] \\ E_1^{p,q+m} = \bigoplus_{i\geqslant \max(0,-p)} H^{q-2i+m}_c(D''^{(p+2i)}, \Lambda(-i+l)) \arrow[r, Rightarrow] & H^{p+q+m}_c(U_{2s}, R\psi_{U_2}\Lambda(l)) \end{tikzcd}$$

Proof. We have a map of quasi-filtered objects

$$\begin{aligned} &(j_{1!}Rj_{12*}R\psi_{U_2}\Lambda, (j_{1!}Rj_{12*}(M_sR\psi_{U_2}\Lambda/M_rR\psi_{U_2}\Lambda))_{s\geqslant r}) \\ &\quad \xrightarrow{\cup\xi} (j_{2!}R\psi_{U_2}\Lambda(l)[m], (j_{2!}(M_sR\psi_{U_2}\Lambda/M_rR\psi_{U_2}\Lambda)(l)[m])_{s\geqslant r}) \end{aligned}$$

and the following map of spectral sequences induced by it.

$$\begin{tikzcd} E_1^{p,q} = H^{p+q}(X_s, j_{1!}Rj_{12*}\operatorname{Gr}^M_{-p} R\psi_{U_2}\Lambda) \arrow[r, Rightarrow] \arrow[d, "\cup\xi"] & H^{p+q}_{!*}(X_s, U_{1s}, U_{2s}; R\psi_{U_2}\Lambda) \arrow[d, "\cup\xi"] \\ E_1^{p,q+m} = H^{p+q+m}(X_s, j_{2!}\operatorname{Gr}^M_{-p} R\psi_{U_2}\Lambda(l)) \arrow[r, Rightarrow] & H^{p+q+m}_c(U_{2s}, R\psi_{U_2}\Lambda(l)) \end{tikzcd}$$

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By Lemma 5.3.2 below, the following diagram is commutative for every r.

$$\begin{tikzcd}
\bigoplus_{p-q=r} j_{1!}Rj_{12*}a''^{(p+q)}_{*}\Lambda(-p)[-p-q] \arrow[r, "\cong"] \arrow[d, "\cup\xi"] & j_{1!}Rj_{12*}\operatorname{Gr}^M_r R\psi_{U_2}\Lambda \arrow[d, "\cup\xi"] \\
\bigoplus_{p-q=r} j_{2!}a''^{(p+q)}_{*}\Lambda(-p+l)[-p-q+m] \arrow[r, "\cong"] & j_{2!}\operatorname{Gr}^M_r R\psi_{U_2}\Lambda(l)[m]
\end{tikzcd}$$

On the other hand, the diagram below is obviously commutative.

$$\begin{tikzcd}
j_{1!}Rj_{12*}a''^{(p+q)}_{*}\Lambda \arrow[r, equal] \arrow[d, "\cup\xi"] & a^{(p+q)}_{*}k_!Rk'_*\Lambda \arrow[d, "\cup\xi|_{D^{(p+q)}}"] \\
j_{1!}Rj_{12*}a''^{(p+q)}_{*}\Lambda(l)[m] \arrow[r, equal] & a^{(p+q)}_{*}(k\circ k')_!\Lambda(l)[m]
\end{tikzcd}$$

This completes the proof. □

Lemma 5.3.2. *Let X be a scheme over a field and $j\colon U \hookrightarrow X$ be an open subscheme. Let $\mathcal{F}$ and $\mathcal{G}$ be objects of $D^b_c(U,\Lambda)$. Then for every morphism $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ and every cohomology class $\xi \in H^m_c(X,\Lambda(l)) = \operatorname{Hom}(\Lambda, j_!\Lambda(l)[m])$, the following diagram is commutative.*

$$\begin{tikzcd}
Rj_*\mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\cup\xi"] & Rj_*\mathcal{G} \arrow[d, "\cup\xi"] \\
j_!\mathcal{F}(l)[m] \arrow[r, "\varphi"] & j_!\mathcal{G}(l)[m]
\end{tikzcd}$$

Proof. This follows from the diagram below, whose rectangles are easily seen to be commutative.

$$\begin{tikzcd}
Rj_*\mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}\otimes\xi"] & Rj_*\mathcal{G} \arrow[d, "\mathrm{id}\otimes\xi"] \\
Rj_*\mathcal{F} \overset{\mathbb{L}}{\otimes} j_!\Lambda(l)[m] \arrow[r] & Rj_*\mathcal{G} \overset{\mathbb{L}}{\otimes} j_!\Lambda(l)[m] \\
j_!j^*(Rj_*\mathcal{F} \overset{\mathbb{L}}{\otimes} j_!\Lambda(l)[m]) \arrow[r] \arrow[u, "\cong"] \arrow[d, "\cong"] & j_!j^*(Rj_*\mathcal{G} \overset{\mathbb{L}}{\otimes} j_!\Lambda(l)[m]) \arrow[u, "\cong"] \arrow[d, "\cong"] \\
j_!(\mathcal{F} \overset{\mathbb{L}}{\otimes} \Lambda(l)[m]) \arrow[r] \arrow[d, "\cong"] & j_!(\mathcal{G} \overset{\mathbb{L}}{\otimes} \Lambda(l)[m]) \arrow[d, "\cong"] \\
j_!\mathcal{F}(l)[m] \arrow[r, "\varphi"] & j_!\mathcal{G}(l)[m]
\end{tikzcd}$$

□

5.4 Functoriality: push-forward

5.4.1 Let X and Y be strictly semistable S-schemes and $f\colon X \longrightarrow Y$ a morphism between them. We assume that X (respectively Y) be purely of relative dimension n (respectively n'). Put $d = n-n'$. We denote the irreducible component of X (respectively Y) by $D_1,\dots,D_m$ (respectively $E_1,\dots,E_{m'}$) and define $D^{(p)}$ (respectively $E^{(p)}$) as in Paragraph 5.1.1.

Proposition 5.4.2. *We have a map of spectral sequences*

$$\begin{array}{ccc} E_1^{p,q+2d} = \bigoplus_{i\geqslant \max(0,-p)} H_c^{q+2d-2i}(D^{(p+2i)}, \Lambda(-i+d)) & \Longrightarrow & H_c^{p+q+2d}(X_s, R\psi_X \Lambda(d)) \\ \Big\downarrow {\scriptstyle \oplus f_*^{(p+2i)}} & & \Big\downarrow {\scriptstyle f_*} \\ E_1'^{p,q} = \bigoplus_{i\geqslant \max(0,-p)} H_c^{q-2i}(E^{(p+2i)}, \Lambda(-i)) & \Longrightarrow & H_c^{p+q}(Y_s, R\psi_Y \Lambda) \end{array}$$

where $f_*^{(p)}$ is defined as in [SaT03, § 2.3].

Proof. This follows immediately from [SaT03, Proposition 2.13]. □

5.5 Functoriality: action of correspondence

5.5.1 Let X and Y be strictly semistable S-schemes and $X \hookrightarrow \overline{X}$, $Y \hookrightarrow \overline{Y}$ compactifications over S. Assume that X (respectively Y) is purely of relative dimension n (respectively n'). Let $D_1, \dots, D_m$ (respectively $E_1, \dots, E_{m'}$) be the irreducible components of X_s (respectively Y_s). Denote $\overline{D}_i$ (respectively $\overline{E}_i$) the closure of D_i in $\overline{X}$ (respectively of E_i in $\overline{Y}$). Write $\mathcal{I}_i$ (respectively $\mathcal{I}'_i$) for the defining ideal of $\overline{D}_i$ (respectively $\overline{E}_i$). Let $\pi\colon \overline{Z} \longrightarrow \overline{X} \times_S \overline{Y}$ be the blow-up of $\overline{X} \times_S \overline{Y}$ by the ideal $\prod_{(i,i')\in\Delta}(\prod_{j=1}^{i} \mathrm{pr}_1^* \mathcal{I}_j + \prod_{j'=1}^{i'} \mathrm{pr}_2^* \mathcal{I}'_{j'})$, where Δ denotes the set $\{1,\dots,m\} \times \{1,\dots,m'\}$. Put $Z = \pi^{-1}(X \times Y)$. Then by [SaT03, Lemma 1.9], Z is strictly semistable over S and the irreducible components of Z_s are indexed by Δ as $\{C_{i,i'}\}_{(i,i')\in\Delta}$ so that $\pi(C_{i,i'}) = D_i \times E_{i'}$. For $I'' \subset \{1,\dots,m\}\times\{1,\dots,m'\}$, put $C_{I''} = \bigcap_{(i,i')\in I''} C_{i,i'}$. We know that $C^{(p)} = \coprod_{\#I''=p+1} C_{I''}$ (see [SaT03, Lemma 1.9]), where I'' runs over all totally ordered subsets of Δ (the order of Δ is the product order).

For $I \subset \{1,\dots,m\}$ and $I' \subset \{1,\dots,m'\}$ satisfying $\#I = \#I' = p+1$, denote by $I \wedge I' \subset \Delta$ the graph of the increasing bijection $I \longrightarrow I'$. Put $C_1^{(p)} = \coprod_{I\subset\{1,\dots,m\}, I'\subset\{1,\dots,m'\}, \#I=\#I'=p+1} C_{I\wedge I'}$. Let $\pi_{I\wedge I'}\colon C_{I\wedge I'} \longrightarrow D_I \times E_{I'}$ be the restriction of π and $\pi^{(p)}\colon C_1^{(p)} \longrightarrow D^{(p)} \times E^{(p)}$ the morphism induced by $\pi_{I\wedge I'}$.

5.5.2 Let $\Gamma \subset X \times_S Y$ be a closed subscheme with purely n'-dimensional generic fiber such that the composite $\Gamma \hookrightarrow X \times_S Y \xrightarrow{\mathrm{pr}_1} X$ is proper. Denote by Γ' the closure of $\Gamma_\eta \subset X_\eta \times Y_\eta = Z_\eta$ in Z and put $\Gamma'^{(p)} \in \mathrm{CH}_{n'-p}(C^{(p)} \cap \Gamma')$ the refined pull-back of Γ' to $C^{(p)}$. By Lemma 3.2.2, there exists a cohomology class $\xi \in H^{2n}_{\Gamma'}(X \times_S Y, \Lambda(n))$ satisfying the following conditions:

(i) $\xi|_{X_\eta\times Y_\eta} = \mathrm{cl}_{X_\eta\times Y_\eta}(\Gamma_\eta)$;

(ii) $\xi|_{C^{(p)}} = \mathrm{cl}^{C^{(p)}}_{C^{(p)}\cap\Gamma'}(\Gamma'^{(p)})$.

Since $\Gamma' \subset \pi^{-1}(\Gamma)$, the composite of $C_1^{(p)} \cap \Gamma' \hookrightarrow C_1^{(p)} \xrightarrow{\pi^{(p)}} D^{(p)} \times E^{(p)} \xrightarrow{\mathrm{pr}_1} D^{(p)}$ is proper. Thus $\Gamma'^{(p)}$ induces the action on cohomology $(\Gamma'^{(p)})^*\colon H_c^q(D^{(p)}, \Lambda) \longrightarrow H_c^q(E^{(p)}, \Lambda)$ (we write $\Gamma'^{(p)}$ again for the restriction of $\Gamma'^{(p)}$ to $C_1^{(p)} \cap \Gamma'$).

On the other hand we have $\Gamma''^{(p)} = \pi_*^{(p)}(\Gamma'^{(p)}) \in \mathrm{CH}_{n'-p}((D^{(p)} \times E^{(p)}) \cap \Gamma)$. As the composite $(D^{(p)} \times E^{(p)}) \cap \Gamma \hookrightarrow D^{(p)} \times E^{(p)} \xrightarrow{\mathrm{pr}_1} D^{(p)}$ is proper, $\Gamma''^{(p)}$ induces the action on cohomology $(\Gamma''^{(p)})^*\colon H_c^q(D^{(p)}, \Lambda) \longrightarrow H_c^q(E^{(p)}, \Lambda)$. By the projection formula, these two maps are equal. Now we state the functoriality result.

Theorem 5.5.3. *Let the notation be the same as above. Then we have a map of spectral sequences as follows.*

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$$\begin{tikzcd}
E_1^{p,q} = \bigoplus_{i\geqslant \max(0,-p)} H_c^{q-2i}(D^{(p+2i)}, \Lambda(-i)) \arrow[r, Rightarrow] \arrow[d, "\oplus(\Gamma''^{(p+2i)})^*"] & H_c^{p+q}(X_s, R\psi_X\Lambda) \arrow[d, "\Gamma^*"] \\
E_1'^{p,q} = \bigoplus_{i\geqslant \max(0,-p)} H_c^{q-2i}(E^{(p+2i)}, \Lambda(-i)) \arrow[r, Rightarrow] & H_c^{p+q}(Y_s, R\psi_Y\Lambda)
\end{tikzcd}$$

Proof. This follows from Corollary 4.2.11 and Propositions 5.2.4, 5.3.1, and 5.4.2. □

6. On ℓ-independence of nearby cycle cohomology

6.1 The ℓ-independence of nearby cycle cohomology

6.1.1 Let K be a complete discrete valuation field with finite residue field $F = \mathbb{F}_q$. We denote the ring of integers of K by $\mathcal{O}_K$ and the characteristic of F by p. Fix a separable closure $\overline{K}$ of K and let $\overline{F}$ be the residue field of the integral closure of $\mathcal{O}_K$ in $\overline{K}$, which is an algebraic closure of F. We denote by G_K (respectively G_F) the Galois group $\mathrm{Gal}(\overline{K}/K)$ (respectively $\mathrm{Gal}(\overline{F}/F)$). We denote by Fr_q the geometric Frobenius element (the inverse of the qth power map) in G_F. The Weil group W_K of K is defined as the inverse image of the subgroup $\langle \mathrm{Fr}_q\rangle \subset G_F$ by the canonical map $G_K \longrightarrow G_F$. For $\sigma \in W_K$, let $n(\sigma)$ be the integer such that the image of σ in G_F is $\mathrm{Fr}_q^{n(\sigma)}$. Put $W_K^+ = \{\sigma \in W_K \mid n(\sigma) \geqslant 0\}$.

Put $S = \operatorname{Spec} \mathcal{O}_K$. For an S-scheme X, we denote its special fiber, geometric special fiber, generic fiber, geometric generic fiber by X_F, $X_{\overline{F}}$, X_K, $X_{\overline{K}}$ respectively.

Let ℓ be a prime number distinct from p.

6.1.2 The main result in this section is the following theorem.

Theorem 6.1.3. *Let X be a flat arithmetic S-scheme with purely d-dimensional smooth generic fiber, and $\Gamma \subset X \times_S X$ a closed subscheme with purely d-dimensional generic fiber. Assume that the composite $\Gamma \hookrightarrow X \times_S X \xrightarrow{\mathrm{pr}_1} X$ is proper. Then for any $\sigma \in W_K^+$, the number*

$$\mathrm{Tr}(\Gamma^* \circ \sigma_*; H_c^*(X_{\overline{F}}, R\psi\mathbb{Q}_\ell)) = \sum_{i=0}^{2d} (-1)^i \,\mathrm{Tr}(\Gamma^* \circ \sigma_*; H_c^i(X_{\overline{F}}, R\psi\mathbb{Q}_\ell))$$

lies in $\mathbb{Z}[1/p]$ and is independent of ℓ.

6.1.4 First we treat the case where X is strictly semistable over S. We need a slight generalization of the above theorem in this case.

Lemma 6.1.5 (cf. [SaT03, Lemma 3.2]). *Let L be a finite quasi-Galois extension of K and put $S' = \operatorname{Spec} \mathcal{O}_L$. We denote the residue field of L by E. Let X be a strictly semistable S'-scheme which is purely of relative dimension d. Take any $\sigma \in W_K^+$. Fix an embedding $\overline{K} \hookrightarrow \overline{L}$ and extend σ uniquely to an automorphism of $\overline{L}$. We put $X^\sigma = X \times_{\mathcal{O}_L \nearrow \sigma} \mathcal{O}_L$. Let $\Gamma \subset X^\sigma \times_{S'} X$ be a closed subscheme with purely d-dimensional generic fiber satisfying that the composite $\Gamma \longrightarrow X^\sigma \times_{S'} X \xrightarrow{\mathrm{pr}_1} X^\sigma$ is proper. Then the number*

$$\mathrm{Tr}(\Gamma^* \circ \sigma_*; H_c^*(X_{\overline{E}}, R\psi\mathbb{Q}_\ell))$$

lies in $\mathbb{Z}[1/p]$ and is independent of ℓ.

Proof. We denote the irreducible components of X_E by $D_1, \ldots, D_m$ as usual. Then the irreducible components of X_E^σ are $D_1^\sigma, \ldots, D_m^\sigma$. We define $\Gamma''^{(s)} \in \mathrm{CH}_{d-s}((D^{\sigma(s)} \times D^{(s)}) \cap \Gamma)$ for each s as in

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Paragraph 5.5.2. Then by Theorem 5.5.3 we have the following map of spectral sequences.

$$\begin{array}{ccc} E_1'^{s,t} = \bigoplus_{i\geqslant \max(0,-s)} H_c^{t-2i}(D_{\overline{E}}^{\sigma(s+2i)}, \mathbb{Q}_\ell(-i)) & \Longrightarrow & H_c^{s+t}(X^\sigma_{\overline{E}}, R\psi_{X^\sigma}\mathbb{Q}_\ell) \\ \Big\downarrow{\scriptstyle \oplus(\Gamma''^{(s+2i)})^*} & & \Big\downarrow{\scriptstyle \Gamma^*} \\ E_1^{s,t} = \bigoplus_{i\geqslant \max(0,-s)} H_c^{t-2i}(D_{\overline{E}}^{(s+2i)}, \mathbb{Q}_\ell(-i)) & \Longrightarrow & H_c^{s+t}(X_{\overline{E}}, R\psi_X\mathbb{Q}_\ell) \end{array}$$

On the other hand, we have the map of spectral sequences induced by σ

$$\begin{array}{ccc} E_1^{s,t} = \bigoplus_{i\geqslant \max(0,-s)} H_c^{t-2i}(D_{\overline{E}}^{(s+2i)}, \mathbb{Q}_\ell(-i)) & \Longrightarrow & H_c^{s+t}(X_{\overline{E}}, R\psi_X\mathbb{Q}_\ell) \\ \Big\downarrow{\scriptstyle \overline{\sigma}_*} & & \Big\downarrow{\scriptstyle \sigma_*} \\ E_1'^{s,t} = \bigoplus_{i\geqslant \max(0,-s)} H_c^{t-2i}(D_{\overline{E}}^{\sigma(s+2i)}, \mathbb{Q}_\ell(-i)) & \Longrightarrow & H_c^{s+t}(X^\sigma_{\overline{E}}, R\psi_{X^\sigma}\mathbb{Q}_\ell) \end{array}$$

where $\overline{\sigma}$ denotes the image of σ in G_E. Let $\sigma^{(s)}_{\text{geom}}\colon D^{\sigma(s)}_{\overline{E}} \longrightarrow D^{(s)}_{\overline{E}}$ be the composition $\varphi^{f\cdot n(\sigma)} \circ \overline{\sigma}^*$, where φ denotes the absolute Frobenius morphism and f is the integer satisfying $q = p^f$. This is a proper morphism over $\overline{E}$. Since φ induces the identity map on étale cohomology, we have $\overline{\sigma}_* = \sigma^{(s)*}_{\text{geom}}$. Therefore we obtain the endomorphism of a spectral sequence.

$$\begin{array}{ccc} E_1^{s,t} = \bigoplus_{i\geqslant \max(0,-s)} H_c^{t-2i}(D_{\overline{E}}^{(s+2i)}, \mathbb{Q}_\ell(-i)) & \Longrightarrow & H_c^{s+t}(X_{\overline{E}}, R\psi_X\mathbb{Q}_\ell) \\ \Big\downarrow{\scriptstyle (\Gamma''^{(s+2i)})^*\circ\sigma^{(s+2i)*}_{\text{geom}}} & & \Big\downarrow{\scriptstyle \Gamma^*\circ\sigma_*} \\ E_1^{s,t} = \bigoplus_{i\geqslant \max(0,-s)} H_c^{t-2i}(D_{\overline{E}}^{(s+2i)}, \mathbb{Q}_\ell(-i)) & \Longrightarrow & H_c^{s+t}(X_{\overline{E}}, R\psi_X\mathbb{Q}_\ell) \end{array}$$

Denote by $\Gamma'''^{(s)} \in \mathrm{CH}_{d-s}((D^{\sigma(s)}_{\overline{E}} \times D^{(s)}_{\overline{E}}) \cap (\sigma^{(s)}_{\text{geom}} \times \mathrm{id})(\Gamma_{\overline{E}}))$ the image of $\Gamma''^{(s)}$ under the map

$$\begin{aligned} \mathrm{CH}_{d-s}((D^{\sigma(s)} \times D^{(s)}) \cap \Gamma) &\longrightarrow \mathrm{CH}_{d-s}((D^{\sigma(s)}_{\overline{E}} \times D^{(s)}_{\overline{E}}) \cap \Gamma_{\overline{E}}) \\ &\xrightarrow{(\sigma^{(s)}_{\text{geom}}\times\mathrm{id})_*} \mathrm{CH}_{d-s}((D^{\sigma(s)}_{\overline{E}} \times D^{(s)}_{\overline{E}}) \cap (\sigma^{(s)}_{\text{geom}} \times \mathrm{id})(\Gamma_{\overline{E}})). \end{aligned}$$

Then $(\Gamma'''^{(s)})^* = (\Gamma''^{(s)})^* \circ \sigma^{(s)*}_{\text{geom}}$ holds. Thus we have equalities

$$\begin{aligned} &\mathrm{Tr}(\Gamma^* \circ \sigma_*; H_c^*(X_{\overline{E}}, R\psi\mathbb{Q}_\ell)) \\ &\quad = \sum_s \sum_{i\geqslant \max(0,-s)} (-1)^s \,\mathrm{Tr}((\Gamma''^{(s+2i)})^* \circ \sigma^{(s+2i)*}_{\text{geom}}; H_c^*(D^{(s+2i)}_{\overline{E}}, \mathbb{Q}_\ell(-i))) \\ &\quad = \sum_s \sum_{i\geqslant \max(0,-s)} (-1)^s q^{n(\sigma)i}\,\mathrm{Tr}((\Gamma'''^{(s+2i)})^*; H_c^*(D^{(s+2i)}_{\overline{E}}, \mathbb{Q}_\ell)) \\ &\quad = \sum_s (-1)^s \frac{q^{n(\sigma)(s+1)} - 1}{q^{n(\sigma)} - 1}\,\mathrm{Tr}((\Gamma'''^{(s)})^*; H_c^*(D^{(s)}_{\overline{E}}, \mathbb{Q}_\ell)). \end{aligned}$$

By Theorem 2.1.2, the number $\mathrm{Tr}((\Gamma'''^{(s)})^*; H_c^*(D^{(s)}_{\overline{E}}, \mathbb{Q}_\ell))$ lies in $\mathbb{Z}[1/p]$ and is independent of ℓ. Therefore $\mathrm{Tr}(\Gamma^* \circ \sigma_*; H_c^*(X_{\overline{E}}, R\psi\mathbb{Q}_\ell))$ lies in $\mathbb{Z}[1/p]$ and is independent of ℓ. $\square$

6.1.6 Next we reduce Theorem 6.1.3 to Lemma 6.1.5 by de Jong's alteration [deJ96]. We may assume that X is connected. Since X is flat over S with smooth generic fiber, it is irreducible and reduced. Therefore by [deJ96, Theorem 6.5] and [SaT03, Lemma 1.11], we have a finite quasi-Galois extension L of K, a scheme Y that is strictly semistable over $\mathcal{O}_L$ with equidimensional generic fiber, and a proper surjective generically finite S-morphism $f\colon Y \longrightarrow X$. Put $S' = \operatorname{Spec}\mathcal{O}_L$ and

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denote the residue field of L by E as in the proof of Lemma 6.1.5. Let K' be the inseparable closure of K in L. Then we have a canonical isomorphism $H_c^i(X'_{\overline{F}}, R\psi_{X'}\mathbb{Q}_\ell) \cong H_c^i(X_{\overline{F}}, R\psi_X\mathbb{Q}_\ell)$, where $X' = X \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$. Moreover, if we fix an embedding $\overline{K} \hookrightarrow \overline{K'}$, the isomorphism above is compatible with an isomorphism $W_{K'} \xrightarrow{\sim} W_K$. Therefore by replacing K and X by K' and X' respectively, we may assume that the extension L/K is separable.

We denote by Y' the scheme Y considered as an S-scheme. Take any $\sigma \in W_K^+$. By Lemma 4.2.9, we have the commutative diagram below

$$\begin{tikzcd}
H_c^i(Y'_{\overline{F}}, R\psi_{Y'}\mathbb{Q}_\ell) \arrow[r, "\sigma_*"] \arrow[d, "f_*"] & H_c^i(Y'^{\sigma}_{\overline{F}}, R\psi_{Y'^{\sigma}}\mathbb{Q}_\ell) \arrow[r, "\Gamma'^*"] \arrow[d, "f_*^{\sigma}"] & H_c^i(Y'_{\overline{F}}, R\psi_{Y'}\mathbb{Q}_\ell) \\
H_c^i(X_{\overline{F}}, R\psi_X\mathbb{Q}_\ell) \arrow[r, "\sigma_*"] & H_c^i(X_{\overline{F}}, R\psi_X\mathbb{Q}_\ell) \arrow[r, "\Gamma^*"] & H_c^i(X_{\overline{F}}, R\psi_X\mathbb{Q}_\ell) \arrow[u, "f^*"]
\end{tikzcd}$$

where $\Gamma' \in Z_d((f^{\sigma} \times f)^{-1}(\Gamma))$ is an element satisfying

$$\Gamma'_K = (f_K^{\sigma} \times f_K)^{!}[\Gamma_K] \in \mathrm{CH}_d((f_K^{\sigma} \times f_K)^{-1}(\Gamma_K)),$$

as in Lemma 4.2.9. Together with Lemma 4.2.3, we have

$$\mathrm{Tr}(\Gamma'^* \circ \sigma_*; H_c^i(Y'_{\overline{F}}, R\psi_{Y'}\mathbb{Q}_\ell)) = \deg f \cdot \mathrm{Tr}(\Gamma^* \circ \sigma_*; H_c^i(X_{\overline{F}}, R\psi_X\mathbb{Q}_\ell))$$

as in the proof of [SaT03, Lemma 3.3].

Let $h\colon \coprod_{\tau \in \mathrm{Gal}(L/K)} Y^{\tau} \longrightarrow Y' \otimes_S S'$ be the morphism induced by the canonical map $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \longrightarrow \prod_{\tau \in \mathrm{Gal}(L/K)} \mathcal{O}_L$. It is finite surjective and induces an isomorphism on generic fibers. Therefore we have an isomorphism $H_c^i(Y'_{\overline{F}}, R\psi_{Y'}\mathbb{Q}_\ell) \cong \bigoplus_{\tau \in \mathrm{Gal}(L/K)} H_c^i(Y^{\tau}_{\overline{E}}, R\psi_{Y^{\tau}}\mathbb{Q}_\ell)$. The map $\sigma_*\colon H_c^i(Y'_{\overline{F}}, R\psi_{Y'}\mathbb{Q}_\ell) \longrightarrow H_c^i(Y'^{\sigma}_{\overline{F}}, R\psi_{Y'^{\sigma}}\mathbb{Q}_\ell)$ is identified with the direct sum of $\sigma_*\colon H_c^i(Y^{\tau}_{\overline{E}}, R\psi_{Y^{\tau}}\mathbb{Q}_\ell) \longrightarrow H_c^i(Y^{\sigma\tau}_{\overline{E}}, R\psi_{Y^{\sigma\tau}}\mathbb{Q}_\ell)$ under this isomorphism. For $\tau, \tau' \in \mathrm{Gal}(L/K)$, let

$$\Gamma'_{\tau,\tau'} \in Z_d((f^{\tau} \times f^{\tau'})^{-1}(\Gamma))$$

be an element such that $(\Gamma'_{\tau,\tau'})_L = \Gamma'_L|_{Y_L^{\tau} \times Y_L^{\tau'}}$, where Γ'_L is the base change of Γ'_K from K to L. By Lemma 4.2.9 again, the (τ,τ')-component of the map

$$\bigoplus_{\tau \in \mathrm{Gal}(L/K)} H_c^i(Y^{\sigma\tau}_{\overline{E}}, R\psi_{Y^{\sigma\tau}}\mathbb{Q}_\ell) \longrightarrow \bigoplus_{\tau' \in \mathrm{Gal}(L/K)} H_c^i(Y^{\tau'}_{\overline{E}}, R\psi_{Y^{\tau'}}\mathbb{Q}_\ell)$$

induced by $\Gamma'^*\colon H_c^i(Y'^{\sigma}_{\overline{F}}, R\psi_{Y'^{\sigma}}\mathbb{Q}_\ell) \longrightarrow H_c^i(Y'_{\overline{F}}, R\psi_{Y'}\mathbb{Q}_\ell)$ is equal to

$$\Gamma'^*_{\sigma\tau,\tau'}\colon H_c^i(Y^{\sigma\tau}_{\overline{E}}, R\psi_{Y^{\sigma\tau}}\mathbb{Q}_\ell) \longrightarrow H_c^i(Y^{\tau'}_{\overline{E}}, R\psi_{Y^{\tau'}}\mathbb{Q}_\ell).$$

Therefore the number

$$\begin{aligned}
\mathrm{Tr}(\Gamma^* \circ \sigma_*; H_c^*(X_{\overline{F}}, R\psi_X\mathbb{Q}_\ell)) &= \frac{1}{\deg f}\,\mathrm{Tr}(\Gamma'^* \circ \sigma_*; H_c^*(Y'_{\overline{F}}, R\psi_{Y'}\mathbb{Q}_\ell)) \\
&= \frac{1}{\deg f} \sum_{\tau \in \mathrm{Gal}(L/K)} \mathrm{Tr}(\Gamma'^*_{\sigma\tau,\tau} \circ \sigma_*; H_c^*(Y^{\tau}_{\overline{E}}, R\psi_{Y^{\tau}}\mathbb{Q}_\ell))
\end{aligned}$$

lies in $(1/\deg f)\mathbb{Z}[1/p]$ and is independent of ℓ by Lemma 6.1.5.

By the same technique as in [SaT03, p. 629], we can derive from the following lemma that the number $\mathrm{Tr}(\Gamma^* \circ \sigma_*; H_c^*(X_{\overline{F}}, R\psi_X\mathbb{Q}_\ell))$ is in $\mathbb{Z}[1/p]$. Now the proof of Theorem 6.1.3 is complete.

Lemma 6.1.7. *Let K be a field of characteristic 0. Let $a_1, \dots, a_r$ be distinct elements of K and $c_1, \dots, c_r$ non-zero integers. Put $s_m = \sum_{i=1}^{r} c_i a_i^m$ for a non-negative integer m. Assume that there exists an integer $N \geqslant 1$ such that $Ns_m \in \mathbb{Z}[1/p]$ for every $m \geqslant 0$. Then $s_m \in \mathbb{Z}[1/p]$ for every $m \geqslant 0$.*

Proof. By [Kle68, Lemma 2.8], a_i is integral over $\mathbb{Z}[1/p]$ for every i. Therefore every s_m is also integral over $\mathbb{Z}[1/p]$, while it is in $\mathbb{Q}$. Since $\mathbb{Z}[1/p]$ is normal, we have $s_m \in \mathbb{Z}[1/p]$. □

Remark 6.1.8. The result of Bloch and Esnault [BE05] implies that the alternating sum of the trace in Theorem 6.1.3 lies in $\mathbb{Z}$ (cf. Remark 2.1.4). For $\Gamma = \Delta_X$ (the diagonal of X), the integrality also follows from [Mie06, Theorem 4.2].

6.2 The ℓ-independence for stalks of nearby cycles

6.2.1 In this section, we give some results on ℓ-independence for stalks of nearby cycles. All of them are immediate consequences of Theorem 6.1.3.

Theorem 6.2.2. *Let X be a flat arithmetic S-scheme with purely d-dimensional smooth generic fiber, and $x \in X_F$ an F-rational point. Choose a geometric point $\overline{x}$ lying over x. Then the Weil group W_K acts on the stalk $(R^i\psi_X\mathbb{Q}_\ell)_{\overline{x}}$. For every $\sigma \in W_K^+$, the number*

$$\operatorname{Tr}(\sigma_*; (R^*\psi_X\mathbb{Q}_\ell)_{\overline{x}}) = \sum_{i=0}^{d}(-1)^i \operatorname{Tr}(\sigma_*; (R^i\psi_X\mathbb{Q}_\ell)_{\overline{x}})$$

is an integer that is independent of ℓ.

Proof. Put $U = X \setminus \{x\}$. Then we have the following W_K-equivariant exact sequence:

$$\longrightarrow H^i_c(U_{\overline{F}}, R\psi_U\mathbb{Q}_\ell) \longrightarrow H^i_c(X_{\overline{F}}, R\psi_X\mathbb{Q}_\ell) \longrightarrow (R^i\psi_X\mathbb{Q}_\ell)_{\overline{x}} \longrightarrow H^{i+1}_c(U_{\overline{F}}, R\psi_U\mathbb{Q}_\ell) \longrightarrow .$$

Therefore we have the equality

$$\operatorname{Tr}(\sigma_*; (R^*\psi_X\mathbb{Q}_\ell)_{\overline{x}}) = \operatorname{Tr}(\sigma_*; H^*_c(X_{\overline{F}}, R\psi_X\mathbb{Q}_\ell)) - \operatorname{Tr}(\sigma_*; H^*_c(U_{\overline{F}}, R\psi_U\mathbb{Q}_\ell)).$$

Since each term of the right-hand side lies in $\mathbb{Z}[1/p]$ and is independent of ℓ, so is the left-hand side $\operatorname{Tr}(\sigma_*; (R^*\psi_X\mathbb{Q}_\ell)_{\overline{x}})$.

The integrality follows from Remark 6.1.8 (note that since we only use the case $\Gamma = \Delta_X$, we do not need the result of Bloch and Esnault). □

Corollary 6.2.3. *Let the notation be the same as in Theorem 6.2.2. Then the integers*

$$\dim_{\mathbb{Q}_\ell}(R^*\psi_X\mathbb{Q}_\ell)_{\overline{x}} = \sum_{i=0}^{d}(-1)^i \dim_{\mathbb{Q}_\ell}(R^i\psi_X\mathbb{Q}_\ell)_{\overline{x}}, \quad \operatorname{Sw}(R^*\psi_X\mathbb{Q}_\ell)_{\overline{x}} = \sum_{i=0}^{d}(-1)^i \operatorname{Sw}(R^i\psi_X\mathbb{Q}_\ell)_{\overline{x}}$$

are independent of ℓ. Here Sw *denotes the Swan conductor.*

Proof. These are immediate consequences of Theorem 6.2.2 (for the part of the Swan conductor, see [Och99, Corollary 2.6]). □

Remark 6.2.4. The above corollary gives weak evidence of Deligne's conjecture on Milnor numbers [DK73, Exposé XVI, Conjecture 1.9]. The statement of the conjecture is the following.

Conjecture 6.2.5. Let K^{ur} be the maximal unramified extension of K and put $S^{\mathrm{ur}} = \operatorname{Spec}\mathcal{O}_{K^{\mathrm{ur}}}$. Let X be a purely d-dimensional flat arithmetic S^{ur}-scheme. Assume that X is regular and that the structure morphism $X \longrightarrow S^{\mathrm{ur}}$ is smooth outside a unique closed point $x \in X_{\overline{F}}$. Put

$$\operatorname{dimtot}_{\mathbb{F}_\ell}(R^*\phi\,\mathbb{F}_\ell)_x = \dim_{\mathbb{F}_\ell}(R^*\phi\,\mathbb{F}_\ell)_x + \operatorname{Sw}(R^*\phi\,\mathbb{F}_\ell)_x,$$
$$\mu(X/S^{\mathrm{ur}}, x) = \operatorname{length}_{\mathcal{O}_{X,x}} \underline{\operatorname{Ext}}^1(\Omega_{X/S}, \mathcal{O}_X)_x.$$

Then the equality

$$\operatorname{dimtot}_{\mathbb{F}_\ell}(R^*\phi\,\mathbb{F}_\ell)_x = \mu(X/S^{\mathrm{ur}}, x)$$

holds. (The original conjecture allows a more general base trait. See [Org03].)

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This conjecture is solved in the cases below:

(i) $d = 0$ (see [DK73, Exposé XVI, Proposition 1.12]);

(ii) the point x is an ordinary double point (see [DK73, Exposé XVI, Proposition 1.13]);

(iii) the characteristic of K is positive (see [DK73, Exposé XVI, Theorem 2.4]);

(iv) $d = 1$ (see [Org03, Corollaire 0.9]).

Moreover, [Org03, Théorème 0.8] Orgogozo proved that the conductor formula of Bloch implies the above conjecture.

Since

$$\begin{aligned}\mathrm{dimtot}_{\mathbb{F}_\ell}(R^*\phi\,\mathbb{F}_\ell)_x &= \dim_{\mathbb{F}_\ell}(R^*\psi\mathbb{F}_\ell)_x + \mathrm{Sw}(R^*\psi\mathbb{F}_\ell)_x - 1\\ &= \dim_{\mathbb{Q}_\ell}(R^*\psi\mathbb{Q}_\ell)_x + \mathrm{Sw}(R^*\psi\mathbb{Q}_\ell)_x - 1\end{aligned}$$

(the last equality follows from the universal coefficient theorem), from Corollary 6.2.3 we see that the left-hand side of the equality in Conjecture 6.2.5 is independent of ℓ, while the right-hand side is obviously independent of ℓ.

6.3 The ℓ-independence for open schemes over local fields

6.3.1 In this section, we consider an analogue of [SaT03, Theorem 0.1] for open schemes over local fields.

Definition 6.3.2. Let X be an arithmetic S-scheme and $H \subset X$ a closed subscheme of X. We may write $H = H_h \cup H'$, where H' is contained in the special fiber of X and $H_h \longrightarrow S$ is flat. The pair (X, H) is called a *strictly semistable pair* if the following conditions hold (cf. [deJ96, 6.3]):

(i) X is strictly semistable over S;

(ii) H is a strict normal crossing divisor of X.

(iii) Let H_i ($i \in I$) be the irreducible components of H_h. For each $J \subset I$, the scheme $H_J = \bigcap_{i\in J} H_i$ is a union of schemes which are strictly semistable over S.

Moreover, if H is flat over S (namely, $H = H_h$), we call (X, H) a *horizontal strictly semistable pair*. For a strictly semistable pair (X, H), the pair (X, H_h) is a horizontal strictly semistable pair.

Lemma 6.3.3. *Let (X, H) be a horizontal strictly semistable pair over S. Put $U = X \setminus H$ and denote the canonical open immersion $U \hookrightarrow X$ by j. Then the canonical morphism*

$$j_{\overline{F}!}R\psi_U\mathbb{Q}_\ell \longrightarrow R\psi_X(j_{\overline{K}!}\mathbb{Q}_\ell)$$

is an isomorphism. In particular, if X is proper over S, we have an isomorphism $H^i_c(U_{\overline{F}}, R\psi_U\mathbb{Q}_\ell) \cong H^i_c(U_{\overline{K}}, \mathbb{Q}_\ell)$.

Proof. Since the problem is étale local, we may assume that

$$X = \mathrm{Spec}\,\mathcal{O}_K[T_1, \dots, T_n]/(T_{r+1}\cdots T_s - \pi), \quad H = V(T_1\cdots T_r) \subset X,$$

where π is a uniformizer of K (cf. [Ill04, 1.5(d)]). Put

$$X_1 = \mathrm{Spec}\,\mathcal{O}_K[T_{r+1}, \dots, T_n]/(T_{r+1}\cdots T_s - \pi).$$

Then $(X, H) \cong (\mathbb{A}^r_S \times_S X_1, Z \times_S X_1)$, where $Z \subset \mathbb{A}^r_S$ is the divisor defined by $T_1 \cdots T_r = 0$. By the Künneth formula for $R\psi$ (see [Ill94, Théorème 4.7]), we may reduce the lemma to the case $(X, H) = (\mathbb{A}^r_S, Z)$. This case is treated in [DK73, Exposé XIII, Proposition 2.1.9]. □

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6.3.4 The following proposition is an analogue of Lemma 6.1.5.

Proposition 6.3.5. *Let L be a finite quasi-Galois extension of K and put $S' = \operatorname{Spec} \mathcal{O}_L$. We denote the residue field of L by E. Let X be an arithmetic S'-scheme with purely d-dimensional generic fiber and assume that there exists a compactification $X \hookrightarrow \overline{X}$ over S' such that $(\overline{X}, \overline{X} \setminus X)$ is a strictly semistable pair over S'. Take any $\sigma \in W_K^+$. Fix an embedding $\overline{K} \hookrightarrow \overline{L}$ and extend σ uniquely to an automorphism of $\overline{L}$. We put $X^\sigma = X \times_{\mathcal{O}_L \nearrow \sigma} \mathcal{O}_L$. Let $\Gamma \subset X^\sigma \times_{S'} X$ be a closed subscheme with purely d-dimensional generic fiber such that the composite $\Gamma \longrightarrow X^\sigma \times_{S'} X \xrightarrow{\mathrm{pr}_1} X^\sigma$ is proper. Then the number*

$$\operatorname{Tr}(\Gamma_L^* \circ \sigma_*; H_c^*(X_{\overline{L}}, \mathbb{Q}_\ell))$$

lies in $\mathbb{Z}[1/p]$ and is independent of ℓ.

Proof. By Lemma 6.3.3, $H_c^i(X_{\overline{L}}, \mathbb{Q}_\ell) \cong H_c^i(X_{\overline{E}}, R\psi_X \mathbb{Q}_\ell)$ and $H_c^i(X^\sigma_{\overline{L}}, \mathbb{Q}_\ell) \cong H_c^i(X^\sigma_{\overline{E}}, R\psi_{X^\sigma} \mathbb{Q}_\ell)$ hold. Moreover we can easily see that the map $\Gamma_L^* \colon H_c^i(X^\sigma_{\overline{L}}, \mathbb{Q}_\ell) \longrightarrow H_c^i(X_{\overline{L}}, \mathbb{Q}_\ell)$ corresponds to the map $\Gamma^* \colon H_c^i(X^\sigma_{\overline{E}}, R\psi_{X^\sigma} \mathbb{Q}_\ell) \longrightarrow H_c^i(X_{\overline{E}}, R\psi_X \mathbb{Q}_\ell)$ (cf. Proposition 4.2.8). Thus the number

$$\operatorname{Tr}(\Gamma_L^* \circ \sigma_*; H_c^*(X_{\overline{L}}, \mathbb{Q}_\ell)) = \operatorname{Tr}(\Gamma^* \circ \sigma_*; H_c^*(X_{\overline{E}}, R\psi_X \mathbb{Q}_\ell))$$

lies in $\mathbb{Z}[1/p]$ and is independent of ℓ by Lemma 6.1.5. □

6.3.6 Let X be a scheme which is smooth and separated of finite type over K. Take a compactification $X \hookrightarrow Z$ over S. Namely, Z is a scheme which is proper and flat over S, containing X as an open subscheme. Put $Y = Z \setminus X$. By de Jong's alteration [deJ96, Theorem 6.5], there exist a finite extension L of K, a connected arithmetic $\mathcal{O}_L$-scheme W, a proper surjective generically finite S-morphism $f \colon W \longrightarrow Z$ such that $(W, f^{-1}(Y))$ is a strictly semistable pair over $S' = \operatorname{Spec} \mathcal{O}_L$. Let H be the horizontal part $f^{-1}(Y)_h$ of $f^{-1}(Y)$. Then (W, H) is a horizontal strictly semistable pair over S' such that $(W \setminus H)_K \longrightarrow X$ is a proper surjective generically finite K-morphism.

By Lemma 6.3.7 below, we can take L as a quasi-Galois extension of K.

Lemma 6.3.7. *Let (X, H) be a horizontal strictly semistable pair over S. Let L be a finite extension of K and put $S' = \operatorname{Spec} \mathcal{O}_L$. Then there exists a blow-up $\pi \colon X' \longrightarrow X \times_S S'$ whose center is contained in the special fiber such that $(X', \pi^{-1}(H))$ is a horizontal strictly semistable pair over S'.*

Proof. We may take the same blow-up as in [SaT03, Lemma 1.11]. □

Theorem 6.3.8. *Let X be a purely d-dimensional scheme which is smooth and separated of finite type over K, and $\Gamma \subset X \times X$ a purely d-dimensional closed subscheme such that the composite $\Gamma \hookrightarrow X \times X \xrightarrow{\mathrm{pr}_1} X$ is proper. Let Z, L, (W, H), $f \colon W \longrightarrow Z$ be as in Paragraph 6.3.6 (we take L as a quasi-Galois extension of K). Put $U = W \setminus H$ and write $g \colon U_L \longrightarrow X$ for the restriction of f. Assume that the composite $\overline{(g \times g)^{-1}(\Gamma)} \hookrightarrow U \times_{S'} U \xrightarrow{\mathrm{pr}_1} U$ is proper ($\overline{(g \times g)^{-1}(\Gamma)}$ denotes the closure of $(g \times g)^{-1}(\Gamma) \subset U_L \times U_L$ in $U \times_{S'} U$). Then for any $\sigma \in W_K^+$, the number*

$$\operatorname{Tr}(\Gamma^* \circ \sigma_*; H_c^*(X_{\overline{K}}, \mathbb{Q}_\ell)) = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}(\Gamma^* \circ \sigma_*; H_c^i(X_{\overline{K}}, \mathbb{Q}_\ell))$$

lies in $\mathbb{Z}[1/p]$ and is independent of ℓ.

Proof. As in Paragraph 6.1.6, we may assume that the extension L/K is separable. Put $V = U_L$. We denote by V' the scheme V considered as a scheme over K. We have $V'_L \cong \coprod_{\tau \in \mathrm{Gal}(L/K)} V^\tau$. Take any $\sigma \in W_K^+$ and put $\Gamma' = (g^\sigma \times g)^![\Gamma] \in \mathrm{CH}_d((g^\sigma \times g)^{-1}(\Gamma))$. For $\tau, \tau' \in \mathrm{Gal}(L/K)$, put $\Gamma'_{\tau,\tau'} = \Gamma'_L|_{V^\tau \times V^{\tau'}}$, where Γ'_L is the base change of Γ' from K to L. As in Paragraph 6.1.6, we have

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the equality

$$\mathrm{Tr}(\Gamma^* \circ \sigma_*; H_c^*(X_{\overline{K}}, \mathbb{Q}_\ell)) = \frac{1}{\deg f} \sum_{\tau \in \mathrm{Gal}(L/K)} \mathrm{Tr}(\Gamma'^{*}_{\sigma\tau,\tau} \circ \sigma_*; H_c^*(V_{\overline{L}}, \mathbb{Q}_\ell)).$$

By the assumption, for each $\tau, \tau' \in \mathrm{Gal}(L/K)$, there exists a cycle $\overline{\Gamma}'_{\tau,\tau'} \in Z_d(U^\tau \times_{S'} U^{\tau'})$ such that $(\overline{\Gamma}'_{\tau,\tau'})_L = \Gamma'_{\tau,\tau'}$ and the composite $|\overline{\Gamma}'_{\tau,\tau'}| \hookrightarrow U^\tau \times_{S'} U^{\tau'} \xrightarrow{\mathrm{pr}_1} U^\tau$ is proper. Therefore we may reduce our theorem to Proposition 6.3.5. □

7. On ℓ-independence for rigid spaces

Let the notation be the same as in the previous section. We consider rigid spaces over a complete discrete valuation field K as adic spaces locally of finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$ (cf. [Hub94]). We denote a scheme by an ordinary italic letter such as X, a formal scheme by a calligraphic letter such as $\mathcal{X}$, and a rigid space by a sans serif letter such as X. For a scheme X over $S = \mathrm{Spec}\,\mathcal{O}_K$, we denote the completion of X along its special fiber by $X^\wedge$. For a formal scheme $\mathcal{X}$ over $\mathrm{Spf}\,\mathcal{O}_K$, we write $\mathcal{X}^{\mathrm{rig}}$ for its Raynaud generic fiber. It is the analytic adic space $d(\mathcal{X})$ in [Hub96, 1.9].

7.1 Smooth case

7.1.1 In this section, we prove our main theorem for smooth rigid spaces. We derive the following consequence from the result in the previous section.

Corollary 7.1.2. *Let X be an arithmetic S-scheme with smooth generic fiber and X the rigid space $(X^\wedge)^{\mathrm{rig}}$. Then for every $\sigma \in W_K^+$, the number*

$$\mathrm{Tr}(\sigma_*; H_c^*(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell)) = \sum_{i=0}^{2\dim \mathsf{X}} (-1)^i \,\mathrm{Tr}(\sigma_*; H_c^i(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell))$$

is an integer that is independent of ℓ.

Proof. We may assume that X is connected and flat over S. We have a W_K-equivariant isomorphism $H_c^i(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell) \cong H_c^i(X_{\overline{F}}, R\psi_X \mathbb{Q}_\ell)$ (see [Hub96, Theorem 5.7.6]). Applying Theorem 6.1.3 to $\Gamma = X \xhookrightarrow{\Delta_X} X \times_S X$, we see that for every $\sigma \in W_K^+$ the number

$$\mathrm{Tr}(\sigma_*; H_c^*(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell)) = \mathrm{Tr}(\sigma_*; H_c^*(X_{\overline{F}}, R\psi_X \mathbb{Q}_\ell))$$

lies in $\mathbb{Z}[1/p]$ and is independent of ℓ. On the other hand, we know that every eigenvalue of the action of $\sigma \in W_K^+$ on $H_c^i(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell)$ is an algebraic integer [Mie06, Theorem 4.2]. Therefore the rational number $\mathrm{Tr}(\sigma_*; H_c^*(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell))$ is an algebraic integer, i.e. an integer. □

Definition 7.1.3. A formal scheme $\mathcal{X}$ of finite type over $\mathcal{O}_K$ is said to be of *type (SA)* (smoothly algebraizable) if there exists an arithmetic S-scheme X with smooth generic fiber such that $\mathcal{X} \cong X^\wedge$. A rigid space X over K is said to be of *type (SA)* if there exists a formal scheme $\mathcal{X}$ of type (SA) over $\mathcal{O}_K$ such that $\mathsf{X} \cong \mathcal{X}^{\mathrm{rig}}$.

Lemma 7.1.4. *Let X be an arithmetic S-scheme with smooth generic fiber. Then the following hold.*

(i) *Every admissible blow-up of $X^\wedge$ is of type (SA).*

(i) *Every open formal subscheme of $X^\wedge$ is of type (SA).*

Proof. (i) Take a uniformizer π of K. Let $\mathcal{I}$ be an open ideal of $\mathcal{O}_{X^\wedge}$. Since the topology of $\mathcal{O}_{X^\wedge}$ is the π-adic topology and $X^\wedge$ is noetherian, there exists an integer n satisfying $\pi^n \mathcal{O}_{X^\wedge} \subset \mathcal{I}$. Denote by $\mathcal{I}'$ the unique ideal of $\mathcal{O}_X$ containing $\pi^n \mathcal{O}_X$ such that $\mathcal{I}/\pi^n \mathcal{O}_{X^\wedge} = \mathcal{I}'/\pi^n \mathcal{O}_X$. It is clear that $\mathcal{I}'\mathcal{O}_{X^\wedge}$

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coincides with $\mathcal{I}$. Then the admissible blow-up of $X^\wedge$ by $\mathcal{I}$ is equal to the π-adic completion of the scheme X' obtained by the blow-up of X by $\mathcal{I}'$. The generic fiber of X' is obviously smooth.

(ii) We can identify the underlying topological space of $X^\wedge$ with that of X_F. Let $\mathcal{U}$ be an open formal subscheme of $X^\wedge$. Then $U = X \setminus (X_F \setminus \mathcal{U})$ is an arithmetic open subscheme of X satisfying $U_F = \mathcal{U}$ as topological spaces. The generic fiber of U is smooth and $U^\wedge = \mathcal{U}$. □

Corollary 7.1.5. *Let $\mathsf{X} = (X^\wedge)^{\mathrm{rig}}$ be a rigid space of type (SA) over K. Then every quasi-compact open subspace U of X is of type (SA).*

Proof. Since U is quasi-compact, there exist an admissible blow-up $\mathcal{Y} \longrightarrow X^\wedge$ and an open formal subscheme $\mathcal{U} \subset \mathcal{Y}$ such that $\mathsf{U} = \mathcal{U}^{\mathrm{rig}}$ (see [BL93, Lemma 4.4]). By Lemma 7.1.4, $\mathcal{Y}$ and $\mathcal{U}$ are of type (SA). This completes the proof. □

Theorem 7.1.6. *Let X be a quasi-compact separated rigid space which is smooth over K. Then for every $\sigma \in W_K^+$, the number*

$$\mathrm{Tr}(\sigma_*; H_c^*(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell))$$

is an integer that is independent of ℓ.

Proof. By [Mie06, Corollary 2.5], there exists a finite open covering $\{\mathsf{U}_i\}_{1\leqslant i\leqslant m}$ of X consisting of rigid spaces of type (SA). Corollary 7.1.5 ensures that each intersection $\mathsf{U}_{i_1} \cap \cdots \cap \mathsf{U}_{i_n}$ is of type (SA). Thus by Corollary 7.1.2, for every $\sigma \in W_K^+$, the number

$$\mathrm{Tr}(\sigma_*; H_c^*((\mathsf{U}_{i_1} \cap \cdots \cap \mathsf{U}_{i_n})_{\overline{K}}, \mathbb{Q}_\ell))$$

is an integer that is independent of ℓ. On the other hand, we have the spectral sequence below:

$$E_1^{-s,t} = \bigoplus_{1\leqslant i_1<\cdots<i_s\leqslant m} H_c^t((\mathsf{U}_{i_1} \cap \cdots \cap \mathsf{U}_{i_s})_{\overline{K}}, \mathbb{Q}_\ell) \Longrightarrow H_c^{-s+t}(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell).$$

Therefore the number

$$\mathrm{Tr}(\sigma_*; H_c^*(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell)) = \sum_{s=1}^{m} (-1)^s \sum_{1\leqslant i_1<\cdots<i_s\leqslant m} \mathrm{Tr}(\sigma_*; H_c^*((\mathsf{U}_{i_1} \cap \cdots \cap \mathsf{U}_{i_s})_{\overline{K}}, \mathbb{Q}_\ell))$$

is also an integer that is independent of ℓ. □

7.1.7 From now on we consider ordinary cohomology. First we establish the analogous result as in [Mie06, Theorem 4.2].

Theorem 7.1.8. *Let X be a quasi-compact separated rigid space which is smooth over K. Then for every $\sigma \in W_K^+$, every eigenvalue $\alpha \in \overline{\mathbb{Q}}_\ell$ of its action on $H^i(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell)$ is an algebraic integer. Moreover, there exists a non-negative integer m such that, for any isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, the absolute value $|\iota(\alpha)|$ is equal to $q^{n(\sigma)\cdot m/2}$.*

Proof. We may assume that X is connected. Put $d = \dim \mathsf{X}$. By the Poincaré duality [Hub96, Corollary 7.5.6], every eigenvalue α of σ_* on $H^i(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell)$ is of the form $q^{n(\sigma)\cdot d}/\beta$, where β is an eigenvalue of σ_* on $H_c^{2d-i}(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell)$. Therefore α is an algebraic number and there exists an integer m such that, for any isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, the absolute value $|\iota(\alpha)|$ is equal to $q^{n(\sigma)\cdot m/2}$.

Thus we have only to show that α is an algebraic integer. By the same method as in [Mie06, § 4], we can reduce the theorem to the case $\mathsf{X} = (X^\wedge)^{\mathrm{rig}}$, where X is strictly semistable scheme over S. Furthermore by using an analogue of weight spectral sequence, we may reduce the claim to Lemma 7.1.9 (cf. [Mie06, proof of Proposition 4.7]). □

Lemma 7.1.9. *Let X be a scheme separated of finite type over $\mathbb{F}_q$. Then every eigenvalue of the action of Fr_q on $H^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ is an algebraic integer (here $\mathrm{Fr}_q \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is the geometric Frobenius element).*

Proof. We may assume that X is irreducible. By de Jong's alteration [deJ96], we may assume that there exist a proper smooth scheme $\overline{X}$ and a strict normal crossing divisor D of $\overline{X}$ such that $X = \overline{X} \setminus D$. Let $D_1, \dots, D_m$ be the irreducible components of D. Put $D_I = \bigcap_{i \in I} D_i$ for $I \subset \{1, \dots, m\}$ ($D_I = \overline{X}$ for $I = \varnothing$) and $D^{(k)} = \coprod_{I \subset \{1,\dots,m\}, \# I = k} D_I$. By the spectral sequence

$$E_1^{-k,n+k} = H^{n-k}(D^{(k)}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(-k)) \Longrightarrow H^n(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell),$$

the eigenvalue α occurs as an eigenvalue of $\mathrm{Fr}^*_{D^{(k)}}$ on $H^{n-k}(D^{(k)}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(-k))$ for some n, k. Since $D^{(k)}$ is proper smooth over $\mathbb{F}_q$, a result from [DK73, Exposé XXI, Corollaire 5.5.3] ensures that α is integral over $\mathbb{Z}$. □

Theorem 7.1.10. *Let* X *be a quasi-compact separated rigid space which is smooth over* K. *Then for every* $\sigma \in W_K^+$, *the number*

$$\mathrm{Tr}(\sigma_*; H^*(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell)) = \sum_{i=0}^{2\dim \mathsf{X}} (-1)^i \,\mathrm{Tr}(\sigma_*; H^i(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell))$$

is an integer that is independent of ℓ.

Proof. By Theorem 7.1.8, it is sufficient to show that the number $\mathrm{Tr}(\sigma_*; H^*(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell))$ is a rational number that is independent of ℓ. We may assume that X is connected. Put $d = \dim \mathsf{X}$. Let $\alpha_{\ell,i,1}, \dots, \alpha_{\ell,i,m_i}$ be the eigenvalues of σ_* on $H^i_c(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell)$. Then the eigenvalues of σ_* on $H^{2d-i}(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell)$ are $q^{n(\sigma)\cdot d}\alpha^{-1}_{\ell,i,1}, \dots, q^{n(\sigma)\cdot d}\alpha^{-1}_{\ell,i,m_i}$ by the Poincaré duality. Therefore it is sufficient to prove that $\sum_{i=0}^{2\dim \mathsf{X}} \sum_{j=1}^{m_i} (-1)^i \alpha^{-1}_{\ell,i,j}$ is a rational number that is independent of ℓ. For every non-negative integer k, by applying Theorem 7.1.6 to $\sigma^k \in W_K^+$, we can see that $\sum_{i=0}^{2\dim \mathsf{X}} \sum_{j=1}^{m_i} (-1)^i \alpha^k_{\ell,i,j}$ is a rational number that is independent of ℓ. As in the proof of Lemma 2.1.3, we may conclude that $\sum_{i=0}^{2\dim \mathsf{X}} \sum_{j=1}^{m_i} (-1)^i \alpha^{-1}_{\ell,i,j}$ is a rational number that is independent of ℓ. □

7.2 General case

7.2.1 In this section, we prove our main theorem for general rigid spaces over local fields of characteristic 0. We need the following continuity theorem of Huber, which is stronger than [Hub98b, Proposition 2.1(iv)] (cf. [Mie06, Theorem 5.3]).

Theorem 7.2.2. *Assume that the characteristic of* K *is equal to* 0. *Let* X *be a quasi-compact separated rigid space over* K *and* Z *a closed analytic subspace of* X. *Write* U *for* $\mathsf{X} \setminus \mathsf{Z}$. *Then for every pair of prime numbers* ℓ, ℓ' *which do not divide* q, *there exists a quasi-compact open subspace* U' *of* U *such that the canonical maps* $H^i_c(\mathsf{U}'_{\overline{K}}, \mathbb{Z}_\ell) \longrightarrow H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Z}_\ell)$ *and* $H^i_c(\mathsf{U}'_{\overline{K}}, \mathbb{Z}_{\ell'}) \longrightarrow H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Z}_{\ell'})$ *are isomorphisms for every* i.

Proof. This is due to [Hub98b, (II) in the proof of Theorem 3.3]. We briefly recall the argument there. By [Hub98a, Corollary 2.7], there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ the canonical map $H^i_c(\mathsf{U}(\varepsilon)_{\overline{K}}, \mathbb{Z}/\ell\mathbb{Z}) \longrightarrow H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Z}/\ell\mathbb{Z})$ is an isomorphism. Here we write $\mathsf{U}(\varepsilon)$ for $P(\varepsilon)$ in [Hub98a, 2.6]. By the long exact sequence of cohomology groups derived from the short exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z}/\ell\mathbb{Z} \xrightarrow{\times \ell^n} \mathbb{Z}/\ell^{n+1}\mathbb{Z} \longrightarrow \mathbb{Z}/\ell^n\mathbb{Z} \longrightarrow 0,$$

we see inductively that the canonical map $H^i_c(\mathsf{U}(\varepsilon)_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) \longrightarrow H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z})$ is an isomorphism for every $0 < \varepsilon < \varepsilon_0$ and n. In the same way, there exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$ and n the canonical map $H^i_c(\mathsf{U}(\varepsilon)_{\overline{K}}, \mathbb{Z}/\ell'^n\mathbb{Z}) \longrightarrow H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Z}/\ell'^n\mathbb{Z})$ is an isomorphism. Put $\varepsilon_2 = \min\{\varepsilon_0, \varepsilon_1\}$ and $\mathsf{U}' = \mathsf{U}(\varepsilon_2)$. Then U' is quasi-compact and both of the canonical maps

$$H^i_c(\mathsf{U}'_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) \longrightarrow H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z}), \quad H^i_c(\mathsf{U}'_{\overline{K}}, \mathbb{Z}/\ell'^n\mathbb{Z}) \longrightarrow H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Z}/\ell'^n\mathbb{Z})$$

are isomorphisms.

On the other hand we have the canonical isomorphisms

$$\varprojlim_n H^i_c(\mathsf{U}'_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong H^i_c(\mathsf{U}'_{\overline{K}}, \mathbb{Z}_\ell), \quad \varprojlim_n H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Z}_\ell),$$
$$\varprojlim_n H^i_c(\mathsf{U}'_{\overline{K}}, \mathbb{Z}/\ell'^n\mathbb{Z}) \cong H^i_c(\mathsf{U}'_{\overline{K}}, \mathbb{Z}_{\ell'}), \quad \varprojlim_n H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Z}/\ell'^n\mathbb{Z}) \cong H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Z}_{\ell'})$$

(see [Hub98b, Theorems 3.1 and 3.3]). Therefore the canonical homomorphisms

$$H^i_c(\mathsf{U}'_{\overline{K}}, \mathbb{Z}_\ell) \longrightarrow H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Z}_\ell), \quad H^i_c(\mathsf{U}'_{\overline{K}}, \mathbb{Z}_{\ell'}) \longrightarrow H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Z}_{\ell'})$$

are isomorphisms. □

Theorem 7.2.3. *Assume that the characteristic of K is equal to 0. Let X be a quasi-compact separated rigid space over K. Then for every $\sigma \in W_K^+$, the number*

$$\mathrm{Tr}(\sigma_*; H^*_c(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell)) = \sum_{i=0}^{2\dim \mathsf{X}} (-1)^i \,\mathrm{Tr}(\sigma_*; H^i_c(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell))$$

is an integer that is independent of ℓ.

Proof. Let ℓ and ℓ' be prime numbers which do not divide q and $\sigma \in W_K^+$. We prove by induction on $\dim \mathsf{X}$ that the numbers

$$\mathrm{Tr}(\sigma_*; H^*_c(\mathsf{X}_{\overline{K}}, \mathbb{Q}_\ell)), \quad \mathrm{Tr}(\sigma_*; H^*_c(\mathsf{X}_{\overline{K}}, \mathbb{Q}_{\ell'}))$$

are integers and are equal. We may assume that X is reduced. Let Z be the singular locus of X. It is a closed analytic subspace whose dimension is strictly less than $\dim \mathsf{X}$. Thus we have only to show our claim on $H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Q}_\ell)$ and $H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Q}_{\ell'})$, where $\mathsf{U} = \mathsf{X} \setminus \mathsf{Z}$. Take an open subspace $\mathsf{U}' \subset \mathsf{U}$ as in Theorem 7.2.2. Then we have the isomorphisms

$$H^i_c(\mathsf{U}'_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Q}_\ell), \quad H^i_c(\mathsf{U}'_{\overline{K}}, \mathbb{Q}_{\ell'}) \xrightarrow{\sim} H^i_c(\mathsf{U}_{\overline{K}}, \mathbb{Q}_{\ell'})$$

by Theorem 7.2.2. Therefore by Theorem 7.1.6 the numbers

$$\mathrm{Tr}(\sigma_*; H^*_c(\mathsf{U}_{\overline{K}}, \mathbb{Q}_\ell)), \quad \mathrm{Tr}(\sigma_*; H^*_c(\mathsf{U}_{\overline{K}}, \mathbb{Q}_{\ell'}))$$

are integers and are equal. This completes the proof. □

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