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Unitary equivalence and reducibility or invertibly weighted shifts

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Let H be a complex Hilbert space and let $\{A_1, A_2, \ldots\}$ be a uniformly bounded sequence of invertible operators on H. The operator S on $l_2(H) = H \oplus H \oplus \ldots$ given by

$$S\langle x_0, x_1, \ldots\rangle = \langle 0, A_1x_0, A_2x_1, \ldots\rangle$$

is called the invertibly weighted shift on $l_2(H)$ with weight sequence $\{A_n\}$. A matricial description of the commutant of S is established and it is shown that S is unitarily equivalent to an invertibly weighted shift with positive weights. After establishing criteria for the reducibility of S the following result is proved: Let $\{B_1, B_2, \ldots\}$ be any sequence of operators on an infinite dimensional Hilbert space K. Then there is an operator T on K such that the lattice of reducing subspaces of T is isomorphic to the corresponding lattice of the W^* algebra generated by $\{B_1, B_2, \ldots\}$. Necessary and sufficient conditions are given for S to be completely reducible to scalar weighted shifts.

1. Introduction

Much attention has been paid recently to shift operators on Hilbert space. If H is a separable complex Hilbert space with orthonormal basis $\{e_0, e_1, \ldots\}$ and $\{\alpha_1, \alpha_2, \ldots\}$ is a bounded sequence of scalars then the operator S defined by $Se_n = \alpha_{n+1}e_{n+1}$ is called the scalar weighted shift with weight sequence $\{\alpha_n\}$. *Most investigations of shifts*

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deal with shifts all of whose weights are non-zero. A natural generalization of weighted shifts is given by considering the Hilbert space $H \oplus H \oplus \ldots$, denoted by $l_2(H)$, and the operator S defined by $S\langle x_0, x_1, \ldots\rangle = \langle 0, A_1x_0, A_2x_1, \ldots\rangle$ where $\{A_1, A_2, \ldots\}$ is a bounded sequence of operators on H. The shift U_+ defined in this way with each $A_n = I$, the identity operator on H, is of great interest and importance originally in investigations of isometries and later in studying general operators. It seems reasonable then to investigate these more general weighted shifts. Two difficulties, both avoided by U_+, are immediately apparent. First, for scalar shifts the product of several weights is independent of order and secondly, if the weights are all non-zero, one may divide by a weight. When the weights are operators order of multiplication is important and a non-zero operator need not be invertible. In this paper only shifts with invertible weights are considered. Commutativity of the weights is not assumed.

In §2 we establish the notation to be used in the remainder of this paper and state without proof some easily verified properties of operator weighted shifts and of operators commuting with such a shift. In §3 we establish necessary and sufficient conditions for two operator weighted shifts to be unitarily equivalent. We then show that every shift is unitarily equivalent to a shift with positive weights. (An operator A is said to be positive if the associated quadratic form (Ax, x) is positive.) We exhibit two shifts whose weights are pairwise unitarily equivalent while the shifts are not unitarily equivalent. We also find necessary and sufficient conditions for an operator weighted shift to have a reducing subspace and characterize all its reducing subspaces. As an application we show that the lattice of invariant subspaces of a countably generated *algebra of operators on an infinite dimensional Hilbert space K is always isomorphic to the lattice of invariant subspaces of a singly generated *algebra of operators on K.

We point out now for future reference that a scalar shift may be thought of as an operator weighted shift on $l_2(\mathbb{C})$.

2. Preliminaries

Throughout this paper H will denote a complex Hilbert space and $l_2(H)$ will be the Hilbert space of all square summable sequences $\langle x_n\rangle_{n=0}^{\infty}$, x_n in H, with inner product $(\langle x_n\rangle, \langle y_n\rangle) = \sum_{n=0}^{\infty} (x_n, y_n)$.

When convenient we will write $\langle x_n\rangle = \sum_{n=0}^{\infty} \oplus x_n$ or $\langle x_n\rangle = \langle x_0, x_1, \ldots\rangle$. For any Hilbert space K, $L(K)$ is the algebra of all bounded linear transformations (operators) from K to K.

For each bounded sequence $A_1, A_2, \ldots$ of operators on H define the shift with weight sequence $\langle A_n\rangle$ to be the linear transformation S on $l_2(H)$ given by $S\langle x_0, x_1, \ldots\rangle = \langle 0, A_1x_0, A_2x_1, \ldots\rangle$. We will sometimes write $S \sim \langle A_n\rangle$ to indicate S is the shift with weight sequence $\langle A_n\rangle$. If $S \sim \langle A_n\rangle$ and each A_n is invertible then we will say S is an invertibly weighted shift. The set of all invertibly weighted shifts on $l_2(H)$ is denoted by $IWl_2(H)$. Let $S \sim \langle A_n\rangle$ be in $IWl_2(H)$. Define the sequence $S_0, S_1, S_2, \ldots$ by $S_0 = I$ (the identity operator on H) and $S_n = A_nA_{n-1} \ldots A_1$, $n \geq 1$. We note that each S_n is invertible and $S_{n+1} = A_{n+1}S_n$. If T is in $L(l_2(H))$ then T can be represented by a matrix $[T_{ij}]_{i,j=0}^{\infty}$ where each T_{ij} is in $L(H)$. The rules for adding, multiplying, and forming the adjoints of operator matrices are the same as in the scalar matrix case. It is also easy to see that $\|T_{ij}\| \leq \|T\|$ for each i, j. The following two lemmas are stated without proof, as their proofs differ only in notation from the proofs in [4] of the corresponding results for scalar matrices.

LEMMA 2.1. *Let $S \sim \langle A_n\rangle$ and $T \sim \langle B_n\rangle$ be in $IWl_2(H)$ and let X be in $L(l_2(H))$ with matrix $[X_{ij}]$. Then $SX = XT$ if and only if*

$$X_{ij} = \begin{cases} 0 & \text{for } i < j , \\ S_i S_{i-j}^{-1} X_{i-j,0} T_j^{-1} & \text{for } i \geq j . \end{cases}$$

LEMMA 2.2. *Let* $S \sim \langle A_n \rangle$ *be in* $IWl_2(H)$. *Then*

(i) $\|S\| = \sup_k \|A_k\|$;

(ii) the spectral radius, $r(S)$, *of* S *is*

$\operatorname{limit}_{n\to\infty} \sup_k \|S_{k+n}S_k^{-1}\|^{1/n}$; *and*

(iii) the spectrum of S *is* $\{\lambda : |\lambda| \leq r(S)\}$. *In addition, the point spectrum of* S *is empty.*

3. Unitary equivalence and reducibility

It is shown in [4] that two scalar weighted shifts are unitarily equivalent if and only if the n-th weights of the two shifts have the same modulus for each n . After establishing some necessary and sufficient conditions for unitary equivalence of operators in $IWl_2(H)$ we will give an example of two such shifts whose weights are pairwise unitarily equivalent but the shifts themselves are not unitarily equivalent. We will also show that every shift in $IWl_2(H)$ is unitarily equivalent to a shift with positive weights.

LEMMA 3.1. *Let* S *and* T *be in* $IWl_2(H)$. *Then* S *and* T *are unitarily equivalent if and only if there is a unitary operator* U *on* H *such that* $T_nUS_n^{-1}$ *is unitary for all* n .

Proof. Suppose U and $T_nUS_n^{-1}$ are unitary for all n . Let $V_n = T_nUS_n^{-1}$ and $V = \sum_{n=0}^{\infty} \oplus V_n$, which is unitary on $l_2(H)$. Since $V_0 = U$, $V_n = T_nV_0S_n^{-1}$. Thus by Lemma 2.1 (with S and T interchanged) $VS = TV$. Conversely, suppose $VS = TV$ where V is unitary with matrix $[V_{ij}]$. Then $SV^* = V^*T$ and so by Lemma 2.1 both V and V^* are lower

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triangular. (We will say an operator $B = [B_{ij}]$ on $l_2(H)$ is lower triangular if $B_{ij} = 0$ for $i < j$.) This implies that V is diagonal, say $V = \sum_{n=0}^{\infty} \oplus V_n$ and each V_n is unitary on H . Moreover, $V_n = T_n V_0 S_n^{-1}$ as required.

COROLLARY 3.2. *If* $S \sim \langle A_n \rangle$ *and* $T \sim \langle B_n \rangle$ *are in* $IWl_2(H)$ *where each* A_n *and each* B_n *is unitary on* H *, then* S *and* T *are unitarily equivalent.*

Proof. $T_n S_n^{-1}$ is unitary for all n . Apply Lemma 3.1 with $U = I$.

The next corollary seems a reasonable generalization of the equal modulus condition for equivalence of scalar shifts.

COROLLARY 3.3. *Let* S *and* T *be in* $IWl_2(H)$. *Then* S *and* T *are unitarily equivalent if and only if there is a unitary operator* U *on* H *such that* $T_n^* T_n U = U S_n^* S_n$ *for all* n .

Proof. If S and T are unitarily equivalent then there is a unitary operator U on H such that $T_n U S_n^{-1}$ is unitary for all n . Thus, for each n ,

$$\begin{aligned} I &= \left(T_n U S_n^{-1}\right)\left(T_n U S_n^{-1}\right)^* \\ &= T_n U\left(S_n^* S_n\right)^{-1} U^* T_n^* , \end{aligned}$$

so that

$$T_n^{-1} T_n^{*-1} = U\left(S_n^* S_n\right)^{-1} U^* ,$$

from which it follows that $\left(T_n^* T_n\right) U = U\left(S_n^* S_n\right)$.

Conversely, suppose U is unitary on H and $T_n^* T_n U = U S_n^* S_n$ for all n . Let $T_n = V_n\left(T_n^* T_n\right)^{1/2}$, $S_n = W_n\left(S_n^* S_n\right)^{1/2}$ be the polar decompositions of T_n and S_n respectively (see [3, p. 68]). Since $\left(T_n^* T_n\right)^{1/2}$ and

$(S_n^*S_n)^{1/2}$ are the uniform limits of sequences $\{P_k(T_n^*T_n)\}$ and $\{P_k(S_n^*S_n)\}$ respectively where $\{P_k\}$ is a sequence of polynomials [3, p. 48], it follows that $(T_n^*T_n)^{1/2}U = U(S_n^*S_n)^{1/2}$, hence

$$\begin{aligned} T_nU &= V_n(T_n^*T_n)^{1/2}U \\ &= V_nU(S_n^*S_n)^{1/2} . \end{aligned}$$

Now, T_n and S_n are invertible so V_n and W_n are unitary. Thus

$$\begin{aligned} T_nU &= V_nUW_n^{-1}W_n(S_n^*S_n)^{1/2} \\ &= V_nUW_n^{-1}S_n , \end{aligned}$$

so that $T_nUS_n^{-1}$ is unitary for all n . By Lemma 3.1, S and T are unitarily equivalent.

Using Corollary 3.3 it is easy to exhibit two non-unitarily equivalent operators in $IWl_2(H)$ whose weights are pairwise unitarily equivalent. Let A be an invertible positive operator on H and let W be a unitary operator on H which does not commute with A . If $S \sim \langle A_n \rangle$ where $A_n = A$ for all n and $T \sim \langle B_n \rangle$ where $B_n = A$ for even n and $B_n = WAW^*$ for odd n , then A_n is unitarily equivalent to B for all n . However, suppose S and T are unitarily equivalent. Then there is a unitary operator U on H such that $T_n^*T_nU = US_n^*S_n$ for all n . In particular, since $T_1 = S_1 = A$, $S_2 = A^2$, and $T_2 = WAW^*A$ we have $A^2U = UA^2$ and

$$\begin{aligned} UA^4 &= US_2^*S_2 \\ &= T_2^*T_2U \\ &= (AWAW^*)(WAW^*A)U \\ &= AWA^2W^*AU . \end{aligned}$$

But $A^4U = UA^4$, so

$$A^2 = WA^2W^* ,$$

hence

$$A^2W = WA^2 .$$

But then $AW = WA$, contradicting the choice of A and W.

A useful form of the unitary equivalence theorem for scalar weighted shifts is that every scalar weighted shift is unitarily equivalent to a shift with non-negative real weights. We now state and prove the analogue of this statement for operators in $IWl_2(H)$.

THEOREM 3.4. *Let $S \sim \langle A_n \rangle$ be in $IWl_2(H)$. Then S is unitarily equivalent to a shift $T \sim \langle B_n \rangle$ in $IWl_2(H)$ where each B_n is positive.*

Proof. Let $A_n = U_nP_n$ be the polar decomposition of A_n, $n = 1, 2, \ldots$. Then P_n is positive and invertible and U_n is unitary. Let $P = \sum_{n=0}^{\infty} \oplus P_{n+1}$ and let $U = \sum_{n=0}^{\infty} \oplus U_{n+1}$. If U_+ denotes the unilateral shift given by $U_+\langle x_0, x_1, \ldots\rangle = \langle 0, x_0, x_1, \ldots\rangle$ then U_+ is in $IWl_2(H)$ and $S = (U_+U)P$. Moreover, this last equation gives the usual polar decomposition of S. Now U_+U is in $IWl_2(H)$ and has unitary weights since $U_+U \sim \langle U_n \rangle$. Thus by Corollary 3.2, there is a unitary operator W such that $U_+U = WU_+W^*$. Moreover, W can be chosen to be diagonal. Thus

$$\begin{aligned} S &= (U_+U)P \\ &= WU_+W^*P \\ &= W(U_+W^*PW)W^* , \end{aligned}$$

that is S is unitarily equivalent to $U_+(W^*PW)$. Now W^*PW is a diagonal operator with positive invertible operators on its diagonal and so $U_+(W^*PW)$ is an operator in $IWl_2(H)$ with positive weights.

The operator U_+ mentioned in the above proof is used extensively in the study of canonical models and isometries (see [1, p. 21], [2]). The lattice of reducing subspaces of U_+ is well known to be the collection of all spaces of the form $\sum_{n=0}^{\infty} \oplus K_n$ where $K_n = K$ for all n and K is a subspace of H [2].

For an arbitrary shift S in $IWl_2(H)$ the lattice of its reducing subspaces need not be as simple as that of U_+, however it does admit a reasonably simple description in terms of the lattice of invariant subspaces for the weakly closed *algebra generated by $\left\{S_n^*S_n\right\}_{n=0}^{\infty}$. We introduce the following notation:

For S in $IWl_2(H)$, $T(S)$ is the weakly closed *subalgebra of $L(H)$ generated by $\left\{S_n^*S_n\right\}_{n=0}^{\infty}$.

For any algebra of operators A, lat$A$ is the lattice of all invariant subspaces of A. By relat$A$ we mean the lattice of all reducing subspaces of A. Similar definitions are assumed for lat$S$ etc.

By "projection" we mean orthogonal projection. Also Q_n is, for each n, the projection of $l_2(H)$ onto $0 \oplus 0 \oplus \ldots \oplus H \oplus 0 \oplus \ldots$, with H in the n-th position.

In this section we will show that relat$S$ is lattice isomorphic to lat$T(S)$. As a corollary we show that for any countable collection $\{B_n\}$ of operators there is a single operator B such that the lattice of reducing subspaces for the *algebra generated by $\{B_n\}$ is isomorphic to relat$B$. We also find a necessary and sufficient condition for an operator S in $IWl_2(H)$ to be decomposable into a direct sum of scalar weighted shifts.

Let $S \sim \langle A_n \rangle$ be in $IWl_2(H)$ and suppose M is a subspace of $l_2(H)$ reducing S. Then for P the projection of $l_2(H)$ onto M, $PS = SP$. Since $P^* = P$ it follows that $P = \sum_{n=0}^{\infty} \oplus P_n$ where each P_n is a projection. Moreover for each n, $P_n = S_nP_0S_n^{-1}$. Let $M_n = P_nH$,

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$n = 0, 1, 2, \ldots$. Then $Pl_2(H) = \sum_{n=0}^{\infty} \oplus M_n$. Also

$$\begin{aligned} S_n M_0 &= S_n P_0 H \\ &= P_n S_n H \subset M_n \\ &= P_n H \\ &= P_n S_n \left(S_n^{-1} H\right) \\ &= S_n P_0 \left(S_n^{-1} H\right) \subset S_n M_0 . \end{aligned}$$

Hence $M_n = S_n M_0$, $n = 0, 1, 2, \ldots$. We have thus proved the following lemma:

LEMMA 3.5. *If* M *reduces* S , *then* $M = \sum_{n=0}^{\infty} \oplus S_n M_0$ *for some subspace* M_0 *of* H . *In particular* $Q_n M$ *is closed for all* n .

Note that for any subspace M_0 of H , $\sum_{n=0}^{\infty} \oplus S_n M_0$ is invariant for S but need not reduce S . We now give some necessary and sufficient conditions for a subspace to reduce S .

THEOREM 3.6. *Let* $M = \sum_{n=0}^{\infty} \oplus S_n M_0$ *be a subspace of* $l_2(H)$. *The following are equivalent:*

(i) M *reduces* S ;

(ii) $S_n M_0$ *is invariant for* $A_{n+1}^* A_{n+1}$, $n = 0, 1, \ldots$;

(iii) $\left(S_n M_0\right)^{\perp} = S_n \left(M_0^{\perp}\right)$, $n = 0, 1, \ldots$;

(iv) $S_n^* S_n M_0 = M_0$, $n = 0, 1, \ldots$.

Proof. We show that *(i)* and *(ii)* are equivalent, *(i)* and *(iii)* are equivalent, and *(iii)* and *(iv)* are equivalent. Suppose M reduces S .

Then $S^*SM \subset M$. It is easily seen that for $\sum_{n=0}^{\infty} \oplus x_n$ in $l_2(H)$,

$S^* \sum_{n=0}^{\infty} \oplus x_n = \sum_{n=0}^{\infty} \oplus A_{n+1}^* x_{n+1}$. Thus for $\sum_{n=0}^{\infty} \oplus S_n x_n$ in M,

$$\begin{aligned} S^*S \sum_{n=0}^{\infty} \oplus S_n x_n &= S^*\left(0 \oplus \sum_{n=1}^{\infty} \oplus A_n S_{n-1} x_{n-1}\right) \\ &= S^*\left(0 \oplus \sum_{n=1}^{\infty} \oplus S_n x_{n-1}\right) \\ &= \sum_{n=0}^{\infty} \oplus A_{n+1}^* S_{n+1} x_n . \end{aligned}$$

Hence for fixed n and x in M_0, $A_{n+1}^* A_{n+1} S_n x$ $\left(= A_{n+1}^* S_{n+1} x\right)$ is in $S_n M_0$, showing that *(i)* implies *(ii)*. But if $A_{n+1}^* A_{n+1} S_n M_0 \subset S_n M_0$ for all n then for $\sum_{n=0}^{\infty} \oplus S_n x_n$ in M,

$$\begin{aligned} S^* \sum_{n=0}^{\infty} \oplus S_n x_n &= \sum_{n=0}^{\infty} \oplus A_{n+1}^* S_{n+1} x_{n+1} \\ &= \sum_{n=0}^{\infty} \oplus A_{n+1}^* A_{n+1} S_n x_{n+1} \subset \sum_{n=0}^{\infty} \oplus S_n M_0 , \end{aligned}$$

showing that *(ii)* implies *(i)*.

To show that *(i)* implies *(iii)* note that if M reduces S then so does $M^\perp$. Thus by Lemma 3.5, $M^\perp = \sum_{n=0}^{\infty} \oplus S_n N_0$ for some $N_0 \subset H$. It is easily seen that

$$\begin{aligned} M^\perp &= \left(\sum_{n=0}^{\infty} \oplus S_n M_0\right)^\perp \\ &= \sum_{n=0}^{\infty} \oplus \left(S_n M_0\right)^\perp \end{aligned}$$

so that $S_n N_0 = \left(S_n M_0\right)^\perp$. In particular $N_0 = M_0^\perp$ and so $S_n\left(M_0^\perp\right) = \left(S_n M_0\right)^\perp$, that is *(iii)* holds. Conversely if $\left(S_nM_0\right)^{\perp} = S_n\left(M_0^{\perp}\right)$ for all n, $M^{\perp} = \sum_{n=0}^{\infty} \oplus \left(S_nM_0\right)^{\perp} = \sum_{n=0}^{\infty} \oplus S_n\left(M_0^{\perp}\right)$ is invariant for S, hence M reduces S.

Finally, to show that *(iii)* and *(iv)* are equivalent note that since $S_n^*S_n$ is an invertible self-adjoint operator, if $S_n^*S_nK \subset K$ for a subspace K, then $S_n^*S_nK = K$. Thus it suffices to show that for any invertible operator A and a subspace K, $(AK)^{\perp} = A\left(K^{\perp}\right)$ if and only if $A^*AK = K$. This is a straightforward calculation and is omitted.

COROLLARY 3.7. *The lattice of reducing subspaces of* S *is isomorphic to* $\operatorname{lat}T(S)$. *In particular* S *is irreducible if and only if* $T(S) = L(H)$.

Proof. Since

$$\operatorname{lat}T(S) = \{M_0 : M_0 \text{ a subspace of } H \text{ and } S_n^*S_nM_0 = M_0, \ n = 0, 1, \ldots\},$$

we have, by Theorem 3.6, the reformulation

$$T(S) = \left\{M_0 \subset H : \left(S_nM_0\right)^{\perp} = S_n\left(M_0^{\perp}\right), \ n = 0, 1, \ldots\right\}$$

and

$$\operatorname{relat}S = \left\{\sum_{n=0}^{\infty} \oplus S_nM_0 : M_0 \text{ in } T(S)\right\}.$$

We now show that the map $\Gamma : M_0 \to \sum_{n=0}^{\infty} \oplus S_nM_0$ is a lattice isomorphism of $\operatorname{lat}T(S)$ onto $\operatorname{relat}S$. Clearly Γ is bijective. Note that

$$\Gamma\left(M_0\right)^{\perp} = \left(\sum_{n=0}^{\infty} \oplus S_n M_0\right)^{\perp}$$
$$= \sum_{n=0}^{\infty} \oplus \left(S_n M_0\right)^{\perp}$$
$$= \sum_{n=0}^{\infty} \oplus S_n\left(M_0^{\perp}\right)$$
$$= \Gamma\left(M_0^{\perp}\right) .$$

We must show that for M_1 and M_2 in $\operatorname{lat}T(S)$,

$$\Gamma(M_1 \vee M_2) = \Gamma(M_1) \vee \Gamma(M_2)$$

and

$$\Gamma(M_1 \cap M_2) = \Gamma(M_1) \cap \Gamma(M_2) .$$

Clearly

$$\Gamma(M_1) + \Gamma(M_2) \subset \Gamma(M_1 \vee M_2) ,$$

so

$$\Gamma(M_1) \vee \Gamma(M_2) \subset \Gamma(M_1 \vee M_2) .$$

Also

$$\left(\Gamma(M_1) \vee \Gamma(M_2)\right)^{\perp} = \left(\Gamma(M_1)\right)^{\perp} \cap \left(\Gamma(M_2)\right)^{\perp}$$
$$= \Gamma\left(M_1^{\perp}\right) \cap \Gamma\left(M_2^{\perp}\right) .$$

But if $\sum_{n=0}^{\infty} \oplus S_n x_n$ is in $\Gamma\left(M_1^{\perp}\right) \cap \Gamma\left(M_2^{\perp}\right)$ then for all n, $S_n x_n$ is in $\left(S_n M_1^{\perp}\right) \cap \left(S_n M_2^{\perp}\right)$. By the invertivility of S_n, x_n is in $M_1^{\perp} \cap M_2^{\perp}$, $n = 0, 1, \ldots$. Thus

$$\left(\Gamma(M_1) \vee \Gamma(M_2)\right)^{\perp} \subset \Gamma\left(M_1^{\perp} \cap M_2^{\perp}\right) = \Gamma\left((M_1 \vee M_2)^{\perp}\right)$$
$$= \left(\Gamma(M_1 \vee M_2)\right)^{\perp}$$

and so $\Gamma(M_1) \vee \Gamma(M_2) = \Gamma(M_1 \vee M_2)$. By repeatedly using the fact that $\Gamma\left(K^{\perp}\right) = \left(\Gamma(K)\right)^{\perp}$ for K in $\operatorname{lat}T(S)$ and by replacing M_1 and M_2 by $M_1^{\perp}$ and $M_2^{\perp}$ respectively in the above calculation we see that

$\Gamma(M_1 \cap M_2) = \Gamma(M_1) \cap \Gamma(M_2)$.

Finally, S is irreducible if and only if $\mathrm{lat}T(S) = \{0, H\}$. Since $T(S)$ is a weakly closed *algebra with identity, $\mathrm{lat}T(S) = \{0, H\}$ if and only if $T(S) = L(H)$.

REMARK. If $S \sim \langle A_n \rangle$ is in $IWl_2(H)$ where each A_n is positive it follows that $\mathrm{lat}T(S)$ equals the lat algebra generated by $\{A_1, A_2, \ldots\}$. But if M_0 is invariant for A_n , $n = 1, 2, \ldots$ then $A_n M_0 = M_0$ and so $S_n M_0 = M_0$. Thus if each A_n is positive then

$$\mathrm{relat}S = \left\{l_2(M_0) : M_0 \text{ invariant for } A_n , \ n = 1, 2, \ldots\right\} .$$

We then have the following curious result.

THEOREM 3.8. *Let* $\left\{B_n\right\}_{n=1}^{\infty}$ *be any sequence of operators on* H *and let* B *be the weakly closed *algebra generated by* $\left\{B_n\right\}_{n=1}^{\infty}$. *Then there is an operator* S *in* $IWl_2(H)$ *such that* $\mathrm{relat}S$ *is isomorphic to* $\mathrm{lat}B$.

Proof. By taking suitable scalar multiples of translations of the real and imaginary parts of each B_n we can find a sequence $\left\{A_n\right\}_{n=1}^{\infty}$ of positive invertible operators with $\|A_n\| \le 1$ for all n which generate B . Then for $S \sim \langle A_n \rangle$, $T(S) = B$, hence $\mathrm{lat}B$ and $\mathrm{relat}S$ are isomorphic.

A long but straightforward calculation shows that if $M = \sum_{n=0}^{\infty} \oplus S_n M_0$ reduces S then the restriction T of S to M is unitarily equivalent to an operator in $IWl_2(M_0)$. It is reasonable then to ask under what conditions S can be decomposed into a direct sum of scalar weighted shifts, that is, when can $l_2(H)$ be written as $\sum_{n=0}^{\infty} \oplus M_n$ in such a way that each M_n reduces S and S restricted to M_n is unitarily equivalent to an operator in some $IWl_2(K_n)$ where K_n is one-dimensional. This question is also motivated by noting that U_+ may be represented as the countable direct sum of the scalar weighted shift all of whose weights are 1. For convenience we will say that an algebra $\mathcal{B}$ of operators is diagonalizable if there is an orthonormal basis for the underlying space such that each operator in $\mathcal{B}$ is diagonal with respect to this basis.

THEOREM 3.9. *Let d be the dimension of H and assume $d \le \aleph_0$. Let $S \sim \langle A_n \rangle$ be in $IWl_2(H)$. Then S is a direct sum of scalar weighted shifts if and only if $T(S)$ is diagonalizable.*

Proof. By Theorem 3.4 we may assume that each A_n is positive and hence that $T(S)$ is the weakly closed *algebra generated by $\{I, A_1, A_2, \ldots\}$. Since each A_n is positive $T(S)$ is diagonalizable if and only if the set of common eigenvectors of $\left\{A_n\right\}_{n=1}^{\infty}$ spans H. Suppose first that $\left\{x_k\right\}_{k=0}^{d}$ is an orthonormal basis for H consisting of common eigenvectors of $\{A_n\}$. Let $[x_k]$ denote the one-dimensional space spanned by x_k. Since $[x_k]$ is invariant for $T(S)$, $l_2([x_k])$ reduces S. Moreover since $H = \sum_{k=0}^{d} \oplus [x_k]$, $l_2(H) = \sum_{k=0}^{d} \oplus l_2([x_k])$. Now the restriction of S to $l_2([x_k])$ is a shift and since $[x_k]$ is one-dimensional, this restriction is a scalar-weighted shift. Thus S is a direct sum of scalar weighted shifts.

Conversely suppose $VSV^{-1} = \sum_{k=0}^{m} \oplus T_k$ where V is unitary on $l_2(H)$, $m \le \aleph_0$, and each T_k is a scalar weighted shift on some separable subspace M_k of $l_2(H)$. Then $l_2(H) = \sum_{k=0}^{m} \oplus V^{-1}M_k$ and the

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restriction of S to $V^{-1}M_k$ is a scalar weighted shift. Let $N_k = V^{-1}M_k$. Since $\{N_k\}_{k=0}^{m}$ is a family of pairwise orthogonal reducing subspaces of S, $\{Q_0N_k\}_{k=0}^{m}$ is a family of pairwise orthogonal reducing subspaces of $T(S)$. (Here we have identified Q_0N_k with the set of its first coordinates since all other coordinates of Q_0N_k are 0. We have also used Lemma 3.5 to say that Q_0N_k is closed.) Moreover, since $l_2(H) = \sum_{k=0}^{m} \oplus N_k$, $H = Q_0 l_2(H) = \sum_{k=0}^{m} \oplus Q_0N_k$. Thus to show that $T(S)$ is diagonalizable it suffices to show that Q_0N_k is one dimensional for $0 \le k \le m$. Fix k and let $\{f_0, f_1, \ldots\}$ be a basis for N_k shifted by S, say $Sf_j = \alpha_{j+1}f_{j+1}$. (Note that $\alpha_{j+1} \neq 0$ for all j since S is kernel free.) For each j let $f_j = \sum_{n=0}^{\infty} \oplus g_{n,j}$ be the decomposition of f_j with respect to $l_2(H)$. Then

$$\begin{aligned} Sf_j &= \alpha_{j+1}f_{j+1} \\ &= \sum_{n=0}^{\infty} \oplus \alpha_{j+1}g_{n,j+1} . \end{aligned}$$

However

$$\begin{aligned} Sf_j &= S \sum_{n=0}^{\infty} \oplus g_{n,j} \\ &= 0 \oplus \sum_{n=1}^{\infty} \oplus A_n g_{n-1,j} , \end{aligned}$$

so that for all j and n

$$\alpha_{j+1}g_{0,j+1} = 0$$

and

$$\alpha_{j+1}g_{n+1,j+1} = A_{n+1}g_{n,j} .$$

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Thus $g_{n,j} = 0$ if $n < j$.

Now,

$$Q_0 f_j = Q_0 \sum_{n=0}^{\infty} \oplus g_{n,j}$$
$$= g_{0,j} ,$$

hence $Q_0 f_j = 0$ unless $j = 0$. Thus

$$Q_0 N_k = Q_0 \bigvee_{j=0}^{\infty} f_j$$
$$= [Q_0 f_0] ,$$

which is one-dimensional.

REMARKS. (1) Only slight modifications of the above proof lead to the following result. If S is in $IWl_2(H)$ then S is unitarily equivalent to an operator of the form $R \oplus T$ where R is a direct sum of scalar weighted shifts and no restriction of T to a reducing subspace is a scalar weighted shift. Here R acts on $\sum_{n=0}^{\infty} \oplus S_n M_0$ where M_0 is the closed span of the common eigenvectors of all the operators in $T(S)$.

(2) If $T_1, T_2, \ldots, T_n$ are shifts on $l_2(H_1), l_2(H_2), \ldots, l_2(H_n)$ respectively $(n < \infty)$ then $T_1 \oplus T_2 \oplus \ldots \oplus T_n$ is unitarily equivalent to a shift on $l_2\left(\sum_{k=0}^{n} \oplus H_k\right)$. This follows easily from the fact that the map from $\sum_{k=0}^{n} \oplus l_2(H_k)$ to $l_2\left(\sum_{k=0}^{n} \oplus H_k\right)$ is a unitary isomorphism.

Shields and Wallen proved in [5] that the weakly closed algebra generated by the identity and an injective scalar weighted shift S is exactly the algebra of all bounded operators commuting with S . Although this no longer holds for operators in $IWl_2(H)$, it seems plausible that the algebra $A''(S)$ of all operators commuting with every bounded operator commuting with S is the weakly closed algebra generated by I and S , S in $IWl_2(H)$. We have not been able to prove this conjecture or find a

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counterexample.

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