BULL. AUSTRAL. MATH. SOC. 31C99, 35J99
VOL. 9 (1973), 1-9.

# A remark on a mean value theorem of Alexander Weinstein in Generalized Axially Symmetric Potential Theory

**J.B. Diaz and John G. Leschen**

Cordially dedicated to Professor Alexander Weinstein,
on the occasion of his Seventyseventh birthday, January 21, 1974.

This note contains the proof of an extension of Alexander Weinstein's mean value theorem for Generalized Axially Symmetric Potential Theory.

## 1. Introduction

In his, now classical, paper [4], on Generalized Axially Symmetric Potential Theory (GASPT), Weinstein proved a mean value theorem, which he states as follows (see page 344 of [4]; the notation (WT) is introduced in the present note).

"MEAN VALUE THEOREM.

$$\phi(0,\ 0) \int_0^\pi \sin^p\theta d\theta = \int_0^\pi \phi(x,\ y)\sin^p\theta d\theta\ , \tag{WT}$$

*where* $\theta$ *denotes the polar angle. The integral on the right-hand side is taken over a half-circle of arbitrary radius* $a$ .

Received 5 February 1973.

*A similar formula holds obviously for* $\phi(x_0, 0)$ *at any regular point* $(x_0, 0)$ *of* $\phi$ *."*

In this quotation from [4], $\phi(x, y)$ is a solution of the partial differential equation (see page 343 of [4]):

$$\text{(GP)} \qquad y\left(\phi_{xx}+\phi_{yy}\right) + p\phi_y = 0 \ ,$$

which is even in $y$, is analytic in the two real variables $(x, y)$ in an open set which intersects the $x$-axis, and is such that the partial derivative $\frac{\partial\phi}{\partial y}$ vanishes for $y = 0$.

(The notations (WT) and (GP) have been introduced in the present note, for convenience of reference.)

## 2. Weinstein's mean value theorem for $p > -1$

Weinstein's development of his mean value theorem (WT) is based on the assumptions that $p > 0$ and that $\phi$ is a solution of (GP) which is even in $y$ and is analytic in a region containing a segment of the $x$-axis. Weinstein also uses the fact that there is an associated "stream function" $\psi$ which satisfies the partial differential equation

$$y\left(\psi_{xx}+\psi_{yy}\right) - p\psi_y = 0 \ .$$

However, it is clear that the definite integral on the right hand side of (WT) exists, for $p > -1$, whenever the integrand function $\phi(x, y)$ is continuous on $0 \leq \theta \leq \pi$. Furthermore, the premise that $\phi$ is an even function of $y$ appears to be somewhat unnecessary, specially when one considers that any solution of (GP) which is "regular" on a portion of the singular line $y = 0$, must be such that, if $p \neq 0$, its partial derivative $\frac{\partial\phi}{\partial y}$ vanishes on the $x$-axis.

We are, therefore, led to ask whether Weinstein's mean value theorem (WT) might hold for $p > -1$, $p \neq 0$, with weakened hypotheses on the function $\phi$. An affirmative answer to this question is provided by the following extension of the Weinstein Theorem (WT):

THEOREM. *Let* $p > -1$, $p \neq 0$. *Let* $G$ *be an open set in the upper half plane* $y > 0$, *and suppose that the boundary of* $G$ *contains a non-*

*degenerate open segment* $I$ *of the* $x$*-axis. In* $G$ *, let* $\phi(x, y) \in C^2$ *, with bounded second derivatives, be a real valued solution of* (GP). *In* $G \cup I$ *, let* $\phi \in C$ *. Let* $x_0$ *and* $R$ *,* $R > 0$ *, be any real numbers such that the semicircle* $(x-x_0)^2 + y^2 \leq R^2$ *,* $y \geq 0$ *, lies entirely in* $G \cup I$ *. Then,* (WT) *holds.*

Proof. (It is true, as follows from a known general theorem, that any solution, of class $C^2$ , of the elliptic partial differential equation (GP), must necessarily be analytic, in $(x, y)$ , in $G$ . However, this fact is not used in the present proof.)

Since (GP) is invariant under $x$-translations, the center of the semicircle may be taken to be the origin; thus, $x_0 = 0$ . With $r$ and $\theta$ the usual polar coordinates, we define the "fan-shaped" region

$$D(\alpha, \beta) = \{(r, \theta) \mid \alpha R < r < R, \beta < \theta < \pi-\beta\} ,$$

where $0 < \alpha < 1$ and $0 < \beta < \pi/2$ . Clearly, $D(\alpha, \beta)$ lies in $G$ .

The proof will be based on the application of two Green's identities for the partial differential operator occurring in equation (GP), which are to be found in Weinstein [4, page 343, equations (4) and (5)]:

$$(4) \qquad \iint_R y^p\left(\phi_{xx}+\phi_{yy}+py^{-1}\phi_y\right)dxdy = \int_C y^p \frac{\partial \phi}{\partial n}\, dx ,$$

$$(5) \qquad \iint_R \left[\phi^*\mathrm{div}\left(y^p\mathrm{grad}\phi\right)-\phi\mathrm{div}\left(y^p\mathrm{grad}\phi^*\right)\right]dxdy = \int_C y^p\left(\phi^* \frac{\partial \phi}{\partial n} - \phi \frac{\partial \phi^*}{\partial n}\right)ds ,$$

where $C$ is the boundary of a domain $R$ , the letter $s$ denotes arc length on $C$ , and $n$ is the exterior normal to $C$ , and $\phi$ and $\phi^*$ are "regular" functions (of class $C^2$ in $G$ ). Identities (4) and (5) were given by Weinstein for $p > 0$ , but they remain valid for any real $p$ .

Several inequalities, which will be needed in the course of the proof, will now be established. Recall that, by hypothesis, the first and second partial derivatives of $\phi$ are bounded in $G$ . Thus, there is a positive number $M$ that is a common upper bound of $|\phi_x|$, $|\phi_y|$, $|\phi_{xx}|$, $|\phi_{yy}|$ , in $G$ . Then, as a consequence of the mean value theorem of the differential calculus for functions of two variables, given $\alpha$ and $\theta$ , there is a "mean value point" $Q$ , interior to the straight line segment joining the origin $(0, 0)$ to the point $(\alpha R, \theta)$ , where $0 < \alpha < 1$ and $0 < \theta < \pi$ , such that (with $\phi_0 = \phi(0, 0)$ ):

$$\phi(\alpha R, \theta) - \phi_0 = (\alpha R\cos\theta) \cdot \phi_x(Q) + (\alpha R\sin\theta) \cdot \phi_y(Q) .$$

Therefore,

$$|\phi(\alpha R, \theta)-\phi_0| \leq 2M\alpha R , \text{ for } 0 < \theta < \pi . \tag{6}$$

Since $\phi$ is continuous on $G \cup I$ , the inequality just written continues to hold on the closed interval $0 \leq \theta \leq \pi$ . Further, one has

$$\lim_{\alpha\to+0} \phi(\alpha R, \theta) = \phi_0 , \tag{7}$$

with the convergence being uniform on the closed interval $0 \leq \theta \leq \pi$ . Also, it follows, directly from (GP), that

$$|\phi_y| = |y(\phi_{xx}+\phi_{yy})/p| \leq 2yM/|p| \tag{8}$$

in $G$ .

Now, consider two points of the closure of $D(\alpha, \beta)$ , with polar coordinates $(r, \beta)$ and $(r, \pi-\beta)$ , and, together with them, the "lens shaped" domain $D(\beta)$ which is bounded above by the closed *circular* arc joining these two points, with center at the origin and radius $r > 0$ , and lying in the upper half plane $y > 0$ ; and is bounded below by the *straight* line segment joining these two points, also lying in the half plane $y > 0$ , and which is a subset of the straight line of equation $y = r\sin\beta$ . If Green's first identity (4) is applied to this domain $D(\beta)$ , then, since $\phi$ is a solution of (GP) in $G$ , the result is

$$\int_{\partial D(\beta)} y^p \frac{\partial\phi}{\partial n} ds = 0 ,$$

where $\partial D(\beta)$ denotes the boundary of $D(\beta)$ . Written out explicitly, this means that

$$r^p \int_\beta^{\pi-\beta} (\sin^p\theta) \frac{\partial\phi}{\partial n} (r, \theta)rd\theta - \int_{-r\cos\beta}^{r\cos\beta} (r^p\sin^p\beta) \frac{\partial\phi}{\partial y} (x, r\sin\beta)dx = 0 .$$

Therefore, in view of the inequality (8),

https://doi.org/10.1017/S0004972700042805 Published online by Cambridge University Press

$$\left|r\int_{\beta}^{\pi-\beta}(\sin^p\theta)\,\frac{\partial\phi}{\partial n}\,(r,\ \theta)d\theta\right| \le \int_{-r\cos\beta}^{r\cos\beta}(\sin^p\beta)\,\left|\frac{\partial\phi}{\partial y}\,(x,\ r\sin\beta)\right|dx$$
$$\le 4Mr\left(\sin^{p+1}\beta\right)(r\cos\beta)/|p|\ ,$$

and this means that (since $p+1 > 0$ , and $p \neq 0$ ),

$$\lim_{\beta\to+0}\int_{\beta}^{\pi-\beta}(\sin^p\theta)\,\frac{\partial\phi}{\partial n}\,(r,\ \theta)d\theta = 0\ . \tag{9}$$

(This last equation holds, in particular, for $r = \alpha R$ and for $r = R$ , and will be used below for these two particular values of $r$ .)

Let us now apply the identity (5) to the closure of the domain $D(\alpha,\ \beta)$ . We take $\phi^*$ to be the function $\phi^*(r) = \left(r^{-p}\right)/p$ , where $r^2 = x^2 + y^2$ and $p \neq 0$ (it is at this point of the proof that the restriction $p \neq 0$ is again essential). This function $\phi^*$ can easily be shown to be a solution of (GP) which is regular for $y > 0$ . Since all points of the closure of $D(\alpha,\ \beta)$ lie above the $x$ axis, and since $\phi$ is regular there, then $\operatorname{div}\left(y^p\operatorname{grad}\phi\right) = \operatorname{div}\left(y^p\operatorname{grad}\phi^*\right) = 0$ on $D(\alpha,\ \beta)$ , and the integral over $D(\alpha,\ \beta)$ vanishes. The final result is:

$$\int_{\partial D(\alpha,\beta)} y^p\left(\phi^*\,\frac{\partial\phi}{\partial n} - \phi\,\frac{\partial\phi^*}{\partial n}\right)ds = 0\ , \tag{10}$$

where $\partial D(\alpha,\ \beta)$ denotes the boundary of $D(\alpha,\ \beta)$ .

Now, along any radius, $\frac{\partial\phi^*}{\partial n} \equiv 0$ ; on the circular arc of radius $R$ , $\frac{\partial\phi^*}{\partial n} = -R^{-p-1}$ ; and, on the circular arc of radius $\alpha R$ , $\frac{\partial\phi^*}{\partial n} = (\alpha R)^{-p-1}$ . Hence,

$$-\int_{\partial D(\alpha,\beta)} y^p\phi\,\frac{\partial\phi^*}{\partial n}\,ds = \int_{\beta}^{\pi-\beta}\phi(R,\ \theta)\sin^p\theta d\theta - \int_{\beta}^{\pi-\beta}\phi(\alpha R,\ \theta)\sin^p\theta d\theta\ . \tag{11}$$

Now, for $p+1 > 0$ , the definite integral

$$\int_0^{\pi}\sin^p\theta d\theta$$

is improper, but convergent, as follows from the well known inequalities

$2\theta/\pi \leq \sin\theta \leq \theta$ , for $0 \leq \theta \leq \frac{\pi}{2}$ . Consequently, making use of the uniform convergence over $0 \leq \theta \leq \pi$ in (7), the desired result (WT) would seem to follow upon first taking the limit as $\beta$ approaches $0$ in (11), and then taking the limit as $\alpha$ approaches $0$ , in succession. That is

$$\lim_{\beta\to+0}\left[-\int_{\partial D(\alpha,\beta)} y^p\phi\,\frac{\partial\phi^*}{\partial n}\,ds\right] = \int_0^\pi \phi(R,\ \theta)\sin^p\theta d\theta - \int_0^\pi \phi(\alpha R,\ \theta)\sin^p\theta d\theta \ ,$$

and

$$(12)\quad \lim_{\alpha\to+0}\ \lim_{\beta\to+0}\left[-\int_{\partial D(\alpha,\beta)} y^p\phi\,\frac{\partial\phi^*}{\partial n}\,ds\right] =$$

$$= \int_0^\pi \phi(R,\ \theta)\sin^p\theta d\theta - \int_0^\pi \phi_0\sin^p\theta d\theta \ .$$

Consequently, in order to complete the proof of (WT), it only remains to verify that (see (10)):

$$(13)\qquad \lim_{\alpha\to+0}\ \lim_{\beta\to+0}\left[\int_{\partial D(\alpha,\beta)} y^p\phi^*\,\frac{\partial\phi}{\partial n}\,ds\right] = 0 \ .$$

Now, when written out explicitly, the integral is

$$(14)\quad \int_{\partial D(\alpha,\beta)} y^p\phi^*\,\frac{\partial\phi}{\partial n}\,ds = \left(\sin^p\beta\right)\int_{\alpha R}^{R} r^p\phi^*(r)\left[\phi_n(r,\ \beta)+\phi_n(r,\ \pi-\beta)\right]dr$$

$$+ (\alpha R)^{p+1}\phi^*(\alpha R)\int_\beta^{\pi-\beta}\phi_n(\alpha R,\ \theta)\sin^p\theta d\theta$$

$$+ R^{p+1}\phi^*(R)\int_\beta^{\pi-\beta}\phi_n(R,\ \theta)\sin^p\theta d\theta \ ,$$

where the definition of the auxiliary function $\phi^*$ is to be kept in mind. As $\beta$ approaches zero, the last two terms on the right hand side of equation (14) approach zero, by equation (9) for the particular values $r = \alpha R$ and $r = R$ . Further, from inequality (8) for $|\phi_y|$ , and the common bound $M$ for all the first and second partial derivatives of $\phi$ in $G$ , we obtain

https://doi.org/10.1017/S0004972700042805 Published online by Cambridge University Press

(15) $$|\phi_n(r,\,\beta)| = |\phi_x(r,\,\beta)\sin\beta - \phi_y(r,\,\beta)\cos\beta| \le M\sin\beta + (2r\sin\beta)M\cos\beta/|p| \le (M\sin\beta)(1+2R/|p|)\ ,$$

and the same upper bound is valid for $|\phi_n(r,\,\pi-\beta)|$ , yielding

(16) $$|\phi_n(r,\,\beta)| + |\phi_n(r,\,\pi-\beta)| \le (2M\sin\beta)(1+2R/|p|)\ .$$

Using the definition of the auxiliary function $\phi^*$ , the inequality (16), and the inequality $p+1 > 0$ , it follows that the first term on the right hand side of (14) also approaches zero as $\beta$ approaches $0$ . Therefore, from (14) we have that

(17) $$\lim_{\beta\to+0} \int_{\partial D(\alpha,\beta)} y^p \phi^* \frac{\partial\phi}{\partial n}\, ds = 0\ ,$$

which means that

(18) $$\lim_{\alpha\to+0}\ \lim_{\beta\to+0} \int_{\partial D(\alpha,\beta)} y^p \phi^* \frac{\partial\phi}{\partial n}\, ds = 0\ .$$

The desired result, (WT), now follows directly from (10), (12) and (18), and the proof is complete.

It is a corollary that the theorem holds for all $p > -1$ , provided we require, for $p = 0$ , that $\phi_y(x,\,0) \equiv 0$ on $I$ . In fact, when $p = 0$ , the harmonic function $\phi$ can then be continued into the lower half-plane as an even function of $y$ , and (WT) is equivalent to the classical Gauss mean value theorem for harmonic functions.

## 3. Weinstein's mean value theorem for $p \le -1$ ?

It is clear from (WT), however, that the theorem cannot hold for $p \le -1$ , unless (possibly) restrictions are placed on $\phi$ . The same conclusion can be reached, in a quite intuitive formal manner, by recalling Weinstein's concept of $\phi$ as an axially symmetric potential in a fictitious space of $(p+2)$ dimensions (for definiteness, if desired, think of $p$ as a positive integer). Indeed, if $dS$ and $r$ are, respectively, the surface element and radius of a sphere, centered at the origin, in $(p+2)$-dimensional space, and, if $\omega_{p+1}$ is the surface area of a $(p+1)$-dimensional unit sphere, then the classical Gauss mean value relation for the sphere (for harmonic functions), becomes, by virtue of the axial symmetry, a relation on a meridian plane:

$$\phi_0 = \frac{\int \phi dS}{\int 1 dS} = \frac{\int_0^\pi \phi y^p \omega_{p+1} d\theta}{\int_0^\pi y^p \omega_{p+1} d\theta}$$

$$= \frac{\omega_{p+1}\int_0^\pi \phi(r\sin\theta)^p d\theta}{\omega_{p+1}\int_0^\pi (r\sin\theta)^p d\theta} = \frac{\int_0^\pi \phi\sin^p\theta d\theta}{\int_0^\pi \sin^p\theta d\theta} .$$

This is precisely (WT), and it is apparent that the Weinstein Theorem concerns the mean value of a symmetric harmonic function on a sphere in $(p+2)$-dimensional space. Now, one of these $(p+2)$ dimensions must be assigned to the axis of symmetry, leaving $(p+1)$ "residual" dimensions at our disposal for constructing the $(p+2)$-dimensional sphere about the axis of symmetry. Unless $p > -1$ , however, no "residual" dimensions are available for constructing the sphere, and this intuitive interpretation of the theorem fails. Such considerations must, of course, be purely formal, without a definition of nonintegral dimensionality.

Can "analytic continuation" (with respect to the real variable $p$ , starting with the interval $p > -1$ , and then proceeding to the interval $p \leq -1$ ; compare, for example, Riesz [3], Diaz and Ludford [1], Diaz and Weinberger [2]); or, alternatively, Weinstein's "correspondence principle" (compare, for example, Weinstein [5]), be employed to provide an answer to the question formulated in the title of this section?

## References

[1] J.B. Diaz and G.S.S. Ludford, "On the Euler-Poisson-Darboux equation, integral operators, and the method of descent", *Proc. Conf. Diff. Equ., University of Maryland, March* 1955, 73-89 (University of Maryland Book Store, College Park, Maryland; 1956).

[2] J.B. Diaz and H.F. Weinberger, "A solution of the singular initial value problem for the Euler-Poisson-Darboux equation", *Proc. Amer. Math. Soc.* 4 (1953), 703-715.

[3] Marcel Riesz, "L'intégrale de Riemann-Liouville et le problème de Cauchy", *Acta Math.* 81 (1949), 1-223.

[4] Alexander Weinstein, "Discontinuous integrals and generalized potential theory", *Trans. Amer. Math. Soc.* 63 (1948), 342-354.

[5] Alexander Weinstein, "Generalized axially symmetric potential theory", *Bull. Amer. Math. Soc.* 59 (1953), 20-38.

Rensselaer Polytechnic Institute,
Troy,
New York,
USA;

1170 Mohawk Road,
Schenectady,
New York,
USA.