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SINGULARITY SUBTRACTION IN THE NUMERICAL SOLUTION OF INTEGRAL EQUATIONS

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Abstract

The singularity subtraction technique described by Kantorovich and Krylov in [11] is designed to reduce or overcome the effect of a weakly singular kernel in the numerical solution of integral equations. First, the equation is rearranged in such a way that the singularity of the kernel is at least partially cancelled by the smoothness of the solution, and then numerical integration is applied. We present convergence results and error bounds under general conditions on the nature of the singularity and the numerical integration procedure. Numerical examples demonstrate the benefit of the singularity subtraction technique.

1. Introduction and Summary

We describe the singularity subtraction technique in the context of a Fredholm integral equation on $C[0, 1]$,

$$x(s) - \int_0^1 k(s, t)x(t)\,dt = y(s), \tag{1.1}$$

where $k(s, t)$ is singular along $s = t$. For example, $k(s, t)$ might have a singular factor $|s - t|^{-1/2}$ or $\ln|s - t|$. Precise hypotheses on $k(s, t)$ will be formulated later. They will imply that the integral operator in (1.1) maps $C[0, 1]$ into $C[0, 1]$ and is compact.

Rearrange (1.1) in the form

$$\left[1 - \int_0^1 k(s, t)\,dt\right]x(s) + \int_0^1 k(s, t)\big[x(s) - x(t)\big]\,dt = y(s). \tag{1.2}$$

The continuity of x mitigates the effect of the singularity in the last integral. So it should be more amenable for numerical integration than the integral in (1.1).

We shall use a numerical procedure on (1.2) which was developed first for the numerical integration of weakly singular functions in [3] and then applied to (1.1) in [2]. It is a double approximation scheme consisting of singularity truncation and numerical integration. First, $k(s, t)$ is approximated by bounded kernels $k_n(s, t)$, $n = 1, 2, \ldots$, which coincide with $k(s, t)$ outside certain neighborhoods of $s = t$. Then an appropriate quadrature rule is applied.

With this procedure, (1.1) and (1.2) are approximated respectively by

$$x_n(s) - \sum_{j=1}^{n} w_{nj} k_n(s, t_{nj}) x_n(t_{nj}) = y(s), \tag{1.3}$$

and

$$\left[1 - \int_0^1 k(s, t)\, dt\right] \hat{x}_n(s) + \sum_{j=1}^{n} w_{nj} k_n(s, t_{nj}) \left[\hat{x}_n(s) - \hat{x}_n(t_{nj})\right] = y(s). \tag{1.4}$$

An equivalent formulation of (1.4) is

$$\left[1 + \sum_{j=1}^{n} w_{nj} k_n(s, t_{nj}) - \int_0^1 k(s, t)\, dt\right] \hat{x}_n(s) - \sum_{j=1}^{n} w_{nj} k_n(s, t_{nj}) \hat{x}_n(t_{nj}) = y(s). \tag{1.5}$$

The hypotheses will imply that the coefficient of $\hat{x}_n(s)$ in (1.5) converges uniformly to unity as $n \to \infty$. Therefore, the equation for $\hat{x}_n$ can be regarded as a perturbation of the equation for x_n when n is large. This fact will be exploited in order to extend to $\hat{x}_n$ results for x_n derived in [2].

The equations for x_n and $\hat{x}_n$ reduce to $n \times n$ linear systems for $x_n(t_{ni})$ and $\hat{x}_n(t_{ni})$, $i = 1, \ldots, n$. If these systems are solved, then solutions x_n and $\hat{x}_n$ of (1.3)–(1.5) are given by

$$x_n(s) = y(s) + \sum_{j=1}^{n} w_{nj} k_n(s, t_{nj}) x_n(t_{nj}), \tag{1.6}$$

and

$$\hat{x}_n(s) = \frac{y(s) + \sum_{j=1}^{n} w_{nj} k_n(s, t_{nj}) \hat{x}_n(t_{nj})}{1 + \sum_{j=1}^{n} w_{nj} k_n(s, t_{nj}) - \int_0^1 k(s, t)\, dt}, \tag{1.7}$$

the latter for n sufficiently large. The replacement of k by k_n makes the calculations in (1.6) and (1.7) more tractable for s near some t_{nj}. The presence of

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$k_n(s, t_{nj})$ in both the numerator and denominator in (1.7) has a stabilizing influence which is absent in (1.6).

Conditions will be given which relate the unique solvability of the equations for x, x_n, and $\hat{x}_n$, and which imply uniform convergence $x_n \to x$ and $\hat{x}_n \to x$ plus error bounds. In general, the convergence $\hat{x}_n \to x$ is faster than $x_n \to x$. This will be demonstrated both theoretically and numerically.

It has been assumed tacitly that the integral in (1.4) can be evaluated in closed form. However, if this integral is approximated by means of a quadrature formula of high order relative to n, perhaps order $2n$, then some of the advantage of the singularity subtraction technique remains. With this modification, (1.4) still reduces to an $n \times n$ system and the subsequent analysis requires only minor changes, which are left to the reader. Convergence results and error bounds for the approximate solutions still follow.

The singularity subtraction technique will be compared with the singularity factorization technique of Atkinson [4], [5]. Recent contributions to the numerical solution of weakly singular integral equations by Bechlars [6], [7], Borer [8], Graham [10], Schneider [16], Sloan [17]–[20], and Volk [21] will be mentioned briefly.

2. The quadrature formula

Assume that the quadrature rule satisfies

$$\sum_{j=1}^{n} w_{nj} x(t_{nj}) \to \int_0^1 x(t)\, dt, \qquad x \in C[0, 1], \tag{2.1}$$

with

$$0 \leqslant t_{n1} < \cdots < t_{nn} \leqslant 1, \qquad w_{nj} \geqslant 0. \tag{2.2}$$

The last condition is a convenience rather than a necessity. Also assume either of the conditions

$$\text{H1:} \quad \sum_{t_{nj} \in E} w_{nj} \leqslant \frac{c}{n}, \qquad E = [a, b) \text{ and } (a, b], \quad b - a \leqslant \frac{1}{n},$$

or

$$\text{H2:} \quad w_{nj}, w_{n,j-1} \leqslant c(x_{nj} - x_{n,j-1}), \qquad j = 2, \ldots, n,$$

with some constant c. Romberg rules and the usual compound rules satisfy both H1 and H2. Gauss and Féjer rules satisfy H2. For further details see [3] and [16].

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3. Singular kernels and bounded approximations

First, consider a kernel $k(s, t)$ with a monotone symmetric singular factor:

$$k(s, t) = g(|s - t|)h(s, t), \tag{3.1}$$

$$h \in C([0, 1] \times [0, 1]), \tag{3.2}$$

$$g \in L^1(0, 1) \cap C(0, 1], \tag{3.3}$$

and

$$g \geqslant 0, \qquad g \text{ nonincreasing on } (0, \delta], \tag{3.4}$$

for some $\delta \in (0, 1]$. Examples include $g(r) = r^{\alpha - 1}$ with $0 < \alpha < 1$ and $g(r) = \ln c/r$ with $c \geqslant 1$.

For $n = 1, 2, \ldots,$ we introduce approximations g_n for g and numbers δ_n such that

$$0 < \delta_n \leqslant \delta, \qquad \delta_n \to 0, \tag{3.5}$$

$$g_n \in C[0, 1], \qquad g_n = g \text{ on } [\delta_n, 1], \tag{3.6}$$

$$0 \leqslant g_n \leqslant g, \qquad g_n \text{ nonincreasing on } [0, \delta_n], \tag{3.7}$$

$$g_n(0) \geqslant g_m(0) \qquad \text{for } n \geqslant m, \tag{3.8}$$

and, depending on whether H1 or H2 holds,

$$\text{H1:} \quad \frac{1}{n} g_n(0) \to 0, \tag{3.9}$$

or

$$\text{H2:} \quad \Big(\max_j w_{nj}\Big) g_n(0) \to 0. \tag{3.10}$$

When g_n is constant or linear or is a completely monotone polynomial on $[0, \delta_n]$, (3.9) and (3.10) are satisfied if (see [3])

$$\text{H1:} \quad \delta_n \geqslant \rho/n, \tag{3.11}$$

or

$$\text{H2:} \quad \delta_n \geqslant \rho \max_j w_{nj}, \tag{3.12}$$

for some $\rho > 0$. In any case, δ_n should not converge too fast to zero.

Continuous approximations k_n for the kernel k in (3.1) are defined by

$$k_n(s, t) = g_n(|s - t|)h(s, t). \tag{3.13}$$

Note that $k_n(s, t) = k(s, t)$ for $|s - t| \geqslant \delta_n$.

More general kernels and kernel approximations are given next. Let

$$k(s, t) = f(s - t)h(s, t), \tag{3.14}$$

$$f(r) \text{ continuous for } r \neq 0, \tag{3.15}$$

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and

$$|f(r)| \leqslant g(|r|) \quad \text{for } 0 < |r| \leqslant \delta, \tag{3.16}$$

where g and h satisfy (3.2)–(3.4). Approximate f by f_n, $n = 1, \ldots,$ such that

$$f_n \in C[-1, 1], \qquad f_n(r) = f(r) \quad \text{for } |r| \geqslant \delta_n, \tag{3.17}$$

and

$$|f_n(r)| \leqslant g_n(|r|) \quad \text{for } 0 \leqslant |r| \leqslant \delta_n, \tag{3.18}$$

where g_n and δ_n satisfy (3.5)–(3.10).

Continuous approximations k_n for k in (3.14) are defined by

$$k_n(s, t) = f_n(s - t)h(s, t). \tag{3.19}$$

As before, $k_n(s, t) = k(s, t)$ for $|s - t| \geqslant \delta_n$. Assume henceforth that k and k_n are given by (3.14) and (3.19) or the special cases (3.1) and (3.13).

Discontinuous kernel approximations, for example with

$$\begin{aligned} k_n(s, t) &= 0, && |s - t| < \delta_n, \\ k_n(s, t) &= k(s, t), && |s - t| \geqslant \delta_n, \end{aligned} \tag{3.20}$$

could be admitted. Merely replace $C[0, 1]$ by the space $R[0, 1]$ of bounded Riemann integrable functions in the subsequent analysis (see [1], [3]).

4. Operator approximation theory

Let $X = C[0, 1]$ with $\|x\| = \max|x(t)|$ and let $[X]$ be the space of bounded linear operators on X. Equations (1.1), (1.3), and (1.4) have the operator forms on X:

$$(I - K)x = y, \qquad (I - K_n)x_n = y, \qquad \text{and} \qquad (I - \hat{K}_n)\hat{x}_n = y. \tag{4.1}$$

The operators K, K_n, and $\hat{K}_n$ are defined on X by

$$Kx(s) = \int_0^1 k(s, t)x(t)\, dt, \tag{4.2}$$

$$K_n x(s) = \sum_{j=1}^{n} w_{nj} k_n(s, t_{nj}) x(t_{nj}), \tag{4.3}$$

$$\hat{K}_n x(s) = \sum_{j=1}^{n} w_{nj} k_n(s, t_{nj}) \big[x(t_{nj}) - x(s) \big] + \int_0^1 k(s, t)x(s)\, dt. \tag{4.4}$$

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The collectively compact operator approximation theory presented in [1] was applied to K and K_n (and more general operators) in [2]. It was established that

$$K_n \to K, \qquad K \text{ compact, } \{K_n\} \text{ collectively compact.} \tag{4.5}$$

Thus, $K_n \to K$ pointwise on X, and the sets $\{Kx\colon \|x\| \leqslant 1\}$ and $\{K_n x\colon \|x\| \leqslant 1, n \geqslant 1\}$ are relatively compact (bounded and equicontinuous). Since K and K_n are compact, $(I - K)^{-1} \in [X]$ and $(I - K_n)^{-1} \in [X]$ whenever the inverses exist. Conditions (4.5) imply

$$\|(K_n - K)K\| \to 0 \qquad \text{and} \qquad \|(K_n - K)K_n\| \to 0. \tag{4.6}$$

According to the theory in [1], if $(I - K_n)^{-1}$ exists and $\|(I - K_n)^{-1}\|$ $\|(K_n - K)K\| < 1$ for some n, then $(I - K)^{-1}$ exists. If $(I - K)^{-1}$ exists and n is so large that $\|(I - K)^{-1}\|\,\|(K_n - K)K_n\| < 1$, then there exist uniformly bounded $(I - K_n)^{-1}$ and the unique solutions of $(I - K)x = y$ and $(I - K_n)x_n = y$ satisfy

$$\|x_n - x\| \leqslant \|(I - K_n)^{-1}\|\,\|K_n x - Kx\| \to 0. \tag{4.7}$$

Other error bounds are given in [1].

Now consider $\hat{K}_n$. From (4.2)–(4.4),

$$\hat{K}_n = K_n - (K_n u - Ku)I, \qquad u \in C[0, 1], u \equiv 1. \tag{4.8}$$

These operators are not compact unless $K_n u \equiv Ku$. By (4.8),

$$\|\hat{K}_n - K_n\| = \|K_n u - Ku\| \to 0. \tag{4.9}$$

It follows from (4.5) and (4.9) that $\hat{K}_n \to K$ and $\{\hat{K}_n\}$ is asymptotically compact, that is, for any bounded sequence $\{x_n\}$, $\{\hat{K}_n x_n\}$ has a convergent subsequence. A study of $(I - K)x = y$ and $(I - \hat{K}_n)\hat{x}_n = y$ can be based on these properties. However we shall proceed along another path which yields more results.

Let n be so large that $\|\hat{K}_n - K_n\| < 1$. Then

$$(I - \hat{K}_n + K_n)^{-1} \in [X],$$

and

$$I - \hat{K}_n = \left[I - K_n(I - \hat{K}_n + K_n)^{-1}\right](I - \hat{K}_n + K_n).$$

Since $K_n(I - \hat{K}_n + K_n)^{-1}$ is compact, it follows that $(I - \hat{K}_n)^{-1} \in [X]$ if the latter inverse exists. From $K_n \to K$, (4.6) and (4.9),

$$\hat{K}_n \to K, \qquad \|(\hat{K}_n - K)K\| \to 0 \qquad \text{and} \qquad \|(\hat{K}_n - K)\hat{K}_n\| \to 0. \tag{4.10}$$

The conclusions for K_n and K given above extend to $\hat{K}_n$ and K by means of (4.10) and Theorem 1.10 of [1]. If $(I - \hat{K}_n)^{-1}$ exists and $\|(I - \hat{K}_n)^{-1}\|$ $\|(\hat{K}_n - K)K\| < 1$ for some n, then $(I - K)^{-1}$ exists. If $(I - K)^{-1}$ exists and n is so large that $\|(I - K)^{-1}\|\,\|(\hat{K}_n - K)\hat{K}_n\| < 1$, then there exist uniformly bounded $(I - \hat{K}_n)^{-1}$ and the solutions of $(I - K)x = y$ and $(I - \hat{K}_n)\hat{x}_n = y$

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satisfy

$$\|\hat{x}_n - x\| \leqslant \|(I - \hat{K}_n)^{-1}\| \, \|\hat{K}_n x - Kx\| \to 0. \tag{4.11}$$

Other error bounds are available from [1].

Convergence results and error bounds for special cases of $\hat{x}_n$ were obtained by a different method, based on the collectively compact theory, by Borer [8].

5. Comparisons of convergence rates

Assume $(I - K)^{-1}$ exists and n is so large that $(I - K_n)^{-1}$ and $(I - \hat{K}_n)^{-1}$ exist. Fix $y \in X$. Let x, x_n, and $\hat{x}_n$ be the unique solutions of $(I - K)x = y$, $(I - K_n)x_n = y$ and $(I - \hat{K}_n)\hat{x}_n = y$. In view of (4.7) and (4.11), the convergence rates of $x_n \to x$ and $\hat{x}_n \to x$ depend primarily on the convergence rates of $K_n x \to Kx$ and $\hat{K}_n x \to Kx$, where $x = (I - K)^{-1}y$.

From (4.2) and (4.3),

$$K_n x(s) - Kx(s) = \sum_{j=1}^{n} w_{nj} k_n(s, t_{nj}) x(t_{nj}) - \int_0^1 k(s, t)x(t)\, dt. \tag{5.1}$$

This is a numerical integration error for $k(s, t)x(t)$ as a function of t for each s. Since k is singular, the convergence of $K_n x \to Kx$ and $x_n \to x$ is expected to be slow.

From (4.2) and (4.4),

$$\hat{K}_n x(s) - Kx(s) = \sum_{j=1}^{n} w_{nj} k_n(s, t_{nj})\big[x(t_{nj}) - x(s)\big] - \int_0^1 k(s, t)\big[x(t) - x(s)\big]\, dt. \tag{5.2}$$

This is a numerical integration error for $k(s, t)[x(t) - x(s)]$ as a function of t for each s. Since the continuity of x should reduce the effect of the singularity of k, the convergence of $\hat{K}_n x \to Kx$ and $\hat{x}_n \to x$ should be faster than $K_n x \to Kx$ and $x_n \to x$. More precise statements will depend on the nature of the singularity, the smoothness of $k(s, t)$ for $s \neq t$, and the smoothness of the solution x of $(I - K)x = y$.

The smoothness of solutions of weakly singular integral equations has been studied by a number of authors. Earlier references include Giraud [9] and Miranda [15]. More recent contributions have been made by Bechlars [7] and Schneider [16]. The latter contains an extensive bibliography.

Let C_α, $0 < \alpha \leqslant 1$, denote classes of Hölder-continuous functions on [0, 1]:

$$x \in C_\alpha \quad \text{if } |x(s) - x(t)| \leqslant A|s - t|^\alpha, \qquad 0 < \alpha < 1,$$

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and

$$x \in C_1 \quad \text{if } |x(s) - x(t)| \leqslant A|s - t|\ln \frac{B}{|s - t|},$$

with constants A and B which depend on x.

We define certain classes K_α, $0 < \alpha \leqslant 1$, of the kernels in (3.1):

$$k \in K_\alpha \quad \text{if } k(s, t) = |s - t|^{\alpha - 1}h(s, t), \qquad 0 < \alpha < 1,$$

and

$$k \in K_1 \quad \text{if } k(s, t) = \ln \frac{c}{|s - t|}\, h(s, t), \qquad c \geqslant 1,$$

where $h(s, t)$ satisfies (3.2) and the derivative $h_s \in C[0, 1]$ as a function of s for each t.

Let $x = (I - K)_y^{-1}$. Then, for $0 < \alpha \leqslant 1$,

$$k \in K_\alpha, \qquad y \in C_\alpha \Rightarrow x \in C_\alpha, \tag{5.3}$$

and

$$k \in K_\alpha, \qquad y \in C_\alpha \cap C'(0, 1) \Rightarrow x \in C_\alpha \cap C'(0, 1). \tag{5.4}$$

For proofs and further results, see [6] and [16]. As (5.4) suggests, x is typically less smooth at the endpoints of [0, 1]. A quadrature formula with the nodes denser near 0 and 1 could help compensate for this lack of smoothness (see [6] and [16]).

In the situation of (5.4), the integrand in (5.2) satisfies

$$|k(s, t)[x(t) - x(s)]| \leqslant A|s - t|^{2\alpha - 1}, \qquad 0 < \alpha < 1, \tag{5.5}$$

and

$$|k(s, t)[x(t) - x(s)]| \leqslant A|s - t|\ln\left(\frac{B}{|s - t|}\right)^2, \qquad \alpha = 1, \tag{5.6}$$

with some A and B. If $\frac{1}{2} \leqslant \alpha \leqslant 1$ then $k(s, t)[x(t) - x(s)]$ is not singular. If $0 < \alpha < \frac{1}{2}$ then the singularity in (5.2) is weaker than the singularity in (5.1). Hence, as already stated, the convergence of $\hat{K}_n x \to Kx$ and $\hat{x}_n \to x$ should be faster than $K_n x \to Kx$ and $x_n \to x$. Numerical examples are given in Section 7.

6. Comparisons with other approximation methods

With $k(s, t)$ given by (3.14), $(I - K)x = y$ has the form

$$x(s) - \int_0^1 f(s - t)h(s, t)x(t)\, dt = y(s). \tag{6.1}$$

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In the singularity factorization technique of Atkinson [4], [5], $(I - K)x = y$ is approximated by an equation $(I - K^n)x^n = y$ of the form

$$x^n(s) - \int_0^1 f(s - t)P_n[h(s, t)x(t)]\,dt = y(s), \tag{6.2}$$

where $P_n \in [X]$ is a projection which acts with respect to t, for example, spline interpolation or approximation. Assume $\dim \bar{P}_n\bar{X} = n$ and $\bar{P}_n \to \bar{I}$. Then $\bar{K}^n$ and K satisfy (4.5), so that $x^n \to x$ when $(I - K)^{-1}$ exists.

Equation (6.2) reduces to an $n \times n$ matrix problem. The entries are integrals which can be evaluated explicitly if f, h and the functions in P_nX are simple enough. Then the singular factor is treated exactly, so $x^n \to x$ should converge faster than $x_n \to x$. Numerical examples in Section 7 demonstrate this. The comparison of $x^n \to x$ and $\hat{x}_n \to x$ is far from clear when the integrals involving the singular factor $f(s, t)$ must be approximated. Then the derivation of each x^n requires much more machine time and the speed of convergence is reduced.

The singularity factorization method has been extended and sharpened by Bechlars [6], [7]. His analysis covers singularities or discontinuities of $k(s, t)$. It includes Volterra equations.

The thesis by Volk [21] on the numerical treatment of integral equations by means of spline functions contains applications to weakly singular equations.

The thesis by Schneider [16] presents a thorough and systematic study of the numerical solution of weakly singular integral equations by a projection method: $(I - K)x = y$ is approximated by $(I - KP_n)x_n = y$, where $P_n \in [X]$ is a projection with $\dim P_nX < \infty$ and $P_n \to I$. Smoothness properties of $x = (I - K)^{-1}$ are used in order to design particularly efficient spline projections.

The recent papers by Graham [10] and Sloan [17]–[20] further refine and extend the effectiveness of singularity factorization.

7. Numerical examples

These results are adapted from Borer [8].

Example 1: $x(s) + \int_0^1 \ln|s - t|x(t)\,dt = y(s)$, $x(s) \equiv s$.

Replace $\ln|s - t|$ by $\ln(1/n)$ for $|s - t| \leqslant 1/n$ and use Simpson's rule to define approximations x_n and $\hat{x}_n$ for x as in Section 1. Table 1 compares x_n, $\hat{x}_n$ and corresponding singularity factorization approximations x^n at $s = \frac{1}{2}$, where $x = \frac{1}{2}$.

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TABLE 1

Results for Example 1

n	x_n	$\hat{x}_n$	x^n
9	1.24844	.50356	.50213
11	1.24886	.50292	.50112
15	1.24922	.50106	.49917
39	1.24953	.49641	.50031
65	1.24955	.48155	.50005

Example 2: $x(s) + \int_0^1 |s - t|^{-1/2} x(t)\, dt = y(s)$, $x(s) \equiv s$.

Define x_n, $\hat{x}_n$, and x^n in the same way as above. At $s = \frac{1}{2}$, where $x = \frac{1}{2}$, we have the following computations.

TABLE 2

Results for Example 2

n	x_n	$\hat{x}_n$	x^n
9	.98471	.52311	.53112
11	.94379	.51132	.51306
15	1.1247	.49371	.50972
39	1.1149	.48116	.50103
65	1.1236	.49612	.50033

In both examples, x_n does not seem to converge, whereas both $\hat{x}_n$ and x^n are quite accurate even for small n. As anticipated, x^n is apparently more accurate than $\hat{x}_n$ since the simplicity of the kernels makes it possible to treat the singularity exactly in the calculation of x^n.

Further numerical comparisons would be desirable, particularly for more complicated singular integral equations.

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References

[1] P. M. Anselone, *Collectively compact operator approximation theory and applications to integral equations* (Prentice-Hall, Englewood Cliffs, NJ, 1971).

[2] P. M. Anselone and W. Krabs, "Approximate solution of weakly singular integral equations", *J. of Integral Equations* 1 (1979), 61–75.

[3] P. M. Anselone and G. Opfer, "Numerical integration of weakly singular functions", in *Numerische integration* (ed. G. Hämmerlin) (Birkhäuser Verlag, Basel, 1979).
[4] K. E. Atkinson, "The numerical solution of Fredholm integral equations of the second kind with singular kernels", *Numer. Math.* 19 (1972), 248–259.
[5] K. E. Atkinson, *A survey of numerical methods for the solution of Fredholm integral equations of the second kind* (SIAM Publications, 1976).
[6] J. Bechlars, "Glattheit und numerische Berechnung der Lösung linearer Integralgleichungen 2. Art mit schwachsingulären Kernen" (Hahn-Meitner-Institut Berlin HMI-B 283, 1978).
[7] J. Bechlars, "Weakly singular integral equations, smoothness properties and numerical solution", *J. of Integral Equations* 2 (1980), 259–263.
[8] D. Borer, "Approximate solution of Fredholm integral equations of the second kind with singular kernels", Thesis, Oregon State University, Corvallis, 1977.
[9] G. Giraud, "Sur certains problèmes non-linéaires de Neumann et sur certain problèmes non-linéaires mixtes", *Ann. Ecole Norm. Supér.* 49 (1932), 3–17.
[10] I. G. Graham, "Singularity expansions for the solutions of second kind integral equations with weakly singular convolution kernels", *J. of Integral Equations*, to appear.
[11] L. V. Kantorovich and V. I. Krylov, *Approximate methods of higher analysis* (Interscience, New York, 1958).
[12] R. E. Kleinman and W. L. Wendland, "On Neumann's method for the exterior Neumann problem for the Helmholtz equation", *J. Math. Anal. Appl.* 57 (1977), 170–202.
[13] F. W. Krechel, "Numerische Lösung Fredholmscher Integralgleichungen 2. Art mit schwach singulären Kernen", Diplomarbeit, Univ. Köln, 1977.
[14] R. Kussmaul and P. Werner, "Fehlerabschätzungen fur ein numerisches Verfahren zur Auflösung linearer Integralgleichungen mit schwachsingulären Kernen", *Computing* 3 (1968), 22–46.
[15] C. Miranda, *Equazioni alle derivate parziali di tipo ellittico* (Springer-Verlag, Berlin, 1955).
[16] C. Schneider, "Beiträge zur numerischen Behandlung schwachsingulärer Fredholmscher Integralgleichungen 2. Art", Dissertation, Univ. Mainz, 1977.
[17] I. H. Sloan, "The numerical solution of Fredholm equations of the second kind by polynomial interpolation", *J. of Integral Equations*, to appear.
[18] I. H. Sloan, "On choosing the points in product integration", *J. Math. Phys.* 21 (1980), 1032–1039.
[19] I. H. Sloan, "Analysis of general quadrature methods for integral equations of the second kind", Univ. of Maryland Tech. Note BN 931, 1979, submitted for publication.
[20] I. H. Sloan and B. J. Burn, "Collocation and polynomials for integral equations of the second kind: a new approach to the theory", *J. of Integral Equations*, 1 (1979), 77–94.
[21] W. Volk, "Die numerische Behandlung Fredholmscher Integralgleichungen zweiter Art mittels Splinefunktionen", Hahn-Meitner-Institut Berlin, HMI-B 286, 1979.
[22] W. L. Wendland, *Elliptic systems in the plane* (Pitman Publishing, Ltd., London, 1979).

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