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CENTRALISERS IN WREATH PRODUCTS

by J. D. P. MELDRUM

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In this paper, the centraliser of an arbitrary element of a wreath product is determined. One application of this is to find the breadth of a wreath product (Theorems 21 and 22), a problem which was raised in discussion with Dr. I. D. Macdonald. Another application is to groups generated by elements generating their own centralisers (Theorem 20).

Let A and B be two groups. Define

$$A^B = \{f : B \to A;\quad f(b) = e \quad \text{for all but a finite number of elements of } B\}$$

to be a group by defining the product pointwise

$$fg(b) = f(b)g(b) \quad \text{for all} \quad b \in B.$$

Then A^B is a restricted direct power of copies of A indexed by elements of B. Define B as a group of automorphisms of A^B by

$$f^b(b_1) = f(b_1 b^{-1}).$$

Then $A \text{ wr } B$ is the semidirect product of A^B by B determined by this definition. A recent paper on wreath products with a good bibliography is C. Wells (3).

If A and B are finite p-groups, then so is $A \text{ wr } B$. The class of a finite p-group will denote its nilpotency class. The breadth of a finite p-group is defined as b where p^b is the size of the largest conjugacy class of the group. So p^b is the index of the smallest centraliser. If c is the class of the group, there is a conjecture that

$$b \geq c - 1.$$

A recent paper dealing with this conjecture is Macdonald (1), which has a good bibliography.

Let $fg \in G = A \text{ wr } B$, where $f \in A^B$, $g \in B$, and let $dh \in C_G(fg)$, where $d \in A^B$, $h \in B$.

Lemma 1. *$dh \in C_G(fg)$ if and only if*
(i) $h \in C_B(g)$,
(ii) $d(xg) = f(x)^{-1} d(x) f(xh)$ *for all* $x \in B$.

Proof. $dh \in C_G(fg)$ if and only if $(dh)^{-1} fg\, dh = fg$ if and only if $h^{-1} d^{-1} fg\, dh = fg$ if and only if $d^{-h} f^h d^{g^{-1}h} h^{-1} gh = fg$ if and only if $g^h = g$ and $d^{-h} f^h d^{g^{-1}h} = f$,

that is

$$h \in C_B(g) \text{ and } d^{-1} f d^{g^{-1}} = f^{h^{-1}}.$$

The latter condition is

$$d(x)^{-1}f(x)d(xg) = f(xh)$$

or

$$d(xg) = f(x)^{-1}d(x)f(xh).$$

Corollary 2. *If $dh \in C_G(fg)$ then*

$$d(xg^{n+1}) = f(xg^n)^{-1}\dots f(x)^{-1}d(x)f(xh)\dots f(xhg^n)$$

for all $n \geq 0$.

Proof. By induction on n, using Lemma 1.

We use $o(g)$ to denote the order of the element g of a group. For reasons that are obvious from Corollary 2, we will define

$$\bar{f}(x,g) = f(xg^{-n})f(xg^{-n+1})\dots f(xg^m)$$

where

(i) if $o(g)$ is infinite, then $f(xg^{-i}) = e$ for $i > n$ and $f(xg^j) = e$ for $j > m$,
(ii) if $o(g)$ is finite, then $n = 0$ and $o(g) - 1 = m$.

Another aspect that will occur several times is that elements of $C_B(g)$ will permute the *left* cosets of $Gp\langle g\rangle$ under the *right* regular representation since $xGp\langle g\rangle h = xhGp\langle g\rangle$. This permutation will be denoted $\rho(h)$.

From now on until further notice, h will always denote an element of $C_B(g)$.

Case 1. $o(g)$ is infinite.

In this case $\bar{f}(x,g)$ is uniquely defined for each left coset of $Gp\langle g\rangle$. We use ρ as defined above. Then ρ is a homomorphism from $C_B(g)$ to the group of permutations of the left cosets of $Gp\langle g\rangle$. Since

$$xh_1Gp\langle g\rangle = xh_2Gp\langle g\rangle$$

if and only if $h_1Gp\langle g\rangle = h_2Gp\langle g\rangle$, the kernel of ρ is $Gp\langle g\rangle$.

Let X be the set of left cosets of $Gp\langle g\rangle$. Partition X into $\{X_i; i \in I\}$ where

$$X_i = \{xGp\langle g\rangle: \bar{f}(x,g) = g_i\}.$$

So X_i consists of all left cosets of $Gp\langle g\rangle$ with a common value for $\bar{f}(x,g)$. Note that I is finite.

Since $\rho(C_B(g))$ permutes the left cosets of $Gp\langle g\rangle$, it permutes the set $\{\bar{f}(x,g)\}$ of values of $\bar{f}(x,g)$ under an obvious extension of the definition of the action of $\rho(h)$. We consider

$$H(fg) = \{h \in C_B(g);\ \rho(h) \text{ stabilizes } X_i \text{ for } i \in I\}.$$

Thus $H(fg)$ consists precisely of those $h \in C_B(g)$ such that $\bar{f}(x,g) = \bar{f}(xh,g)$ for all $x \in B$.

Theorem 3. *Let $o(g)$ be infinite. Then $dh \in C_G(fg)$ if and only if*
(i) $h \in H(fg)$,
(ii) $d(xg^{-k+n}) = f(xg^{-k+n-1})^{-1} \dots f(xg^{-k})^{-1} f(xhg^{-k}) \dots f(xhg^{-k+n-1})$ *for all* $n \geq 1$ *and* k *is defined by* $f(xg^{-l}) = e = f(xhg^{-l})$ *for* $l > k$, $d(xg^{-l}) = e$ *if* $l \geq k$.

Proof. By definition $H(fg) \subseteq C_B(g)$. So $h \in C_B(g)$. Since $h \in H(fg)$, we have $\bar{f}(x, g) = \bar{f}(xh, g)$. Hence for sufficiently large n, $d(xg^{-k+n}) = e$. Also for only finitely many left cosets of $Gp\langle g\rangle$ do we have $f(xg^i) \neq e$ for any i. So d is well defined as an element of A^B. From (ii), it is obvious that $d(xg) = f(x)^{-1}d(x)f(xh)$ for all $x \in B$. Thus dh satisfies the conditions of Lemma 1 and we have sufficiency.

We now consider necessity. We can assume the results of Lemma 1 and Corollary 2. Let k be defined as in (ii) of Theorem 3. Since $d(xg^{-l}) = e$ for sufficiently large values of l, we can use Corollary 2 and the definition of k to deduce that $d(xg^{-l}) = e$ for $l \geq k$. For sufficiently large values of n, we have by definition

$$\bar{f}(x, g) = f(xg^{-k}) \dots f(xg^{-k+n-1}),$$
$$\bar{f}(xh, g) = f(xhg^{-k}) \dots f(xhg^{-k+n-1}).$$

Also by Corollary 2,

$$\begin{aligned} d(xg^{-k+n}) &= f(xg^{-k+n-1})^{-1} \dots f(xg^{-k})^{-1} d(xg^{-k}) f(xhg^{-k}) \dots f(xhg^{-k+n-1}) \\ &= \bar{f}(x, g)^{-1} \bar{f}(xh, g) \end{aligned}$$

for sufficiently large values of n, and this must be e as $d \in A^B$. Hence $\bar{f}(x, g) = \bar{f}(xh, g)$. This must hold for all values of x. Thus $h \in H(fg)$. We have also shown above that $d(xg^{-l}) = e$ if $l \geq k$ and then the necessity of (ii) follows from Corollary 2.

Corollary 4. *Let $g \in B$ have infinite order. Then $C_G(fg)$ is isomorphic to $H(fg)$.*

Proof. From Theorem 3, using the map $dh \to h$ and noting that d is uniquely defined, given f and h.

Corollary 5. *Let $g \in B$ have infinite order, and satisfy $C_B(g) = Gp\langle g\rangle$. Then $C_G(fg) = Gp\langle fg\rangle$.*

Proof. Immediate from Corollary 4.

Lemma 6. *Let $K \subseteq C_B(g)$. Then the orbits of $\rho(K)$ consist of left cosets of $KGp\langle g\rangle$.*

Proof. This is verified easily directly from the definition of ρ.

Corollary 7. *Let $g \in B$ have infinite order. Let $f \in A^B$ satisfy $\bar{f}(x, g) = e$ for all $x \in B$. Then $C_G(fg)$ is isomorphic to $C_B(g)$.*

Proof. This follows immediately from Theorem 3 and Lemma 6.

Corollary 8. *Let $g \in B$ have infinite order. Let $f \in A^B$ satisfy $|\{\bar{f}(x, g); x \in B\}| \geq 2$. If $C_B(g)/Gp\langle g\rangle$ is torsion-free, then $C_G(fg) = Gp\langle fg\rangle$.*

Proof. If $X_i = \{xGp\langle g\rangle; \bar{f}(x, g) = g_i\}$ and $g_i \neq e$, then X_i is finite since $f \in A^B$. So X_i cannot satisfy Lemma 6 for any $K \subseteq C_B(g)$ such that $K \supset Gp\langle g\rangle$ if $C_B(g)/Gp\langle g\rangle$ is torsion-free. Now apply Corollary 4.

Corollary 9. *Let $g \in B$ have infinite order. If $X_i = \{xGp\langle g\rangle; \bar{f}(x, g) = g_i\}$ consists of a single coset of $Gp\langle g\rangle$ for some g_i, then $C_G(g) = Gp\langle fg\rangle$.*

Proof. If $|X_i| = 1$, then X_i cannot be a union of left cosets of $Gp\langle g\rangle$ of the form $xKGp\langle g\rangle$ with $K \supset Gp\langle g\rangle$. Now apply Lemma 6 and Corollary 4.

Corollary 10. *Let B have a set of generators of infinite order. Then G can be generated by a set of elements which generate their own centralisers.*

Proof. Let $B = Gp\langle b_i; i \in I, o(b_i)$ is infinite$\rangle$.

By choosing suitable f_{ij} of the type $f_{ij}(x) = e$ for all but $x = b_i$, $f_{ij}(b_i) = a_j$, where a_j runs through a generating set of A, we can apply Corollary 9 to get the result.

There are a number of results along these lines which could be stated. But we will turn to the next case now.

Case 2. $o(g)$ is finite.

Let $o(g) = m$. In this case $\bar{f}(x, g)$ is not uniquely defined for a given coset of $Gp\langle g\rangle$. For this case we have that

$$\bar{f}(x, g) = f(x)f(xg)\dots f(xg^{m-1}).$$

Note that

$$\begin{aligned}\bar{f}(xg, g) &= f(x)^{-1}\bar{f}(x, g)f(xg^m)\\ &= f(x)^{-1}\bar{f}(x, g)f(x).\end{aligned}$$

Lemma 11. *Let $g \in B$ have finite order m. Let $dh \in C_G(fg)$. Then $\bar{f}(xh, g) = d(x)^{-1}\bar{f}(x, g)d(x)$.*

Proof. By Corollary 2, and using the fact that $g^m = e$,

$$\begin{aligned}d(x) = d(xg^m) &= f(xg^{m-1})^{-1}\dots f(x)^{-1}d(x)f(xh)\dots f(xhg^{m-1})\\ &= \bar{f}(x, g)^{-1}d(x)\bar{f}(xh, g)\end{aligned}$$

giving the result we want, after a slight rearrangement.

Define $K(fg)$ by

$$K(fg) = \{h \in C_B(g); \bar{f}(xh, g) \text{ is a conjugate of } \bar{f}(x, g) \text{ for all } x \in B\}.$$

Lemma 12. *$K(fg)$ is a subgroup of B, and $g \in K(fg)$.*

Proof. Since $\bar{f}(xg, g) = f(x)^{-1}\bar{f}(x, g)f(x)$ and $g \in C_B(g)$, so $g \in K(fg)$ and $K(fg)$ is not empty. Let $h_1, h_2 \in K(fg)$. Suppose

$$\bar{f}(xh_i, g) = b_i(x)^{-1}\bar{f}(x, g)b_i(x), \; i = 1, 2.$$

Then $\bar{f}(x, g) = b_i(x)\bar{f}(xh_i, g)b_i(x)^{-1}$, $i = 1, 2$. So

$$\begin{aligned}\bar{f}(xh_1h_2^{-1}, g) &= b_2(xh_1)\bar{f}(xh_1, g)b_2(xh_1)^{-1}\\ &= b_2(xh_1)b_1(x)^{-1}\bar{f}(x, g)b_1(x)b_2(xh_1)^{-1}.\end{aligned}$$

Hence $h_1h_2^{-1} \in K(fg)$ and we have proved the lemma.

Lemma 13. *Let g have finite order. Let $dh \in C_G(fg)$. Then $\{\bar{f}(x, g)\,;\, x \in yGp\langle h\rangle$ for some fixed $y\}$ forms a conjugacy class in A.*

Proof. Directly from Lemma 11.

Lemma 14. *If $\bar{f}(x, g) \neq e$ for some $x \in B$ and $g \in B$ has finite order, then $dh \in C_G(fg)$ satisfies $o(h)$ is finite.*

Proof. This follows quickly from Lemma 13.

Theorem 15. *Let $o(g)$ be finite. Then $dh \in C_G(fg)$ if and only if*

(i) $h \in K(fg)$,

(ii) $d(xg) = f(x)^{-1}d(x)f(xh)$,

(iii) *if $\bar{f}(x, g) \neq e$ for some $x \in B$, then $o(h)$ is finite,*

(iv) $\bar{f}(xh, g) = d(x)^{-1}\bar{f}(x, g)d(x)$.

Proof. We prove necessity first. The definition of $K(fg)$ and Lemma 11 show that (i) and (iv) are necessary. Lemma 14 shows the necessity of (iii). Lemma 1 shows the necessity of (ii).

Since $K(fg) \subseteq C_B(g)$, (i) and (iv) give sufficiency by Lemma 1.

Theorem 15 is just a restatement of earlier results which enables us to specify exactly the elements of $C_G(fg)$. Given $fg \in G$, we first determine $K(fg)$, which we know contains $Gp\langle g\rangle$. If $\bar{f}(x, g) \neq e$ for some $x \in B$, then we can only choose elements h in $K(fg)$ of finite order. Any power of g is such an element. We can now determine $d \in A^B$ such that $dh \in C_G(fg)$. Theorem 15 (iv) determines the coset of $C_A(\bar{f}(x, g))$ to which $d(x)$ belongs. Then Theorem 15 (ii) determines the values of $d(xg^i)$ for $1 \leq i < o(g)$. If $\bar{f}(x, g) = e$ for all $x \in B$, then there is no restriction on the choice of h in $K(fg)$. Note that any possibility of a double definition for $d(x)$ due to Theorem 15 (ii) is taken care of by Theorem 15 (iv) and if $d(x)$ is chosen to satisfy Theorem 15

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(iv), then $d(xg)$ defined by (ii) satisfies

$$\begin{aligned} d(xg)^{-1}\bar{f}(xg,g)d(xg) &= f(xh)^{-1}d(x)^{-1}f(x)\bar{f}(xg,g)f(x)^{-1}d(x)f(xh) \\ &= f(xh)^{-1}d(x)^{-1}\bar{f}(x,g)d(x)f(xh) \\ &= f(xh)^{-1}\bar{f}(xh,g)f(xh) \\ &= \bar{f}(xgh,g) \end{aligned}$$

namely Theorem 15 (iv) with xg replacing x.

Let X be the set of left cosets of $Gp\langle g\rangle$ in B.

Theorem 16. *Let $fg \in G$, and let $o(g)$ be finite. Then there is a homomorphism from $C_G(fg)$ onto $K(fg)$ sending $dh \to h$, whose kernel is isomorphic to $\Pi_{x\in T} C_A(\bar{f}(x,g))$, where T is a left transversal of $Gp\langle g\rangle$ in B and Π denotes restricted direct product.*

This result follows from the remarks above. We look at two special cases.

Lemma 17. *Let $g \in B$ have finite order. Then $C_G(g)$ is isomorphic to $(\Pi_{x\in T} A)C_B(g)$.*

This is a well-known result, namely $dh \in C_G(g)$ if and only if $h \in C_B(g)$ and d is constant on left cosets of $Gp\langle g\rangle$.

Lemma 18. *Let $e \neq f \in A^B$. Then $K(f)$ is a torsion group and $C_G(f)$ is isomorphic to $(\Pi_{x\in B} C_A(f(x)))K(f)$.*

Proof. This follows directly from Theorems 15 and 16 once we remember that $\bar{f}(x,e) = f(x)$ and $\bar{f}(x,g) \neq e$ for some x, since $e \neq f$.

Lemma 19. *Let A and B be non-trivial groups. Let $g \in B$ have finite order. Then $C_G(fg) > Gp\langle fg\rangle$.*

Proof. Let fg satisfy $C_G(fg) = Gp\langle fg\rangle$. Then we must have $K(fg) = Gp\langle g\rangle$ by Theorem 16. Also given $dh \in C_G(fg)$, d must be uniquely determined by h, as $h = g^i$ for some i and then $dh = (fg)^i$. So the kernel of the homomorphism described in Theorem 16 must be the identity. But this is obviously impossible. If $g = e$, the result follows directly from Lemma 18.

Theorem 20. *Let $G = A$ wr B, A and B be non-trivial groups. Then G is generated by a set of elements which generate their own centralisers if and only if B can be generated by a set of elements all of which have infinite order.*

Proof. The sufficiency follows from Corollary 10 and the necessity from Lemma 19.

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As an application of this work we consider the breadth of the wreath product of two finite p-groups.

We first note that the breadth of a finite p-group G is given by $b(G)=b$, where p^b is the index of the smallest centraliser in G. Let $G=A \operatorname{wr} B$ where A and B are finite p-groups. By Theorem 16

$$C_G(fg) \cong \prod_{x \in T} C_A(\bar{f}(x,g)) \cdot K(fg)$$

where $fg \in G$. So we seek to make $C_A(\bar{f}(x,g))$, $K(fg)$ and T as small as possible. But there is a conflict between the first and the last two of these.

Let A have order p^a, B have order p^b and exponent p^e. Let the breadth of A be w.

Theorem 21. *Let $G=A \operatorname{wr} B$ have constants as defined above. Then the breadth of G is*

(i) $ap^b-(a-w)p^{b-e}+b-e$

if A has two distinct conjugacy classes of maximal size,

(ii) $ap^b-(a-w)p^{b-e}+\max\{y, b-e-x\}$

if A has only one conjugacy class of maximal size, x is defined by p^{w-x} is the size of the second largest conjugacy class in A, y is defined as $p^{b-y}=\min(|C_B(g)|;\ O(g)=p^e)$.

Proof. Let $g \in B$ have order p^j. Then $j \leqslant e$, and the transversal T of $Gp\langle g\rangle$ has order p^{b-j}.

(i) Suppose A has two distinct conjugacy classes of maximal size. Then choose f such that $\bar{f}(x,g)$ lies in one of these conjugacy classes for all but one of the cosets of $Gp\langle g\rangle$, and in the other one for the remaining coset of $Gp\langle g\rangle$. This will ensure that $K(fg)=Gp\langle g\rangle$. So if we choose g to have maximal order, namely p^e we have made $C_G(fg)$ as small as possible. Hence the breadth of G is the exponent of

$$p^{ap^b+b}/p^{(a-w)p^{b-e}} \cdot p^e = p^{ap^b-(a-w)p^{b-e}+b-e},$$

i.e., the breadth of A is $ap^b-(a-w)p^{b-e}+b-e$.

(ii) Suppose A has only one conjugacy class of maximal size, and let x, y be defined as in the statement of the theorem. If we try to follow the same process as in (i), we find that either all the $\bar{f}(x,g)$ lie in the same (maximal) conjugacy class, and then $K(fg)=C_B(g)$, or one of $\bar{f}(x,g)$ lies in the second largest maximal class and then $K(fg)=Gp\langle g\rangle$. In the first of these cases we get a conjugacy class size

$$p^{ap^b+b}/p^{(a-w)p^{b-e}} \cdot p^{b-y} = p^{ap^b-(a-w)p^{b-e}+y}.$$

Note that $b-y \geqslant e$, i.e. $b-e \geqslant y$ by definition of y. In the second case we get a conjugacy class of size

$$p^{ap^b+b}/p^{(a-w)(p^{b-e}-1)} \cdot p^{a-w+x} \cdot p^e$$
$$= p^{ap^b-(a-w)p^{b-e}-x+b-e}.$$

So the breadth of G is at least

$$ap^b-(a-w)p^{b-e}+\max\{y, b-e-x\}.$$

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If we choose g not to be of maximal order then we would replace e by $j < e$ in the formula, and increase the last term by max $\{e-j, y'-y\}$, where $p^{b-y'} = \min(|C_B(g)|; o(g) = p^j)$.

The second term would decrease by

$$(a-w)p^{b-j} - (a-w)p^{b-e} = (a-w)p^{b-e}(p^{e-j}-1).$$

Obviously $e - j \leqslant (a-w)p^{b-e}(p^{e-j}-1)$. But $b - y' \geqslant j$ and so $y' \leqslant b - j$. Hence $y' - y \leqslant b - j$. It is easy to check that $p^z - z \geqslant p^{z-1}$ for positive integral values of z. Hence $(a-w)p^{b-j} - (y'-y) \geqslant (a-w)p^{b-j-1}$ and thus

$$(a-w)p^{b-j} - (a-w)p^{b-e} - (y'-y) \geqslant (a-w)p^{b-j-1} - (a-w)p^{b-e} \geqslant 0.$$

So by choosing g not to be of maximal order we get a smaller conjugacy class. This finishes the proof of the theorem.

Theorem 22. *Let A be a cyclic group of order p^a, B a cyclic group of order p^b. Then the breadth of A* wr *B is equal to the class of A* wr *B less one if and only if $a = 1$ or $b = 1$.*

Proof. The class of A wr B is $p^b + (a-1)p^{b-1}(p-1)$, and the breadth of A wr B is $ap^b - a$. Then

$$\begin{aligned} ap^b - a = p^b + (a-1)p^{b-1}(p-1) - 1 \quad &\text{if and only if} \\ (a-1)(p^b - p^{b-1}(p-1)) = a - 1 \quad &\text{if and only if} \\ (a-1)p^{b-1} = a - 1 \quad &\text{if and only if} \\ a = 1 \text{ or } p^{b-1} = 1. \quad & \end{aligned}$$

This gives the result.

The case $a = 1$ might have been expected. But the case $b = 1$ is somewhat surprising. The class of general wreath products is given by D. Shield (2). To do a precise comparison between this and the breadth of A wr B would involve a good deal of analysis which would not be in character with the rest of this paper.

REFERENCES

(1) I. D. MacDonald, The breadth of finite p-groups *I*, *Proc. A. Royal Soc. Edinburgh.* **78** (1977–78), 31–39.

(2) D. Shield, Power and commutator structure of groups, *Bull. Austral. Math. Soc.* **15** (1976), 315–317.

(3) C. Wells, Some applications of the wreath product construction, *American Math. Monthly* **83** (1976), 317–338.

University of Edinburgh