OVALS IN A FINITE PROJECTIVE PLANE

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1. Let $\mathfrak{P}$ be a finite projective plane **(8**, §17**)**, i.e. a projective space of dimension 2 over a Galois field γ. We suppose that γ has *characteristic* $p \neq 2$, hence order $q = p^h$, where p is an odd prime and h is a positive integer. It is well known that every straight line and every non-singular conic of $\mathfrak{P}$ then contains $q + 1$ points exactly.

Using the term *oval* to designate any set of $q + 1$ distinct points of $\mathfrak{P}$ no three of which are collinear, we shall prove the following theorem, already surmised by Järnefelt and Kustaanheimo **(3)** (deemed "implausible" in Math. Rev., *14* (1953), p. 1008):

THEOREM I. *If* $p \neq 2$, *every oval of* $\mathfrak{P}$ *is a conic* (*i.e., can be represented by an equation of the second degree*).

This result fills up a gap in the finite congruence axiomatics set up by Kustaanheimo **(4)**, and has important implications if we accept the idea, advanced by Järnefelt **(2)**, of a possible connection between the physical world and the geometry of a finite linear space (cf. also **1, 5, 6, 7)**.

2. Let $\mathscr{C}$ denote any given oval of $\mathfrak{P}$, and B be an arbitrary point of $\mathscr{C}$. Then $\mathscr{C}$ has a *tangent* at B, uniquely defined as the line of $\mathfrak{P}$ which contains B and no other point of $\mathscr{C}$; moreover, no three tangents of $\mathscr{C}$ meet at a point **(7**, Theorem 3**)**. We begin by proving

THEOREM II. *Every inscribed triangle of* $\mathscr{C}$ *and its circumscribed triangle are perspective.*

It is not restrictive to identify the given inscribed triangle with the triangle of reference for homogeneous coordinates (x_1, x_2, x_3):

$$A_1:(1, 0, 0), \quad A_2:(0, 1, 0), \quad A_3:(0, 0, 1);$$

then we may denote by

$$a_1 : x_2 = k_1 x_3, \quad a_2 : x_3 = k_2 x_1, \quad a_3 : x_1 = k_3 x_2$$

the tangents of $\mathscr{C}$ at A_1, A_2, A_3 respectively, where k_1, k_2, k_3 are three non-zero elements of the field γ. If $B:(c_1, c_2, c_3)$ is any of the $q-2$ points of $\mathscr{C}$ distinct from A_1, A_2, A_3, then $c_1 c_2 c_3 \neq 0$; moreover, the lines $A_1 B, A_2 B, A_3 B$ have equations of the form

$$x_2 = \lambda_1 x_3, \quad x_3 = \lambda_2 x_1, \quad x_1 = \lambda_3 x_2,$$

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where the coefficients $\lambda_1, \lambda_2, \lambda_3$ are distinct from k_1, k_2, k_3 respectively, as well as from zero. Since these coefficients are given precisely by

$$\lambda_1 = c_2 c_3^{-1}, \quad \lambda_2 = c_3 c_1^{-1}, \quad \lambda_3 = c_1 c_2^{-1},$$

they satisfy the equation

(1) $$\lambda_1 \lambda_2 \lambda_3 = 1.$$

Conversely, if λ_1 denotes any of the $q-2$ elements of γ distinct from zero and from k_1, the line $x_2 = \lambda_1 x_3$ meets $\mathscr{C}$ at A_1 and at a further point, B say, distinct from A_1, A_2, A_3; hence the coefficients λ_2, λ_3 in the equations $x_3 = \lambda_2 x_1$, $x_1 = \lambda_3 x_2$ of the lines $A_2 B$, $A_3 B$ are functions of λ_1, connected by (1), which take once each of the non-zero values of γ distinct from k_2, k_3 respectively. On multiplying the $q-2$ equations (1) thus obtained, we see that

$$\Pi^3 = k_1 k_2 k_3,$$

where Π denotes the product of the $q-1$ non-zero elements of γ; whence

(2) $$k_1 k_2 k_3 = -1,$$

as it is well known (**8**, §59) that $\Pi = -1$.

From the equation (2), Theorem II follows at once. In fact the points

$$a_2 \cdot a_3 : \ (k_3, 1, k_2 k_3), \quad a_3 \cdot a_1 : \ (k_3 k_1, k_1, 1), \quad a_1 \cdot a_2 : \ (1, k_1 k_2, k_2)$$

are joined to A_1, A_2, A_3 respectively by the lines:

$$x_3 = k_2 k_3 x_2, \quad x_1 = k_3 k_1 x_3, \quad x_2 = k_1 k_2 x_1;$$

by virtue of (2), these lines concur at the point K:$(1, k_1 k_2, -k_2)$, which is therefore a centre of perspective of the triangles $A_1 A_2 A_3$ and $a_1 a_2 a_3$.

3. We can now prove Theorem I. For this purpose we use the notation of §2, assuming, as it is not restrictive, that K coincides with the unit point (1, 1, 1); this is tantamount to supposing

$$k_1 = k_2 = k_3 = -1.$$

If B:(c_1, c_2, c_3) is any of the $q-2$ points of $\mathscr{C}$ distinct from A_1, A_2, A_3, we denote by

$$b : \ b_1 x_1 + b_2 x_2 + b_3 x_3 = 0$$

the tangent of $\mathscr{C}$ at it. This line contains B, but none of the points A_1, A_2, A_3, $a_2 \cdot a_3$, $a_3 \cdot a_1$, $a_1 \cdot a_2$; hence, if we put

$$\beta_1 = b_1 - b_2 - b_3, \quad \beta_2 = -b_1 + b_2 - b_3, \quad \beta_3 = -b_1 - b_2 + b_3,$$

we have

(3) $$b_1 c_1 + b_2 c_2 + b_3 c_3 = 0$$

and

(4) $$b_1 b_2 b_3 \beta_1 \beta_2 \beta_3 \neq 0.$$

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By virtue of Theorem II, the triangles BA_2A_3 and ba_2a_3 are perspective; this—as is immediately seen—is expressed algebraically by the equation

$$\begin{vmatrix} c_3 - c_2 & c_1 + c_3 & -c_1 - c_2 \\ b_1 - b_3 & b_2 & 0 \\ b_1 - b_2 & 0 & b_3 \end{vmatrix} = 0,$$

i.e., on suppressing the non-zero factor β_1:

$$b_2(c_1 + c_2) = b_3(c_1 + c_3).$$

Likewise, the consideration of the inscribed triangles BA_3A_1, BA_1A_2 and their circumscribed triangles gives:

$$b_3(c_2 + c_3) = b_1(c_2 + c_1), \quad b_1(c_3 + c_1) = b_2(c_3 + c_2).$$

The last three equations imply:

$$b_1 : b_2 : b_3 = (c_2 + c_3) : (c_3 + c_1) : (c_1 + c_2);$$

hence from (3), using also (4) and the hypothesis $p \neq 2$, we deduce the equality

$$c_2c_3 + c_3c_1 + c_1c_2 = 0.$$

This equality means that each of the $q-2$ points B lies on the conic

$$x_2x_3 + x_3x_1 + x_1x_2 = 0.$$

Since this conic obviously contains in addition the three points A_1, A_2, A_3, and its points are precisely $q + 1$ in number, thus $\mathscr{C}$ must coincide with it, which proves Theorem I.

4. We remark, in conclusion, that Theorem I does not hold on a finite plane of *characteristic* $p = 2$, if $q > 4$. For, as it is well known, the $q + 1$ tangents of a non-singular conic then meet at a point; this point and q of the $q + 1$ points of the conic constitute an oval, which, however, is clearly not a conic.

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