THE CAUCHY PROBLEM FOR A HYPERBOLIC SECOND ORDER EQUATION WITH DATA ON THE PARABOLIC LINE

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1. Introduction. In this paper we consider the Cauchy problem for the equation

$$h(x, y)\, K(y)\, v_{xx} - v_{yy} + a(x, y)\, v_x + b(x, y)\, v_y + c(x, y)\, v + f(x, y) = 0 \tag{1}$$

with initial values prescribed on a segment of the x-axis. The coefficients in (1) are assumed to possess two continuous derivatives with respect to x and one continuous derivative with respect to y in the closure of the domain under consideration.[1] The function $K(y)$ is a monotone increasing function of y with $K(0) = 0$ and we suppose $h(x, y)$ is positive in the closure of the domain. Equation (1) is hyperbolic for positive values of y and parabolic on the line $y = 0$. The characteristics of (1) are given by the two families of curves

$$\frac{dy}{dx} = \pm \frac{1}{\sqrt{Kh}}. \tag{2}$$

Frankl **(4)** solved the Cauchy problem for the equation

$$yu_{xx} - u_{yy} + a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u = 0 \tag{3}$$

under the assumption that the coefficients $a(x, y)$, $b(x, y)$, and $c(x, y)$ are analytic. Berezin **(1)** treated the same problem for the equation

$$h(x, y)\, y^{\alpha}\, u_{xx} - u_{yy} + a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u + f(x, y) = 0 \tag{4}$$

with restrictions on the coefficients similar to those for (1), but with the condition $0 < \alpha < 2$. Starting from a different point of view Bers **(2)** solved the Cauchy problem for the equation

$$K(y)\, u_{xx} - u_{yy} = 0 \tag{5}$$

where $K(y)$ is a continuous monotone increasing function of y with $K(0) = 0$. A solution to the same problem has been obtained for equation (5) by Germain and Bader **(5)**. They make the additional assumption that $K(y) \sim cy$ as $y \to 0$ and thus make use of Riemann's method. The result of Bers shows that if the lower order terms are absent in an equation such as (4) there is no restriction on the rate of growth of the coefficient of u_{xx}. On the other hand

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[1]The smoothness conditions on the coefficients can be weakened slightly.

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Berezin gives an example to show that for $\alpha > 2$ the Cauchy problem is not correctly set for equation (4). In solving the initial value problem for equation (1) we shall impose such conditions on the coefficients as to encompass (except for slight differences in smoothness requirements) the previous results on this problem.

Let D be the domain bounded by a segment $a_0 \leqslant x \leqslant a_1$ of the x-axis and the characteristics Γ_1 and Γ_2 of the families (2) emanating from $(a_0, 0)$ and $(a_1, 0)$ respectively, and which intersect. The initial values are given by two functions $\tau(x)$, $\nu(x)$, $a_0 \leqslant x \leqslant a_1$ which are assumed to have continuous fourth derivatives.[2] That is, we seek a solution of (1) in D satisfying the conditions

$$v(x, 0) = \tau(x), \quad v_y(x, 0) = \nu(x), \quad a_0 \leqslant x \leqslant a_1. \tag{6}$$

With the change of variable

$$w(x, y) = v(x, y) - y\nu(x) - \tau(x)$$

equation (1) takes the form

$$h(x, y)K(y)w_{xx} - w_{yy} + a(x, y)w_x + b(x, y)w_y + c(x, y)w + F(x, y) = 0 \tag{7}$$

where

$$F(x, y) = hK(y\nu'' + \tau'') + a(y\nu' + \tau') + b\nu + c(y\nu + \tau) + f.$$

The initial conditions (6) become

$$w(x, 0) = w_y(x, 0) = 0, \qquad a_0 \leqslant x \leqslant a_1. \tag{8}$$

We restrict our considerations to equation (7) and inquire under what circumstances the Cauchy problem is correctly set. We shall show that the Cauchy problem is indeed correctly set if the condition

$$\frac{y\, a(x, y)}{\sqrt{K(y)}} \to 0 \text{ as } y \to 0, \qquad a_0 \leqslant x \leqslant a_1, \tag{9}$$

is satisfied.

This condition is automatically fulfilled in the case of (5) while for equation (4) it makes no additional requirement on $a(x, y)$ if $0 < \alpha < 2$. On the other hand we find as a special case that for the equation

$$h(x, y)\, K(y)\, u_{xx} - u_{yy} + b(x, y)\, u_y + c(x, y)\, u + f(x, y) = 0$$

the Cauchy problem is correctly set for all monotone $K(y)$ as (9) is clearly satisfied in this case. This is an example of a result not obtainable from any of the previous works on the singular Cauchy problem.

THEOREM. *Assume that in the closure of D the coefficients of equation* (7) *are twice continuously differentiable with respect to x, once continuously differentiable*

[2]Assuming the third derivatives satisfy a Lipschitz condition would be sufficient.

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with respect to y and that $h(x, y) > 0$. Suppose $K(y)$ is a monotone increasing function of y with $K(0) = 0$. Then if condition (9) is satisfied the Cauchy problem for equation (7) is correctly set.

In §2 equation (7) is transformed to a system of integral equations. The above theorem is proved in §3, and in §4 some remarks are made about more general equations.[3]

2. Reduction to a System of Integral Equations. We introduce the new unknown functions

$$u_1(x, y) = w(x, y), \quad u_2(x, y) = \sqrt{Kh}\, w_x + w_y, \quad u_3(x, y) = -\sqrt{Kh}\, w_x + w_y.$$

Then (7) may be written as the system

$$\begin{aligned} u_{1y} &= \tfrac{1}{2}(u_2 + u_3), \\ u_{2y} - \sqrt{Kh}\, u_{2x} &= cu_1 + \tfrac{1}{2}\left(\frac{a}{\sqrt{Kh}} + b + \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} - (\sqrt{Kh})_x\right)u_2 \\ &\quad + \tfrac{1}{2}\left(-\frac{a}{\sqrt{Kh}} + b - \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} + (\sqrt{Kh})_x\right)u_3 + F(x, y), \\ u_{3y} + \sqrt{Kh}\, u_{3x} &= cu_1 + \tfrac{1}{2}\left(\frac{a}{\sqrt{Kh}} + b - \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} + (\sqrt{Kh})_x\right)u_2 \\ &\quad + \tfrac{1}{2}\left(-\frac{a}{\sqrt{Kh}} + b + \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} + (\sqrt{Kh})_x\right)u_3 + F(x, y), \end{aligned} \tag{10}$$

subject to the initial conditions

$$u_1(x, 0) = u_2(x, 0) = u_3(x, 0) = 0, \quad a_0 \leqslant x \leqslant a_1.$$

The characteristics of (10) are the lines $x =$ const. and the two families of curves given by (2). Let $P(x, y)$ be a point in D and construct the three characteristics of (10) passing through P. The left side of each of the equations in (10) represents a derivative in a characteristic direction. If we denote by s_2 the member of the family

$$\frac{dy}{dx} = \frac{-1}{\sqrt{Kh}}$$

passing through P and by s_3 the member of the family

$$\frac{dy}{dx} = \frac{1}{\sqrt{Kh}}$$

[3]The smoothness conditions on the coefficients can be weakened slightly.

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passing through P we can write (10) in the form

$$\frac{du_1}{dy} = \tfrac{1}{2}(u_2 + u_3),$$

$$\frac{du_2}{ds_2} = \frac{c}{\sqrt{(1+Kh)}}u_1 + \frac{1}{2\sqrt{(1+Kh)}}\left(\frac{a}{\sqrt{Kh}} + b + \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} - (\sqrt{Kh})_x\right)u_2 + \frac{1}{2\sqrt{(1+Kh)}}\left(-\frac{a}{\sqrt{Kh}} + b - \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} + (\sqrt{Kh})_x\right)u_3 + \frac{F}{\sqrt{(1+Kh)}}$$

(11)

$$\frac{du_3}{ds_3} = \frac{c}{\sqrt{(1+Kh)}}u_1 + \frac{1}{2\sqrt{(1+Kh)}}\left(\frac{a}{\sqrt{Kh}} + b - \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} - (\sqrt{Kh})_x\right)u_2 + \frac{1}{2\sqrt{(1+Kh)}}\left(-\frac{a}{\sqrt{Kh}} + b + \frac{(\sqrt{Kh})_y}{\sqrt{Kh}} + (\sqrt{Kh})_x\right)u_3 + \frac{F}{\sqrt{(1+Kh)}}.$$

To simplify the notation we define the quantities

$$A = \frac{c}{\sqrt{(1+Kh)}}, \qquad B_2 = \frac{1}{2\sqrt{(1+Kh)}}\left(b - (\sqrt{Kh})_x + \frac{h_y}{2h}\right)$$

$$C_2 = \frac{1}{2\sqrt{(1+Kh)}}\left(b + (\sqrt{Kh})_x - \frac{h_y}{2h}\right),$$

$$D_2 = \frac{1}{2\sqrt{(1+Kh)}}\left(\frac{a}{\sqrt{Kh}} + \frac{K'}{2K}\right)$$

$$B_3 = \frac{1}{2\sqrt{(1+Kh)}}\left(b - (\sqrt{Kh})_x - \frac{h_y}{2h}\right),$$

$$C_3 = \frac{1}{2\sqrt{(1+Kh)}}\left(b + (\sqrt{Kh})_x + \frac{h_y}{2h}\right)$$

$$D_3 = \frac{1}{2\sqrt{(1+Kh)}}\left(\frac{a}{\sqrt{Kh}} - \frac{K'}{2K}\right), \qquad E = \frac{F}{\sqrt{(1+Kh)}}.$$

The system (11) then becomes

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$$(12)\qquad \begin{aligned} \frac{du_1}{dy} &= \tfrac{1}{2}(u_2 + u_3) \\ \frac{du_i}{ds_i} &= Au_1 + B_i u_2 + C_i u_3 + D_i(u_2 - u_3) + E \qquad (i = 2, 3). \end{aligned}$$

Integrating (12) along the characteristics we obtain the system of integral equations

$$(13)\qquad \begin{aligned} u_1(x, y) &= \tfrac{1}{2}\int_0^y [u_2(x, y) + u_3(x, y)]\, dy \\ u_i(x, y) &= \int_0^{s_i} [Au_1 + B_i u_2 + C_i u_3 + D_i(u_2 - u_3) + E]\, ds_i \quad (i = 2, 3). \end{aligned}$$

Any solution $u_1(x, y)$ of (13) with the proper differentiability properties will clearly satisfy (7).

3. Proof of Theorem. It suffices to prove the theorem for an arbitrarily small segment $0 \leqslant y \leqslant \eta$. For, once the solution is determined in such a domain the standard Cauchy problem may be solved on the line $y = \eta$ yielding the result in D. We select initially for $K(y)$ the function y^α, $\alpha > 0$, since the main argument of the proof is exhibited in this case. In the last paragraph of this section the case where $K(y)$ does not behave like y^α is discussed.

Let $P(\xi, \eta)$ be a point of D and suppose $x = x_2(y)$, $x = x_3(y)$ are the characteristics of (2) passing through P. We have the inequality for $0 \leqslant y \leqslant \eta$

$$(14)\qquad |x_2 - x_3| \leqslant 2\int_0^y |\sqrt{Kh}|\, dy \leqslant My^{\frac{1}{2}\alpha+1}$$

where M is an upper bound in D for $4\sqrt{h}/(\alpha + 2)$. The quantity M will denote throughout a positive constant that dominates in absolute value the coefficients of (7) and their first derivatives with respect to x. That is, we require that M be so large that

$$(15)\qquad |A|, |A_x|, |B_i|, |B_{ix}|, |C_i|, |C_{ix}|, |h^{-1}|, |h_y h^{-1}|, |h_x h^{-1}|, |E|, |E_x| \leqslant M$$

for all x, y in D and $i = 2, 3$. From condition (9) we have

$$(16)\qquad a(x, y) \sim \delta(y)\, y^{\frac{1}{2}\alpha-1}$$

where $\delta(y) \to 0$ as $y \to 0$. We select $\gamma(0 < \gamma < 1)$ and $\eta(>0)$ so that

$$(17)\qquad \begin{aligned} &\frac{3}{2}M\eta + \left[\delta(\eta)\, M + \frac{\alpha}{2}\right]\frac{1}{\alpha + 2}\eta^{\frac{1}{2}\alpha} < \gamma, \\ &\frac{6\, M^2\eta^2}{\alpha + 6} + \frac{8M\eta}{\alpha + 2} + \left[\delta(\eta) + \frac{\alpha}{2M}\right]\frac{2M}{\alpha + 2} < \gamma. \end{aligned}$$

It is easy to see that if γ is taken sufficiently close to 1 and η sufficiently small the inequalities (17) can always be satisfied.

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To establish the existence of a solution of the system (13) we proceed by iterations. We define $u_i^{(0)}(x, y) \equiv 0$ $(i = 1, 2, 3)$, and the quantities $u_i^{(k)}(x, y)$ by the relations

$$u_1^{(k)} = \tfrac{1}{2}\int_0^y [u_2^{(k-1)} + u_3^{(k-1)}]\, dy$$

$$u_i^{(k)} = \int_0^{s_i} [A u_1^{(k-1)} + B_i u_2^{(k-1)} + C_i u_3^{(k-1)} + D_i (u_2^{(k-1)} - u_3^{(k-1)}) + E]\, ds_i \qquad (i = 2, 3).$$

We shall show that the sequences $\{u_i^{(k)}(x, y)\}$ $(i = 1, 2, 3)$ converge uniformly in that part of D contained in the strip $0 \leqslant y \leqslant \eta$. We first establish some inequalities for the $\{u_i^{(k)}(x, y)\}$. To do this it is necessary to examine the characteristics (2) and inequality (14). If $P(\xi_0, \eta_0)$ is a point of D with $\eta_0 < \eta$ then the characteristics through P are given by the solutions of (2) which we write in the form

$$x = x_2(y;\ \xi_0, \eta_0), \quad x = x_3(y;\ \xi_0, \eta_0).$$

Let D_1 be the domain bounded by these characteristics and the line $y = 0$. Then it is clear that an inequality such as (14) (with perhaps M somewhat larger) will hold for any two points $P_1(x_2, y)$ and $P_2(x_3, y)$ in D.

LEMMA 1. *For all k the inequalities*

$$(18)\quad \begin{aligned} &|u_i^{(k)}(x, y)| \leqslant M\sum_{j=0}^{k} \gamma^j y, \quad |u_i^{(k)}(x_2, y) - u_i^{(k)}(x_3, y)| \leqslant M\sum_{j=0}^{k} \gamma^j y^{\frac{1}{2}\alpha+1},\\ &|u_2^{(k)}(x, y) - u_3^{(k)}(x, y)| \leqslant M\sum_{j=0}^{k} \gamma^j y^{\frac{1}{2}\alpha+1} \qquad (i = 1, 2, 3).\end{aligned}$$

hold in D_1.

Proof. We proceed by induction, establishing all inequalities simultaneously. That is, we show that all inequalities (18) hold for $n = 1$, and then assuming they *all* hold for $n = k$ we establish each inequality for $n = k + 1$. Clearly $u_1^{(1)}(x, y)$ vanishes and

$$|u_i^{(1)}(x, y)| \leqslant \int_0^y |E|\, dy \leqslant My \leqslant M\sum_{j=0}^{1} \gamma^j y \qquad (i = 2, 3).$$

Further, for $i = 2, 3$

$$\begin{aligned} |u_i^{(1)}(x_2, y) - u_i^{(1)}(x_3, y)| &\leqslant \int_0^y |E(x_i(t; x_2, y), t) - E(x_i(t; x_3, y), t)|\, dt\\ &\leqslant \int_0^y |E_x||x_i(t; x_2, y) - x_i(t; x_3, y)|\, dt\\ &\leqslant M\int_0^y |x_2 - x_3|\, dy \leqslant M\int_0^y My^{\frac{1}{2}\alpha+1}\, dy\\ &\leqslant M\sum_{j=0}^{1} \gamma^j y^{\frac{1}{2}\alpha+1}\end{aligned}$$

and

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$$|u_2^{(1)}(x, y) - u_3^{(1)}(x, y)| \leqslant \int_0^y |E(x_2, y) - E(x_3, y)|\, dy \leqslant M \sum_{j=0}^{1} \gamma^j y^{\frac{1}{2}\alpha+1}.$$

Assume the result holds for $n = k$. Then for $n = k + 1$ we find

$$|u_1^{(k+1)}(x, y)| \leqslant \tfrac{1}{2} \int_0^y [|u_2^{(k)}| + |u_3^{(k)}|]\, dy$$

$$\leqslant \tfrac{1}{2} \int_0^y 2M \sum_{j=0}^{k} \gamma^j y dy \leqslant M \sum_{j=0}^{k} \gamma^j \tfrac{1}{2} y^2 \leqslant M \sum_{j=0}^{k+1} \gamma^j y,$$

as we can add to (17) the condition that η be less than 2γ. For $i = 2, 3$ we obtain

$$|u_i^{(k+1)}(x, y)| \leqslant \int_0^y \Big\{ |A| \sum_{j=0}^{k} M\gamma^j y + |B_i| \sum_{j=0}^{k} M\gamma^j y + |C_i| \sum_{j=0}^{k} M\gamma^j y + |D_i| \sum_{j=0}^{k} M\gamma^j y^{\frac{1}{2}\alpha+1} + |E| \Big\}\, dy$$

$$\leqslant M \int_0^y \Big\{ 3M \sum_{j=0}^{k} \gamma^j y + \tfrac{1}{2}\Big(\frac{\delta(y)}{y|h|} + \frac{\alpha}{2y}\Big) y^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k} \gamma^j + 1 \Big\}\, dy$$

$$\leqslant My \Big[1 + \Big\{ \frac{3}{2} My + \Big(\frac{\delta(y)}{|h|} + \frac{\alpha}{2}\Big) \frac{y^{\frac{1}{2}\alpha}}{\alpha + 2} \Big\} \sum_{j=0}^{k} \gamma^j \Big]$$

$$\leqslant M \sum_{j=0}^{k+1} \gamma^j y,$$

the last inequality being valid because of the first of the inequalities in (17). We also have

$$|u_2^{(k+1)}(x, y) - u_3^{(k+1)}(x, y)| \leqslant \int_0^y |A(x_2, y)\, u_1^{(k)}(x_2, y) - A(x_3, y)\, u_1^{(k)}(x_3, y) + B_2(x_2, y)\, u_2^{(k)}(x_2, y) - B_3(x_3, y)\, u_2^{(k)}(x_3, y) + C_2(x_2, y)\, u_3^{(k)}(x_2, y) - C_3(x_3, y)\, u_3^{(k)}(x_3, y) + D_2(x_2, y)[u_2^{(k)}(x_2, y) - u_3^{(k)}(x_2, y)] - D_3(x_3, y)[u_2^{(k)}(x_3, y) - u_3^{(k)}(x_3, y)] + E(x_2, y) - E(x_3, y)|\, dy.$$

To get a bound for the integral on the right side we have the following estimates

$$|A(x_2, y)\, u_1^{(k)}(x_2, y) - A(x_3, y)\, u_1^{(k)}(x_3, y)|$$

$$\leqslant |A(x_2, y)\, u_1^{(k)}(x_2, y) - A(x_3, y)\, u_1^{(k)}(x_2, y)| + |A(x_3, y)[u_1^{(k)}(x_2, y) - u_1^{(k)}(x_3, y)]|$$

$$\leqslant M \sum_{j=0}^{k} \gamma^j\, y |A(x_2, y) - A(x_3, y)| + M^2 \sum_{j=0}^{k} \gamma^j\, y^{\frac{1}{2}\alpha+1}.$$

Now applying the theorem of the mean to the first term on the right we find

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$$|A(x_2, y)\, u_1^{(k)}(x_2, y) - A(x_3, y)\, u_1^{(k)}(x_3, y)|$$
$$\leqslant M^2 \sum_{j=0}^{k} \gamma^j\, y|x_2 - x_3| + M^2 \sum_{j=0}^{k} \gamma^j\, y^{\frac{1}{2}\alpha+1}$$
$$\leqslant M^2 \sum_{j=0}^{k} \gamma^j\, yMy^{\frac{1}{2}\alpha+1} + M^2 \sum_{j=0}^{k} \gamma^j\, y^{\frac{1}{2}\alpha+1}.$$

We also have the inequality

$$\begin{aligned}(19)\quad &|B_2(x_2, y)\, u_2^{(k)}(x_2, y) - B_3(x_3, y)\, u_2^{(k)}(x_3, y) \\ &\qquad + C_2(x_2, y)\, u_3^{(k)}(x_2, y) - C_3(x_3, y)\, u_3^{(k)}(x_3, y)| \\ &\leqslant |B_2(x_2, y) - B_2(x_3, y)||u_2^{(k)}(x_2, y)| \\ &+ |[B_2(x_3, y) - B_3(x_3, y)]\, u_2^{(k)}(x_2, y) \\ &\qquad + [C_2(x_3, y) - C_3(x_3, y)]\, u_3^{(k)}(x_2, y)| \\ &+ |B_3(x_3, y)||u_2^{(k)}(x_2, y) - u_2^{(k)}(x_3, y)| \\ &\qquad + |C_3(x_3, y)||u_3^{(k)}(x_2, y) - u_3^{(k)}(x_3, y)|.\end{aligned}$$

Taking into account the definitions of B_2, B_3, C_2, C_3 in the second term on the right above, we obtain after an application of the theorem of the mean the following upper bound for the right side:

$$M|x_2 - x_3|M \sum_{j=0}^{k} \gamma^j\, y + \left|\frac{h_y}{4h}\right|\, |u_2^{(k)}(x_2, y) - u_3^{(k)}(x_2, y)|$$
$$+ M|u_2^{(k)}(x_2, y) - u_2^{(k)}(x_3, y)| + M|u_3^{(k)}(x_2, y) - u_3^{(k)}(x_3, y)|$$
$$\leqslant M^3\, y^{\frac{1}{2}\alpha+2} \sum_{j=0}^{k} \gamma^j + 3M^2\, y^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k} \gamma^j.$$

Hence we have

$$\begin{aligned}|u_2^{(k+1)}(x, y) - u_3^{(k+1)}(x, y)| \leqslant \int_0^y \Big\{ &M^3\, y^{\frac{1}{2}\alpha+2} \sum_{j=0}^{k} \gamma^j + M^2\, y^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k} \gamma^j \\ &+ M^3 y^{\frac{1}{2}\alpha+2} \sum_{j=0}^{k} \gamma^j + 3M^2 y^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k} \gamma^j + |D_2| M y^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k} \gamma^j + |D_3| M y^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k} \gamma^j \\ &+ M^2 y^{\frac{1}{2}\alpha+1} \Big\}\, dy\end{aligned}$$
$$\leqslant My^{\frac{1}{2}\alpha+1}\left[\frac{2M}{\alpha+4}\, y + \left(\sum_{j=0}^{k} \gamma^j\right)\left\{\frac{4M^2y^2}{\alpha+6} + \frac{8My}{\alpha+2} + \left(\delta(y)M + \frac{\alpha}{2}\right)\frac{2}{\alpha+2}\right\}\right].$$

And taking inequality (17) into account we finally find

$$|u_2^{(k+1)}(x, y) - u_3^{(k+1)}(x, y)| \leqslant My^{\frac{1}{2}\alpha+1} \sum_{j=0}^{k+1} \gamma^j.$$

The proof for the cases

$$|u_i^{(k+1)}(x_2, y) - u_i^{(k+1)}(x_3, y)| \qquad (i = 1, 2, 3)$$

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is completely analogous and may be omitted. The only change required is that for $i = 2, 3$ the inequality

$$\frac{6M^2y^2}{\alpha+6} + \frac{6My}{\alpha+2} + \left(M\delta(y) + \frac{\alpha}{2}\right)\frac{2}{\alpha+2} < \gamma$$

is employed. However this follows from (17) and the induction is complete.

Lemma 2. *For all k the inequalities*

$$|u_i^{(k+1)}(x, y) - u_i^{(k)}(x, y)| \leqslant M\gamma^k y \qquad (i = 1, 2, 3)$$

$$(20) \quad |u_2^{(k+1)}(x, y) - u_3^{(k+1)}(x, y) - u_2^{(k)}(x, y) + u_3^{(k)}(x, y)| \leqslant M\gamma^k y^{\frac{1}{2}\alpha+1}$$

$$|u_i^{(k+1)}(x_2, y) - u_i^{(k+1)}(x_3, y) - u_i^{(k)}(x_2, y) + u_i^{(k)}(x_3, y)| \leqslant M\gamma^k y^{\frac{1}{2}\alpha+1} \qquad (i = 1, 2, 3)$$

hold in D_1.

Proof. We proceed by induction. It is clear that each of the inequalities holds for $n = 1$. Assume they are valid for $n = k$. For $n = k + 1$ we have

$$|u_i^{(k+1)}(x, y) - u_i^{(k)}(x, y)| \leqslant \int_0^y \Big\{ |A(x_2, y)||u_1^{(k)} - u_1^{(k-1)}| + |B_i||u_2^{(k)} - u_2^{(k-1)}| + |D_i||u_2^{(k)} - u_3^{(k)} - u_2^{(k-1)} + u_3^{(k-1)}| \Big\} dy$$

$$\leqslant M\gamma^{k-1}\left[My^2 + \left(M\delta(y) + \frac{\alpha}{2}\right)\frac{1}{\alpha+2}y^{\frac{1}{2}\alpha+1}\right] \leqslant M\gamma^k y \qquad (i = 2, 3),$$

and similarly for $i = 1$. Also,

$$|u_2^{(k+1)}(x, y) - u_3^{(k+1)}(x, y) - u_2^{(k)}(x, y) + u_3^{(k)}(x, y)| \leqslant \int_0^y |A(x_2, y)\, u_1^{(k)}(x, y)$$
$$+ B_2(x_2, y)u_2^{(k)}(x_2, y) + C_2(x_2, y)u_3^{(k)}(x_2, y) + D_2(x_2, y)[u_2^{(k)}(x_2, y) - u_3^{(k)}(x_2, y)]$$
$$- A(x_3, y)\, u_1^{(k)}(x_3, y) - B_3(x_3, y)\, u_2^{(k)}(x_3, y)$$
$$- C_3(x_3, y)\, u_3^{(k)}(x_3, y) - D_3(x_3, y)[u_2^{(k)}(x_3, y) - u_3^{(k)}(x_3, y)]$$
$$- A(x_2, y)\, u_1^{(k-1)}(x_2, y) - B_2(x_2, y)\, u_2^{(k-1)}(x_2, y) - C_2(x_2, y)\, u_3^{(k-1)}(x_2, y)$$
$$- D_2(x_2, y)[u_2^{(k-1)}(x_2, y) - u_3^{(k-1)}(x_2, y)] + A(x_3, y)\, u_1^{(k-1)}(x_3, y)$$
$$+ B_3(x_3, y)\, u_2^{(k-1)}(x_3, y) + C_3(x_3, y)\, u_3^{(k-1)}(x_3, y)$$
$$+ D_3(x_3, y)[u_2^{(k-1)}(x_3, y) - u_3^{(k-1)}(x_3, y)|\, dy.$$

We make the estimate

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$$|A(x_2, y)[u_1^{(k)}(x_2, y) - u_1^{(k-1)}(x_2, y)] - A(x_3, y)[u_1^{(k)}(x_3, y) - u_1^{(k-1)}(x_3, y)]$$
$$\leqslant |A(x_2, y) - A(x_3, y)||u_1^{(k)}(x_2, y) - u_1^{(k-1)}(x_2, y)|$$
$$+ |A(x_3, y)||u_1^{(k)}(x_2, y) - u_1^{(k-1)}(x_2, y) - u_1^{(k)}(x_3, y) + u_1^{(k-1)}(x_3, y)|$$
$$\leqslant M|x_2 - x_3|\, M\gamma^{k-1}y + M^2\gamma^{k-1}y^{\frac{1}{2}\alpha+1} \leqslant M\gamma^{k-1}y^{\frac{1}{2}\alpha+1}(My + M).$$

Similar bounds are found for the remaining terms B_i, C_i, D_i. We have only to be sure to combine the terms involving B_2, B_3, C_2, C_3 as in the estimate (19). This yields the inequality

$$|B_2(x_2, y)[u_2^{(k)}(x_2, y) - u_2^{(k-1)}(x_2, y)] - B_3(x_3, y)[u_2^{(k)}(x_3, y) - u_2^{(k-1)}(x_3, y)]$$
$$+ C_2(x_2, y)[u_3^{(k)}(x_2, y) - u_3^{(k-1)}(x_2, y)] - C_3(x_3, y)[u_3^{(k)}(x_3, y) - u_3^{(k-1)}(x_3, y)|$$
$$\leqslant 2M^2\gamma^{k-1}y|x_2 - x_3| + \left|\frac{2h_y}{h}\right| M\gamma^{k-1}y^{\frac{1}{2}\alpha+1} + 2M^2\gamma^{k-1}y^{\frac{1}{2}\alpha+1}$$
$$\leqslant y^{\frac{1}{2}\alpha+1}M\gamma^{k-1}(2M^2y + 3M).$$

With the aid of these estimates inequalities (20) follow.

From Lemma 2 it is clear that the sequences $\{u_i^{(k)}(x, y)\}$ $(i = 1, 2, 3)$ converge uniformly. Since each $u_i^{(k)}(x, y)$ is continuous, so are the limits, which we denote by $u_i(x, y)$. Inequalities (18) yield

$$|u_i(x, y)| \leqslant M_1 y, \qquad (i = 1, 2, 3)$$
$$|u_2(x, y) - u_3(x, y)| \leqslant M_1 y^{\frac{1}{2}\alpha+1} \tag{21}$$

where

$$M_1 = M\sum_{j=0}^{\infty}\gamma^j.$$

The limit functions obtained with the aid of Lemma 2 are easily seen to satisfy the system of integral equations (13) and the initial conditions $u_1(x, 0) = u_2(x, 0) = u_3(x, 0) = 0$, $a_0 \leqslant x \leqslant a_1$.

The uniqueness of the solution follows from the fact that the difference of two solutions would have to satisfy the homogeneous system

$$v_1 = \tfrac{1}{2}\int_0^y (v_2 + v_3)\, dy,$$
$$v_i = \int_0^{s_i} \{Av_1 + B_i v_2 + C_i v_3 + D_i(v_2 - v_3)\}\, ds_i \qquad (i = 2, 3).$$

The functions v_i satisfy the inequalities (20) and repeated insertion of these in the right side above shows that each v_i must satisfy an inequality of the form $|v_i| \leqslant M_2\gamma^k$ for arbitrary k. Hence $v_i \equiv 0 (i = 1, 2, 3)$.

It remains to be shown that $w(x, y) = u_1(x, y)$ satisfies equation (7) and depends continuously on the given data. From the relation $u_{1y}(x, y) = \frac{1}{2}(u_2 + u_3)$ we see that w possesses a derivative with respect to y. Also,

$$w_x = \frac{u_2 - u_3}{2\sqrt{Kh}},$$

and from the basic inequality for $u_2 - u_3$ it is clear that w_x exists for $y \geqslant 0$. To obtain the existence of the second derivatives of w we consider the system of integral equations

$$u_{1x} = \tfrac{1}{2}\int_0^{y_0} (u_{2x} + u_{3x})\, dy$$

(22) $$u_{ix}(x_0, y_0) = \int_0^{y_0} \{Au_{1x} + B_i u_{2x} + C_i u_{3x} + D_i(u_{2x} - u_{3x}) + E_x + A_x u_1 + B_{ix}u_2 + C_{ix}u_3 + D_{ix}(u_2 - u_3)\}\frac{d\gamma_i}{dx_0}\, dy, \qquad (i = 2, 3)$$

where $y = \gamma_i(x; x_0, y_0)$ $(i = 2, 3)$ are the equations of the characteristics through $P(x_0, y_0)$. The above system is obtained from (13) by differentiation with respect to x. An iteration process can be set up and a solution found by the same method employed in solving (13). It is in the solution of this system that the bounds for the second derivatives of the coefficients are employed. The solution of (22) yields the existence of the second derivatives of w. Since w satisfies (13) and has the required differentiability properties, it is the solution of (7) satisfying initial conditions (8). The continuous dependence on the given data follows at once from inequalities (21).

If $K(y)$ tends to zero more rapidly than any power of y we have the inequality

$$|x_2 - x_3| \leqslant 2\int_0^y \sqrt{Kh}\, dy \leqslant \theta(y)\sqrt{K}\, y,$$

where $\theta(y) \to 0$ as $y \to 0$. This is easily seen by considering the ratio

$$\int_0^y \sqrt{Kh}\, dy / \sqrt{K}\, y,$$

and noting that this approaches zero as $y \to 0$. Hence the estimate for $|x_2 - x_3|$, which is basic, is better in this case than the case $K(y) \sim y^\alpha$. Should $K(y) \to 0$ slower than any power of y, the argument used for the case $0 < \alpha < 2$ applies and condition (9) is unnecessary.

4. Other Equations. Conti **(3)** has shown that the Cauchy problem for the equation

(23) $$h(x, y)y^\alpha u_{xx} - u_{yy} = f(x, y, u, u_x, u_y)$$

is correctly set for the range $0 < \alpha < 2$. The discussion of equation (7) can be modified to include equation (23). In this case condition (9) is replaced by the condition

$$\frac{y f_{u_x}(x, y, u, u_x, u_y)}{\sqrt{K}} \to 0, \quad \text{as } y \to 0, \quad a_0 \leqslant x \leqslant a_1,$$

and otherwise the arguments are analogous.

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