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TWIN SQUAREFUL NUMBERS

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Abstract

A number is squareful if the exponent of every prime in its prime factorization is at least two. In this paper, we give, for a fixed l, the number of pairs of squareful numbers n, $n + l$ such that n is less than a given quantity.

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1. Introduction

Recall that a positive integer n is a squareful number when, if a prime number p divides n, then p^2 also divides n. In other words, the exponents e_i in the prime factorization $p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r}$ of n are all at least two. Hence all numbers of the form $a^2 b^3$ are squareful. In fact, any squareful number n can be written uniquely as $a^2 b^3$ for some positive integers a and b, with b squarefree. Here squarefree means that, in the prime factorization of $n = p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r}$, all the exponents e_i are equal to one. It is well known (see, for example, [7]) that there are asymptotically $Cx^{1/2}$ squareful numbers up to x for some positive constant C. Similar to the concept of twin primes, one can talk about twin squareful numbers, namely when both n and $n + 1$ are squareful. By looking at the Pell equation $x^2 - 8y^2 = 1$, one can see that there are infinitely many twin squareful numbers. In the summer of 2009, Koo posed the following question.

QUESTION 1. How many twin squareful numbers n, $n + 1$ are there with $n \leq x$? Do they have 'zero density' among all squareful numbers up to x?

More generally, we consider the following question.

QUESTION 2. For a given positive integer l, how many twin squareful numbers n, $n + l$ are there with $n \leq x$? Do they have 'zero density' among all squareful numbers up to x?

Let $N(x; l)$ denote the number of positive integers $n \leq x$ such that n and $n + l$ are both squareful.

We will prove the following result.

THEOREM 3. *If* $x \geq 2$ *and* $l \geq 1$, *then*

$$N(x; l) \ll d_3(l) x^{2/5} (\log x)^2,$$

where $d_3(l)$ *is the number of ways to write* l *as a product of three positive integers.*

Since $2/5 < 1/2$, this shows that twin squareful numbers indeed have 'zero density' among all squareful numbers if l is not too big. For a fixed integer l, we have a slight improvement.

THEOREM 4. *If* $x \geq 2$ *and* $l \geq 1$, *then*

$$N(x; l) \ll_l x^{7/19} \log x.$$

Note that $7/19 = 0.36842\ldots$.

We suspect that the following conjecture is true.

CONJECTURE 5. For any positive ϵ, there exists a positive constant C_ϵ such that

$$N(x; l) \leq C_\epsilon x^\epsilon$$

for all $x, l \geq 1$.

Towards Conjecture 5, we have the following conditional result.

THEOREM 6. *Assume the abc-conjecture. Then for any positive integer* l *and any positive* ϵ,

$$N(x; l) \ll_{\epsilon,l} x^\epsilon.$$

The paper is organized as follows. We prove Theorems 3 and 6 first, then the more involved Theorem 4. Throughout the paper, we write $F(x) \ll G(x)$ or $F(x) = O(G(x))$ to mean that $|F(x)| \leq c\, G(x)$ for some constant $c > 0$, while $F(x) \ll_{\lambda_1,\lambda_2,\ldots,\lambda_n} G(x)$ and $F(x) = O_{\lambda_1,\lambda_2,\ldots,\lambda_n}(G(x))$ mean that the implicit constant may depend on $\lambda_1, \lambda_2, \ldots, \lambda_n$. Also, $|S|$ stands for the number of elements in a set S.

2. Proof of Theorem 3

To begin, let us define the divisor function

$$d_{2,3}(n) = \sum_{\substack{a,b \\ a^2b^3=n}} 1.$$

In particular $d_{2,3}(n)$ is supported on squareful numbers only. Clearly,

$$N(x,l) \leq \sum_{n \leq x} d_{2,3}(n) d_{2,3}(n+l). \tag{1}$$

This looks like the divisor sum

$$\sum_{n \leq x} d(n)d(n+l) \tag{2}$$

where $d(n)$ is the usual divisor function, which counts the number of divisors of n. Many people have studied (2), starting with Ingham [6]. Our inspiration comes from Ingham's work.

Proof of Theorem 3. The sum $\sum_{n \leq x} d_{2,3}(n) d_{2,3}(n+l)$ in (1) can be rewritten as counting the number of quadruples of positive integers

$$S = \{(a,b,c,d) : a^2b^3 - c^2d^3 = l, c^2d^3 \leq x\}.$$

We will switch our focus to the variables a, b, c, d, just as Ingham did. Observe that

$$a^2c^2b^3d^3 = (c^2d^3)(c^2d^3 + l) \leq x(x+l) \leq 2x^2 =: X.$$

Let $0 < \lambda < 1$ be a parameter, which we will choose later. Clearly either $a^2c^2 \leq X^\lambda$ or $b^3d^3 \leq X^{1-\lambda}$. Let S_1 be the subset of S satisfying the extra condition $a^2c^2 \leq X^\lambda$ and S_2 be the subset of S satisfying the extra condition $b^3d^3 \leq X^{1-\lambda}$. Then

$$|S_1| = \sum_{ac \leq X^{\lambda/2}} N_1(a,c) \quad \text{and} \quad |S_2| = \sum_{bd \leq X^{(1-\lambda)/3}} N_2(b,d),$$

where

$$N_1(a,c) = |\{(b,d) : a^2b^3 - c^2d^3 = l \text{ and } d^3 \leq x/c^2\}|$$

and

$$N_2(b,d) = |\{(a,c) : b^3a^2 - d^3c^2 = l \text{ and } c^2 \leq x/d^3\}|.$$

We have a Thue equation of the form $Ax^3 - By^3 = l$ in $N_1(a,c)$. A uniform bound on the number of solutions, depending on the degree and l only, was first obtained by Evertse [4]. Here we use a later improvement by Bombieri and Schmidt [2] and have $N_1(a,c) \leq C3^{\omega(l)}$ for some positive absolute constant C, where $\omega(l)$ denotes the number of distinct prime factors of l. Hence

$$|S_1| \ll 3^{\omega(l)} X^{\lambda/2} \log X$$

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by a standard result on divisor sums. It is worth mentioning that when $l = 1$, a remarkable result of Bennett [1] gives $N_1(a, c) \le 1$.

As for $N_2(b, d)$, here we are counting the number of solutions to a Pell equation. By Estermann [3, Hilfssatz 2], $N_2(b, d) \le 2d(l)(\log X + 1)$. Hence

$$|S_2| \ll d(l)X^{(1-\lambda)/3}(\log X)^2.$$

On choosing $\lambda = \frac{2}{5}$,

$$|S| \le |S_1| + |S_2| \ll d_3(l)X^{1/5}(\log X)^2,$$

where $d_3(l)$ is the number of ways to write l as a product of three positive integers and $3^{\omega(l)} \le d_3(l)$. Therefore $N(x) \le |S| \ll d_3(l)x^{2/5}(\log x)^2$, as $X \ll x^2$, which gives Theorem 3. □

3. Proof of Theorem 6

First, let us recall the famous *abc*-conjecture. For a positive integer n, define $R(n)$, the kernel of n, by $R(n) = \prod_{p|n} p$, where the product is over all the prime numbers that divide n. For example, $R(8) = 2$ and $R(72) = 6$. Considering the equation $a + b = c$, the *abc*-conjecture states that for every $\epsilon > 0$,

$$c \ll_\epsilon R(abc)^{1+\epsilon}$$

for any relatively prime integers a, b, c.

Proof of Theorem 6. As in the previous section, we consider the set

$$S = \{(a, b, c, d) : a^2b^3 - c^2d^3 = l, c^2d^3 \le x\}.$$

Rearranging the equation,

$$c^2d^3 + l = a^2b^3.$$

Suppose that k is the greatest common divisor of c^2d^3, l and a^2b^3. There are at most $d(l)$ possibilities for k. For each fixed k, let $k = p_1^{e_1}p_2^{e_2}\ldots p_r^{e_r}$ be its prime factorization. Observe that if a^2b^3 is divisible by k, then $a^2b^3/k = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}a'^2b'^3$ for some a', b' and $\alpha_1, \alpha_2, \ldots, \alpha_r \in \{0, 1\}$ where $\alpha_i = 1$ when the exponent of p_i in the prime factorization of a^2b^3/k is exactly one, and $\alpha_i = 0$ otherwise. Similarly, $c^2d^3/k = p_1^{\beta_1}p_2^{\beta_2}\ldots p_r^{\beta_r}c'^2d'^3$ for some c', d' and $\beta_1, \beta_2, \ldots, \beta_r \in \{0, 1\}$ where $\beta_i = 1$ when the exponent of p_i in the prime factorization of c^2d^3/k is exactly one, and $\beta_i = 0$ otherwise. So we are reduced to counting the number of solutions in a', b', c', d' to

$$p_1^{\beta_1}p_2^{\beta_2}\ldots p_r^{\beta_r}c'^2d'^3 + \frac{l}{k} = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}a'^2b'^3.$$

By the *abc*-conjecture and the definition of $R(n)$, for fixed $k, \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r$,

$$\begin{aligned} c'^2d'^3, a'^2b'^3 &\ll_\epsilon R\left((p_1^{\beta_1}p_2^{\beta_2}\ldots p_r^{\beta_r}c'^2d'^3)\left(\frac{l}{k}\right)(p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}a'^2b'^3)\right)^{1+\epsilon/2} \\ &\le l^{1+\epsilon/2}(a'b'c'd')^{1+\epsilon/2} \end{aligned}$$

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because $R(mn) \leq R(m)R(n)$. Thus

$$a'^2 b'^3 c'^2 d'^3 \ll_\epsilon l^{2+\epsilon}(a'b'c'd')^{2+\epsilon}$$

which implies that $b'd' \ll_{\epsilon,l} (a'c')^{\epsilon/(1-\epsilon)} \ll x^{\epsilon/(1-\epsilon)} \leq x^{2\epsilon}$ for $\epsilon < 1/2$. Hence there are $O_{\epsilon,l}(x^{2\epsilon} \log x)$ choices for the pair (b', d'). For each such pair of b' and d', the Pell equation

$$p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} b'^3 a'^2 - p_1^{\beta_1} p_2^{\beta_2} \ldots p_r^{\beta_r} d'^3 c'^2 = l/k$$

has at most $O(d(l/k) \log x)$ solutions in (a', c') by [3, Hilfssatz 2]. Consequently, taking into account all the possibilities for k, $\alpha_1, \ldots, \alpha_r$, $\beta_1, \ldots, \beta_r$, there can be at most $O_{\epsilon,l}(d(l)2^r 2^r d(l/k) x^{2\epsilon} \log^2 x) = O_{\epsilon,l}(x^{3\epsilon})$ quadruples in S. This completes the proof of Theorem 6 as ϵ can be arbitrarily small. □

4. Proof of Theorem 4

We will prove Theorem 4 for the case where $l = 1$ and indicate how to modify the proof for general l at the end of this section. We need a result of Huxley [5] on rational points close to a curve.

THEOREM 7. *Suppose that f is defined on the interval $[0, M]$ and is $2l + 2$ times continuously differentiable with*

$$\left| \frac{f^{(r)}(x)}{r!} \right| \leq \frac{\lambda C^{r+1}}{M^r} \quad \forall r = 0, 1, 2, \ldots, 2l + 2.$$

Assume that

$$|D_{l+1,s}(f(x))| \geq \left(\frac{\lambda}{C^{l+2} M^{l+1}} \right)^s \quad \forall s = 1, 2, \ldots, l + 1,$$

where

$$D_{k,n}(f(x)) = \det\left(\frac{f^{(k+i-j)}}{(k + i - j)!} \right)_{n\times n}.$$

Let

$$\mathcal{R} = \left\{ \left(m, \frac{r}{q} \right) : 0 \leq m \leq M, 1 \leq q \leq Q, (r, q) = 1, \left| f(m) - \frac{r}{q} \right| \leq \frac{\Delta}{q^2} \right\}.$$

Let $T = \lambda Q^2$ and $\Delta < 1/2$, $C \geq 1$, $M \geq 2$, $Q \geq 2$, $T \geq 4$. Then

$$|\mathcal{R}| \ll_d (C^{l+2} M^l T)^{1/(2l+1)} + (C^{2l^3+8l^2+11l+4} \Delta^{l+1} T^l)^{1/(2(l+1)^2)} M.$$

In particular, when $l = 2$, the above theorem gives

$$|\mathcal{R}| \ll (C^4 M^2 T)^{1/5} + (C^{74} \Delta^3 T^2)^{1/18} M. \tag{3}$$

PROOF OF THEOREM 4. Recall from the previous section that we want to count the number of quadruples of positive integers

$$S = \{(a, b, c, d) : a^2 b^3 - c^2 d^3 = 1, x/2 < c^2 d^3 \leq x\}.$$

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Note that $(a, c) = 1 = (b, d)$ automatically. So we want $|a^2b^3 - c^2d^3| = 1$. Divide everything by c^2b^3, and then

$$\left|\frac{a^2}{c^2} - \frac{d^3}{b^3}\right| = \frac{1}{c^2b^3}.$$

Upon factoring, we see that

$$\left|\frac{a}{c} - \frac{d^{3/2}}{b^{3/2}}\right|\left|\frac{a}{c} + \frac{d^{3/2}}{b^{3/2}}\right| = \frac{1}{c^2b^3}.$$

Hence

$$\left|\frac{a}{c} - \frac{d^{3/2}}{b^{3/2}}\right| \leq \frac{1}{c^2b^3}\,\frac{1}{a/c} = \frac{1}{acb^3}. \tag{4}$$

Suppose that $1 \leq R_1 \leq a \leq 2R_1$ and $1 \leq R_2 \leq c \leq 2R_2$. Define

$$f_b(d) = \frac{d^{3/2}}{b^{3/2}}$$

where

$$\frac{M}{2} \leq d \leq M \leq \left(\frac{x}{R_2^2}\right)^{1/3}$$

since $c^2d^3 \leq x$. Based on (4), we will apply Theorem 7 to count the set

$$\mathcal{R}_{b,M} = \left\{\left(d, \frac{a}{c}\right) : \frac{M}{2} \leq d \leq M, R_2 \leq c \leq 2R_2, (a, c) = 1, \left|f_b(d) - \frac{a}{c}\right| \leq \frac{\Delta}{c^2}\right\}$$

where $\Delta = 4R_2/R_1b^3$. Now with $l = 2$, $C = 100$, $\lambda = M^{3/2}/b^{3/2}$, the reader can check that f is six times continuously differentiable and satisfies

$$\left|\frac{f^{(r)}(x + M/2)}{r!}\right| \leq \frac{\lambda 100^{r+1}}{(M/2)^r} \quad \text{for } r = 0, 1, 2, \ldots, 6 \text{ and } x \in [0, M/2].$$

As for the determinant conditions in Theorem 7, let $g(x) = c(x + M/2)^{\alpha}$ with $\alpha \notin \mathbb{Z}$. Then

$$\frac{g^{(k)}(x)}{k!} = (\alpha)_k c(x + M/2)^{\alpha-k}$$

where $(\alpha)_k = \alpha(\alpha - 1) \ldots (\alpha - k + 1)/k!$. Thus

$$D_{3,1}(g(x)) = \frac{g^{(3)}(x)}{3!} = c(x + M/2)^{\alpha-3}(\alpha)_3,$$

$$D_{3,2}(g(x)) = \begin{vmatrix} \frac{g^{(3)}(x)}{3!} & \frac{g^{(2)}(x)}{2!} \\ \frac{g^{(4)}(x)}{4!} & \frac{g^{(3)}(x)}{3!} \end{vmatrix} = c^2(x + M/2)^{2(\alpha-3)} \begin{vmatrix} (\alpha)_3 & (\alpha)_2 \\ (\alpha)_4 & (\alpha)_3 \end{vmatrix},$$

$$D_{3,3}(g(x)) = \begin{vmatrix} \frac{g^{(3)}(x)}{3!} & \frac{g^{(2)}(x)}{2!} & \frac{g^{(1)}(x)}{1!} \\ \frac{g^{(4)}(x)}{4!} & \frac{g^{(3)}(x)}{3!} & \frac{g^{(2)}(x)}{2!} \\ \frac{g^{(5)}(x)}{5!} & \frac{g^{(4)}(x)}{4!} & \frac{g^{(3)}(x)}{3!} \end{vmatrix} = c^3(x + M/2)^{3(\alpha-3)} \begin{vmatrix} (\alpha)_3 & (\alpha)_2 & (\alpha)_1 \\ (\alpha)_4 & (\alpha)_3 & (\alpha)_2 \\ (\alpha)_5 & (\alpha)_4 & (\alpha)_3 \end{vmatrix}.$$

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In particular, if $g(x) = f_b(x + M/2)$, then

$$D_{3,1}(f_b(x + M/2)) = \frac{-1/16}{(b(x + M/2))^{3/2}},$$
$$D_{3,2}(f_b(x + M/2)) = \frac{-5/2^{10}}{(b(x + M/2))^{6/2}}$$

and

$$D_{3,3}(f_b(x + M/2)) = \frac{-35/2^{15}}{(b(x + M/2))^{9/2}}.$$

The determinant conditions can be easily seen to be true.

In our situation, $T = \lambda(2R_2)^2$. To ensure that $\Delta < 1/2$, note that

$$\Delta = \frac{4R_2}{R_1 b^3} = \frac{16R_2R_1}{(2R_1)^2b^3} \le \frac{16R_2R_1}{a^2b^3} \le \frac{32R_1R_2}{x}$$

as $a^2b^3 \ge x/2$. Hence to ensure that $\Delta < 1/2$, we need the condition

$$R_1R_2 < x/64.$$

To ensure that $T = 4(M^{3/2}/b^{3/2})R_2^2 \ge 4$, we require $M \ge b/R_2^{4/3}$. What happens when $M < b/R_2^{4/3}$? From the definition of $\mathcal{R}_{b,M}$,

$$\frac{1}{2R_2} \le \frac{a}{c} \le \frac{d^{3/2}}{b^{3/2}} + \frac{\Delta}{c^2} < \frac{1}{R_2^2} + \frac{4}{R_1R_2b^3} < \frac{1}{R_2^2} + \frac{4}{R_2^5}, \tag{5}$$

which is impossible when $R_2 \ge 3$. When $R_2 < 3$, at most a finite number of a/c satisfy (5) and $R_2 \le c \le 2R_2$. Hence, when $M < b/R_2^{4/3}$,

$$|\mathcal{R}_{b,M}| \ll M. \tag{6}$$

Now by (3), when $b/R_2^{4/3} \le M \le (x/R_2^2)^{1/3}$,

$$|\mathcal{R}_{b,M}| \ll \left(M^2 \frac{M^{3/2}}{b^{3/2}} R_2^2\right)^{1/5} + \left(\Delta^3 \left(\frac{M^{3/2}}{b^{3/2}} R_2^2\right)^2\right)^{1/18} M = \frac{M^{7/10}R_2^{2/5}}{b^{3/10}} + \frac{M^{7/6}R_2^{7/18}}{R_1^{1/6}b^{2/3}}. \tag{7}$$

Summing over all dyadic intervals over M for (6) and (7),

$$|\mathcal{R}_b| \ll \frac{b}{R_2^{4/3}} + \frac{x^{7/30}}{R_2^{1/15}b^{3/10}} + \frac{x^{7/18}}{R_1^{1/6}R_2^{7/18}b^{2/3}},$$

where

$$\mathcal{R}_b = \left\{\left(d, \frac{a}{c}\right) : 1 \le d \le \left(\frac{x}{R_2}\right)^{1/3}, R_2 \le c \le 2R_2, (a, c) = 1, \left|f_b(d) - \frac{a}{c}\right| \le \frac{\Delta}{c^2}\right\}.$$

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Now summing over $b \le ((x+1)/R_1^2)^{1/3}$, we see that the set of quadruples in S with the extra conditions $R_1 \le a \le 2R_1$ and $R_2 \le c \le 2R_2$, which we denote by S_{R_1,R_2}, satisfies

$$|S_{R_1,R_2}| \ll \frac{x^{2/3}}{R_1^{4/3}R_2^{4/3}} + \frac{x^{7/15}}{R_2^{1/15}R_1^{7/15}} + \frac{x^{1/2}}{R_1^{7/18}R_2^{7/18}}.$$

By symmetry, we also have

$$|S_{R_1,R_2}| \ll \frac{x^{2/3}}{R_1^{4/3}R_2^{4/3}} + \frac{x^{7/15}}{R_1^{1/15}R_2^{7/15}} + \frac{x^{1/2}}{R_1^{7/18}R_2^{7/18}}.$$

Therefore, since $\min(a,b) \le \sqrt{ab}$,

$$|S_{R_1,R_2}| \ll \frac{x^{2/3}}{R_1^{4/3}R_2^{4/3}} + \frac{x^{7/15}}{R_1^{4/15}R_2^{4/15}} + \frac{x^{1/2}}{R_1^{7/18}R_2^{7/18}}. \tag{8}$$

We now finish the proof of Theorem 4. By the result of Bennett [1], the equation $a^2b^3 - c^2d^3 = 1$ has at most one solution for each pair of a and c. Hence

$$|S_{R_1,R_2}| \ll R_1R_2. \tag{9}$$

When $R_1R_2 \ge x/64$,

$$\left(\frac{x}{64}\right)^2 (bd)^3 \le (R_1R_2)^2(bd)^3 \le a^2b^3c^2d^3 \le x(x+1) \le 2x^2$$

which implies that $bd \le 2^{13/3}$. So there are at most finitely many Pell equations $a^2b^3 - c^2d^3 = 1$, each having $O(\log x)$ solutions in a and c. Together with (8) and (9), this gives, by summing over $R_1 = 2^i$ and $R_2 = 2^j$,

$$\begin{aligned}
|S| &\ll \sum_{\substack{i,j \\ 2^{i+j} \le x^{7/19}}} |S_{2^i,2^j}| + \sum_{\substack{i,j \\ x^{7/19} < 2^{i+j} < x/64}} |S_{2^i,2^j}| + \sum_{\substack{i,j \\ 2^{i+j} \ge x/64}} |S_{2^i,2^j}| \\
&\ll \sum_{\substack{i,j \\ 2^{i+j} \le x^{7/19}}} 2^{i+j} + \sum_{\substack{i,j \\ x^{7/19} < 2^{i+j} < x/64}} \left(\frac{x^{2/3}}{2^{4(i+j)/3}} + \frac{x^{7/15}}{2^{4(i+j)/15}} + \frac{x^{1/2}}{2^{7(i+j)/18}} \right) + \log x \\
&\ll x^{7/19} \log x.
\end{aligned}$$

Finally summing over dyadic intervals $x/2^{i+1} < c^2d^3 \le x/2^i$, where $i = 0, 1, 2, \ldots$, gives Theorem 4.

For general l, one notes that the solutions to $a^2b^3 - c^2d^3 = l$ may not satisfy $(a,c) = 1$. But they can be divided into classes of solutions to $a'^2b^3 - c'^2d^3 = l/f^2$ with $(a',c') = 1$ according to different divisors f^2 of l. For each such modified equation the above proof works, except that the implicit constants may depend on l. One should also replace the use of Bennett's result with Bombieri and Schmidt's result on the Thue equation. □

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