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METRIC SPACES WHICH CANNOT BE ISOMETRICALLY EMBEDDED IN HILBERT SPACE

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Let $A_1A_2A_3A_4$ be a planar convex quadrangle with diagonals A_1A_3 and A_2A_4. Is there a quadrangle $B_1B_2B_3B_4$ in Euclidean space such that $A_1A_3 < B_1B_3$, $A_2A_4 < B_2B_4$ but $A_iA_j > B_iB_j$ for other edges?

The answer is "no". It seems to be obvious but the proof is more difficult. In this paper we shall solve similar more complicated problems by using a higher dimensional geometric inequality which is a generalisation of the well-known Pedoe inequality (*Proc. Cambridge Philos. Soc.* 38 (1942), 397-398) and an interesting result by L.M. Blumenthal and B.E. Gillam (*Amer. Math. Monthly* 50 (1943), 181-185).

1. Definitions and main result

DEFINITION 1. Let $G = \{A_1, A_2, \ldots, A_{n+2}\}$ be an $(n+2)$-tuple in E^n. An edge A_iA_j of G is called "red" or "blue" if there exists uniquely a hyperplane $\pi_{ij}(G)$ containing $G\backslash\{A_i, A_j\}$ such that A_i and A_j lie to the opposite sides or the same side of $\pi_{ij}(G)$, respectively.

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Some edges, of course, may be neither red nor blue.

DEFINITION 2. Let G be an $(n+2)$-tuple in E^n , (M, d) a semi-metric space. A mapping $f : G \to (M, d)$, satisfying

(i) $|A_i - A_j| \le d(f(A_i), f(A_j))$ if $A_i A_j$ is a red edge of G ,

(ii) $|A_i - A_j| \ge d(f(A_i), f(A_j))$ if $A_i A_j$ is a blue edge of G ,

and the strict inequality holding at least for one edge red or blue, is called a "skew mapping" of G into (M, d) . $f(G)$ is called a "skew image" of G , and G is called a "skew inverse image" of $f(G)$.

The following theorem gives a geometric condition under which a metric space (M, d) cannot be isometrically embedded in Hilbert space.

THEOREM 1. *If a metric space (M, d) contains a finite subset R which has a skew inverse image in Euclidean space, then (M, d) cannot be isometrically embedded in Hilbert space l^2 .*

We shall prove this assertion in Section 3. Furthermore, its converse theorem is true for separable metric spaces. In fact, the authors have proved in [6] that a separable metric space which cannot be isometrically embedded in l^2 must contain a finite subset which has a skew inverse image in Euclidean space.

The proof [6] of the converse theorem, however, is very long and much more difficult than Theorem 1 itself so we need not repeat it here. The purpose of this note is only to prove Theorem 1 which is sufficient to answer the type of problems analogous to the one posed at the beginning of the present paper.

2. Notations and lemmas

Let $G = \{A_1, A_2, \ldots, A_{n+2}\}$ and $R = \{B_1, B_2, \ldots, B_{n+2}\}$ be two $(n+2)$-tuples in E^{n+1} , $a_{ij} = |A_i - A_j|$, $b_{ij} = |B_i - B_j|$ $(i, j = 1, 2, \ldots, n+2)$. By A, B denote the values of the determinants of the following two bordered matrices, respectively:

$$(1)\qquad A = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & & & & & \\ 1 & & & & & \\ \vdots & & & -\tfrac{1}{2}a_{ij}^2 & & \\ 1 & & & & & \end{vmatrix}, \quad B = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & & & & & \\ 1 & & & & & \\ \vdots & & & -\tfrac{1}{2}b_{ij}^2 & & \\ 1 & & & & & \end{vmatrix} .$$

By A_{ij} and B_{ij} denote the cofactors of $-\frac{1}{2}a_{ij}^2$ in A and $-\frac{1}{2}b_{ij}^2$ in B $(i, j = 1, 2, \ldots, n+2)$, respectively.

LEMMA 1.

$$(2)\qquad \sum_{i=1}^{n+2}\sum_{j=1}^{n+2} a_{ij}^2 B_{ij} \ge 0 , \quad \sum_{i=1}^{n+2}\sum_{j=1}^{n+2} b_{ij}^2 A_{ij} \ge 0 .$$

Proof. If G and R span two non-degenerate simplices in E^{n+1}, denoting by $V(G)$ and $V(R)$ the volumes of G and R, we have ([4], p. 204, Theorem 1, or [5])

$$(3)\qquad \sum_{i=1}^{n+2}\sum_{j=1}^{n+2} a_{ij}^2 B_{ij} \ge 2(n+1)\big((n+1)!\big)^2 V(G)^{2/(n+1)} V(R)^{2-2/(n+1)} .$$

This is a generalisation of the Neuberg-Pedoe inequality which is the case $n = 1$ in (3).

It is obvious by continuity that (3) holds still when G or R is degenerate; hence

$$\sum_{i=1}^{n+2}\sum_{j=1}^{n+2} a_{ij}^2 B_{ij} \ge 0 ,$$

analogously

$$\sum_{i=1}^{n+2}\sum_{j=1}^{n+2} b_{ij}^2 A_{ij} \ge 0 .$$

LEMMA 2. *If* $G = \{A_1, A_2, \ldots, A_{n+2}\}$ *is an* $(n+2)$*-tuple in* E^n *and some cofactor* A_{ij} *in* A *is non-vanishing, then* A_i *and* A_j *lie to the opposite sides or the same side of the hyperplane* $\pi_{ij}(G)$ *when* $A_{ij} < 0$ *or* $A_{ij} > 0$.

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This lemma is due to Blumenthal and Gillam ([2], p. 183, Theorem 3.1). There are merely a few differences of notation between the two statements.

LEMMA 3. *Let* $G = \{A_1, A_2, \ldots, A_{n+2}\}$ *be an* $(n+2)$*-tuple in* E^n. *If an edge* A_iA_j *is red or blue, then the corresponding cofactor* A_{ij} *is non-vanishing.*

Proof. We apply the following algebraic identity (4) which is very useful in distance geometry ([1], §41, p. 100). Let D be a symmetric determinant, D_{ii}, D_{jj} and D_{ij} be the corresponding cofactors in D, and D_{jj}^{ii} be the sub-determinant obtained by deleting the ith row, the ith column, the jth row and the jth column from D. Then, for $i \neq j$,

$$D_{ii}D_{jj} - D_{ij}^2 = D \cdot D_{jj}^{ii} . \tag{4}$$

Now we apply this well-known identity to determinant A. It has been shown ([4], p. 206, (1.10)) that

$$A = -\left((n+1)!V(G)\right)^2 \tag{5}$$

where $V(G)$ denotes the $(n+1)$=dimensional volume of the simplex spanned by G. Since G is an $(n+2)$-tuple in E^n this simplex must be degenerate; hence $V(G) = 0$ and so $A = 0$. It follows that

$$A_{ii}A_{jj} - A_{ij}^2 = 0 . \tag{6}$$

Suppose $A_{ij} = 0$ for a certain i and a certain j; then either $A_{ii} = 0$ or $A_{jj} = 0$. Hence either A_j or A_i lies in the hyperplane $\pi_{ij}(G)$. (Since, by analogue with (5) we have $A_{ii} = -\left(n!V\left(G\backslash\{A_i\}\right)\right)^2$, $A_{ii} = 0$ implies that the simplex spanned by $G\backslash\{A_i\}$ is degenerate and the points of $G\backslash\{A_i\}$ including A_j lie in the same hyperplane which is just $\pi_{ij}(G)$.)

But, in this case, according to Definition 1, the edge A_iA_j is neither red nor blue, contradicting the hypothesis, and Lemma 3 has been proved.

3. Proof of Theorem 1

We use reduction to absurdity. Suppose a metric space (M, d) has been isometrically embedded in l^2 and there exists a finite subset R of M with a skew inverse image G in Euclidean space. From this we conclude that there exists $G = \{A_1, A_2, \ldots, A_{n+2}\}$ in E^n and $R = \{B_1, B_2, \ldots, B_{n+2}\}$ in l^2 such that

(i) $|A_i - A_j| \le |B_i - B_j|$ if A_iA_j is red,

(ii) $|A_i - A_j| \ge |B_i - B_j|$ if A_iA_j is blue,

and the strict inequality holds at least for one edge A_iA_j red or blue.

Clearly, $G \subset E^n \subset E^{n+1}$ and $R \subset E^{n+1}$ because the widest position occupied by $n + 2$ points of l^2 is only $(n+1)$-dimensional. We use the same notation as in Lemma 1: $a_{ij} = |A_i - A_j|$, $b_{ij} = |B_i - B_j|$, and so on.

Since $G \subset E^n$ implies $A = 0$ (by formula (5)), by simple calculation we have

$$\sum_{i=1}^{n+2} \sum_{j=1}^{n+2} a_{ij}^2 A_{ij} = 0 , \tag{7}$$

and applying Lemma 1 we obtain

$$\sum_{i=1}^{n+2} \sum_{j=1}^{n+2} b_{ij}^2 A_{ij} \ge 0 = \sum_{i=1}^{n+2} \sum_{j=1}^{n+2} a_{ij}^2 A_{ij} ;$$

that is

$$\sum_{i=1}^{n+2} \sum_{j=1}^{n+2} \left(b_{ij}^2 - a_{ij}^2\right) A_{ij} \ge 0 . \tag{8}$$

First it is easy to verify that every term of the left side of (8) is non-positive:

when $A_{ij} = 0$, $\left(b_{ij}^2 - a_{ij}^2\right) A_{ij} = 0$ and when $A_{ij} > 0$, by Lemma 2 we know that A_iA_j is blue and by hypothesis $a_{ij} \ge b_{ij}$, so we

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have $\left(b_{ij}^2-a_{ij}^2\right)A_{ij} \le 0$;

when $A_{ij} < 0$, A_iA_j is red and by hypothesis $a_{ij} \le b_{ij}$ and we have $\left(b_{ij}^2-a_{ij}^2\right)A_{ij} \le 0$.

Then, according to the hypothesis of Theorem 1 and Definition 2, there exists at least one red or blue edge A_iA_j such that $a_{ij} \ne b_{ij}$. By Lemma 3 there exists at least one non-vanishing term of the left side of (8). We obtain

$$\sum_{i=1}^{n+2}\sum_{j=1}^{n+2}\left(b_{ij}^2-a_{ij}^2\right)A_{ij} < 0 , \tag{9}$$

which contradicts (8). This contradiction shows that (M, d) cannot be isometrically embedded in l^2 and the proof of Theorem 1 is complete.

4. A type of problem involving two metric point sets

Now let us answer the quadrangles problem which was posed at the beginning of the paper. Clearly, the mapping $A_1A_2A_3A_4 \to B_1B_2B_3B_4$ is a skew mapping. According to Theorem 1, it is not possible to realize such a quadrangle in Euclidean space.

Of course, Theorem 1 may be applied to solve more complicated problem problems. For example: let Ω be a convex 6-faced polyhedron with vertices A_1, A_2, A_3, A_4, A_5 in E^3 , such that Ω can be dissected into two tetrahedrons $A_1A_2A_3A_4$ and $A_1A_2A_3A_5$. Is there a 5-tuple $\Omega^* = \{B_1, B_2, B_3, B_4, B_5\}$ in E^4 such that $A_1A_2 < B_1B_2$, $A_2A_3 < B_2B_3$, $A_3A_1 < B_3B_1$, $A_4A_5 < B_4B_5$ but $A_iA_j > B_iB_j$ for other edges?

It can be seen easily that A_1A_2, A_2A_3, A_3A_1, A_4A_5 are red edges of Ω and other edges of Ω are blue. The mapping $A_1A_2A_3A_4A_5 \to B_1B_2B_3B_4B_5$, therefore, is a skew mapping. By Theorem 1 we can assert that it is impossible to realize such a 5-tuple Ω^* in E^4 .

There are a variety of conditions, each of which is necessary and sufficient to embed isometrically a metric space in Euclidean or Hilbert space; nevertheless, it is usually difficult to decide practically whether some given metric point set is embeddable or not. Inequalities involving two metric point sets are often of great use for our work.

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