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A RIGHT CONTINUOUS RIGHT WEAKLY SI-RING IS SEMISIMPLE

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It is shown that a projective CS right module M over a ring R is a direct sum of uniform modules of composition lengths at most 2 if (i) every finitely generated direct summand of M is continuous and (ii) every non-zero M-singular right R-module contains a non-zero M-injective submodule. In particular, a right continuous ring R is semisimple if R is right weakly SI, that is, if every non-zero singular right R-module contains a non-zero injective submodule.

1. INTRODUCTION

Right (left) SI-rings, that is, rings all of whose singular right (left) modules are injective, were introduced and investigated in detail by Goodearl [**9**]. Since then SI-rings have drawn much attention from several authors, see for example, [**7, 10, 13, 14, 15, 16, 17, 20**]. In a similar way, SI-modules have been defined and considered in [**20**] and [**10**] where corresponding properties were obtained.

A weaker form of SI-rings and SI-modules was considered recently in [**15**]: A right R-module M is called *weakly* SI (briefly, WSI) if every non-zero M-singular right R-module contains a non-zero M-injective submodule. A ring R is called a right WSI-ring if R_R is WSI.

As shown in [**15**], WSI-modules have some properties similar to those of SI-modules. However, in general the structure of them still remains unknown. It is clear that any right semiartinian right V-ring is right WSI. By [**2**], [**3**, Theorem 2.2] or [**6**, Corollary 21] there exists a right semiartinian right V-ring R such that the right Loewy series of R has (Loewy) length at least 3. By [**9**, Theorem 3.11] such a ring R is not right SI. Hence right WSI-rings are not necessarily right SI, in general. On the other hand, a right PCI-domain constructed in [**5**] is right SI, and so it is right WSI but not right semiartinian. This means that right WSI-rings need not be right semiartinian. Therefore WSI-modules and WSI-rings seem to be interesting subjects in the area.

The purpose of this note is to prove the following results. For the definition of the category $\sigma[M]$ we refer to the next section.

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Theorem 1. *Let M be a WSI right R-module which is projective in $\sigma[M]$. Then M is M-nonsingular. Assume furthermore that M is CS, then*

(a) M is a direct sum of finitely generated modules M_i, where each M_i has either zero socle or M_i is a semiartinian module and $\overline{M}_i = M_i/Soc(M_i)$ is a V-module, that is, every simple module in $\sigma[\overline{M}_i]$ is $\overline{M}_i$-injective.

(b) If every finitely generated direct summand of M is quasi-continuous, then M is quasi-continuous and $M = \bigoplus_{i\in\Omega} U_i$ where each U_i is a finitely generated uniform submodule of M. Moreover, if $Soc(U_i) \neq 0$ for some $i \in \Omega$, then U_i has composition length $\leqslant 2$, and each U_j with $Soc(U_j) = 0$ is a fully invariant submodule of M. Therefore, in this case, M is a direct sum of a fully invariant SI-submodule with essential socle and fully invariant uniform submodules with zero socles.

(c) If every finitely generated direct summand of M is continuous, then M is a continuous SI-module which is a direct sum of uniform submodules with composition lengths $\leqslant 2$. Moreover, in this case, if N_R is a finitely generated direct summand of M, then $End_R(N)$ is a semisimple ring.

The following consequence of Theorem 1 improves [**6**, Lemma 10] which stated that a right self-injective ring R is semisimple if every non-zero right R-module contains a non-zero injective submodule.

Corollary 2. *Any right continuous right WSI-ring is semisimple.*

For quasi-continuous rings we have:

Corollary 3. *Every right quasi-continuous right WSI-ring is the ring direct sum of a semisimple ring and finitely many right Ore domains which are not division rings. In particular, any right quasi-continuous, right semiartinian right V-ring is semisimple.*

The last statement of Corollary 3 gives the possibility of producing several von Neumann regular right V-rings with zero right socle.

The following result is an easy consequence of (a) in Theorem 1.

Corollary 4. *A right CS right WSI-ring R has a ring direct decomposition $R = A \oplus B$ where A is a right semiartinian ring such that $A/Soc(A_A)$ is a right V-ring and $Soc(B_B) = 0$.*

2. Preliminaries

Throughout this note all rings are associative with identity and all modules are unitary modules.

For a module M over a ring R we write M_R to indicate that M is a right R-module. The socle and the Jacobson radical of M are denoted respectively by $Soc(M)$

and $J(M)$. A module M is called semisimple if $M = Soc(M)$, and a ring R is said to be a semisimple ring if R_R is semisimple, or equivalently, if R is a semiprime right (or left) Artinian ring. A submodule N of a module M_R is called a fully invariant submodule of M if for each $f \in End_R(M)$, $f(N) \subseteq N$.

For a given module M_R we consider the following properties:

(C_1) Every submodule of M is contained essentially in a direct summand of M.

(C_2) If A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is also a direct summand of M.

(C_3) If C is a submodule of M isomorphic to a direct summand of M, then C is itself a direct summand of M.

A module M_R is said to be a CS-module if it satisfies (C_1); M is called quasi-continuous if M satisfies (C_1) and (C_2) and finally, if M satisfies (C_1) and (C_3) then M is said to be a continuous module. We have the following implications:

injective $\Rightarrow$ quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ CS.

In general these classes are distinct.

A ring R is called right CS (right quasi-continuous, right continuous) if R_R is CS (quasi-continuous, continuous). For a detailed study of these classes of rings and modules we refer to Dung-Huynh-Smith-Wisbauer [7] and Mohamed-Müller [11].

For a module M_R (over a ring R) we denote by $\sigma[M]$ the full subcategory of Mod-R (the category of all right R-modules) whose objects are submodules of M-generated modules (see Wisbauer [19]). A module $P \in \sigma[M]$ is called projective in $\sigma[M]$ if P is N-projective for every $N \in \sigma[M]$. A module U_R is called M-singular if there is a module $A \in \sigma[M]$ containing an essential submodule E such that $U \simeq A/E$. Hence any M-singular module is contained in $\sigma[M]$. For $M = R$ the notion of R-singularity is identical to the usual definition of singular R-modules in Mod-R (see Goodearl [9]).

The class of M-singular right R-modules is closed under taking submodules, homomorphic images and direct sums (for example, [19, 17.3, 17.4]). Hence any $N \in \sigma[M]$ contains a largest M-singular submodule which is denoted by $Z_M(N)$. If $Z_M(N) = 0$, then N is called M-nonsingular.

A module M is called an SI-module if every M-singular module is M-injective, and M is called *weakly* SI (briefly WSI) if every non-zero M-singular module contains a non-zero M-injective submodule. Clearly, any SI-module is WSI. However, in general the converse is not true, as mentioned in the Introduction. A ring R is right WSI if R_R is a WSI-module.

The texts by Anderson - Fuller [1], Chatters - Hajarnavis [4], Faith [8], Goodearl [9], Mohamed - Müller [11], Stenström [18] and Wisbauer [19] are general references for module and ring theoretic notions not defined in this note.

3. The Proofs

The following special case of a recent result of Osofsky [**12**, Theorem B] is the key lemma of our proof of Theorem 1.

Lemma 4. *Let M_R be a finitely generated, quasi-continuous and quasi-projective module such that no non-zero element of $End_R(M)$ has essential kernel. Then for each set $\{e_i\}_{i=1}^{\infty}$ of orthogonal idempotents e_i in $End_R(M)$ with $\bigoplus_{i=1}^{\infty} e_iM$ essential in M, the factor module $M\Big/\left(\bigoplus_{i=1}^{\infty} e_iM\right)$ cannot contain a non-zero quasi-continuous direct summand.*

Proof of Theorem 1: Let M_R be a WSI-module such that M is projective in $\sigma[M]$, where R is a ring.

If $Z_M(M) \neq 0$, then $Z_M(M)$ contains a non-zero M-injective submodule N. Hence N is a direct summand of M, and therefore N is projective in $\sigma[M]$, a contradiction. Thus $Z_M(M) = 0$, that is, M is M-nonsingular.

From now on we assume in addition that M is a CS-module.

Claim 1. Any finitely generated submodule U of M is essential in a finitely generated direct summand of M.

In fact this claim can be derived from [**7**, Proposition 2.7], however we give a proof here (with a similar argument) for the sake of completeness. Since M is CS, there is a direct summand U^* of M such that U is essential in U^*. (We assume $U \neq 0$, since for $U = 0$ the statement is clear). To verify Claim 1 we shall show that U_R^* is finitely generated.

Clearly, U^* is projective in $\sigma[M]$ and M-nonsingular. Let $\{x_\lambda,\ \lambda \in \Lambda\}$ be a generating set of U^*, that is,

$$U^* = \sum_{\lambda\in\Lambda} x_\lambda R.$$

Then there exists an epimorphism g from $\bigoplus_{\lambda\in\Lambda} x_\lambda R$ onto U_R^*. Since U^* is projective in $\sigma[M]$ and $\bigoplus_{\lambda\in\Lambda} x_\lambda R \in \sigma[M]$, the map g splits (see [**19**, 18.3]), that is, there exists a submodule H of $\bigoplus_{\lambda\in\Lambda} x_\lambda R$ with $H \simeq U^*$ and

$$\bigoplus_{\lambda\in\Lambda} x_\lambda R = H \oplus Ker(g). \tag{1}$$

It is clear that H also contains a finitely generated essential submodule K (since $H \simeq U^*$). Let $K = y_1R + \ldots + y_nR$ ($y_i \in H$) and for each $\mu \in \Lambda$ let e_μ be the canonical projection of $\bigoplus_{\lambda\in\Lambda} x_\lambda R$ onto $x_\mu R$. Since for each $x \in H$, $e_\mu(x) \neq 0$ only for finitely many $\mu \in \Lambda$, the set

$$\Gamma = \{\gamma \in \Lambda,\ \ e_\gamma(y_i) \neq 0 \text{ for some } y_i,\ 1 \leqslant i \leqslant n\ \}$$

is finite. Hence for every $\lambda \in \Lambda \setminus \Gamma$, $e_\lambda(K) = 0$. This shows that K is contained in $Ker(e_\lambda)$ for each $\lambda \in \Lambda \setminus \Gamma$. Therefore for such λ, $e_\lambda(H)$ is an M-singular submodule of $x_\lambda R$. But $x_\lambda R \subseteq M$ and M is M-nonsingular. It follows that $e_\lambda(H) = 0$ for all $\lambda \in \Lambda \setminus \Gamma$. Hence

$$H \subseteq \bigoplus_{\gamma \in \Gamma} x_\gamma R.$$

This together with (1) shows that H is a direct summand of $\bigoplus_{\gamma\in\Gamma} x_\gamma R$, and so H_R is finitely generated. Thus U_R^* is finitely generated, as desired.

(a) Since M is projective in $\sigma[M]$, by Kaplansky's Theorem (see [**19**, 8.10, 18.4]) M is a direct sum of countably generated modules. Hence to prove (a), we may assume that M is countably generated, say

$$M = \sum_{i=1}^{\infty} x_i R.$$

Note that M is M-nonsingular as shown before Claim 1. Now let M_1 be a maximal essential extension of $x_1 R$ in M. By hypothesis we have

$$M = M_1 \oplus M_1'.$$

By Claim 1, M_1 is finitely generated. Assume inductively that for some positive integer $n \geqslant 1$, we already found finitely many independent submodules $M_1, \dots, M_n$ each of which is finitely generated and

$$M = (M_1 \oplus \cdots \oplus M_n) \oplus M_n'$$

such that $x_1 R + \cdots + x_n R \subseteq M_1 \oplus \cdots \oplus M_n$. Let π be the projection of M onto M_n' and let $x_{n+1}' = \pi(x_{n+1})$. Since M_n' is also a CS and WSI-module which is projective in $\sigma[M]$, we may use the first step above to find a finitely generated direct summand M_{n+1} of M_n' such that $x_{n+1}' R$ is essential in M_{n+1}. Thus

$$M = M_1 \oplus \cdots \oplus M_{n+1} \oplus M_{n+1}'$$

with $x_1 R + \cdots + x_{n+1} R \subseteq M_1 \oplus \cdots \oplus M_{n+1}$. This induction argument shows that M contains an independent set $\{M_i\}_{i=1}^{\infty}$ of finitely generated submodules such that $\sum_{i=1}^{\infty} x_i R \subseteq \bigoplus_{i=1}^{\infty} M_i$. Therefore $M = \bigoplus_{i=1}^{\infty} M_i$ as desired.

Put $M = \bigoplus_{i \in I} M_i$ where each M_i is finitely generated and let S_i be the socle of M_i. Assume that for some $i \in I$, $S_i \neq 0$. Since M is CS we have $M_i = A_i \oplus B_i$ where $Soc(B_i) = 0$ and S_i is essential in A_i. Hence $\overline{A}_i = A_i/S_i$ is an M-singular module. Therefore each non-zero subfactor of $\overline{A}_i$ contains a non-zero M-injective submodule. Moreover, since S_i is a fully invariant submodule of A_i, $\overline{A}_i$ is quasi-projective. Hence we may use the main result of [**6**] to see that $\overline{A}_i$ is a semiartinian V-module, and it follows that A_i is semiartinian. This fact verifies (a).

(b) Assume that every finitely generated direct summand of M is quasi-continuous.

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CLAIM 2. Any finitely generated submodule of M has finite uniform dimension.

Let V be a finitely generated submodule of M. Then V is an essential submodule of a finitely generated direct summand V^* of M by Claim 1. Assume on the contrary that V does not have finite uniform dimension. Then V contains an infinite independent set $\{V_i\}_{i=1}^{\infty}$ of non-zero submodules V_i. Put $W = \bigoplus_{i=1}^{\infty} V_i$. Since V^* is also a CS-module, W is contained as an essential submodule in a direct summand W^* of V^*. Clearly W^* is finitely generated and a direct summand of M. Hence W^* is quasi-continuous. Moreover W^* is projective in $\sigma[M]$.

Let V_i^* be a maximal essential extension of V_i in W^* for each $i = 1, 2, \ldots$. Then, since W^* is quasi-continuous, any finite direct sum $\bigoplus_{i=1}^{m} V_i^*$ is a direct summand of W^*. Let e_i be the canonical projection of W^* onto V_i^*. Then $\{e_i\}_{i=1}^{\infty}$ is a family of orthogonal idempotents in $S = End_R(W^*)$ with $e_iW^* = V_i^*$ and so $\bigoplus_{i=1}^{\infty} e_iW^*$ is essential in W^*. Note that $\bigoplus_{i=1}^{\infty} e_iW^* \neq W^*$, since W^* is finitely generated, and that for each $0 \neq f \in S$, $\mathrm{Ker}(f)$ is not essential in W^* since W^* is M-nonsingular. Now we may apply Lemma 4 to see that the non-zero M-singular module $W^* \big/ \left(\bigoplus_{i=1}^{\infty} e_iW^*\right)$ does not contain a non-zero M-injective submodule. However this is a contradiction to the assumption that M is WSI. Thus V must have finite uniform dimension, proving Claim 2.

Now by (a) M is a direct sum of finitely generated modules. Then from the assumption of (b) and Claim 2 it is easy to derive a decomposition of M as a direct sum of finitely generated uniform modules U_i:

$$M = \bigoplus_{i \in \Omega} U_i. \tag{2}$$

Next we show that M is quasi-continuous. By [**11**, Theorem 2.13], it is enough to show that each $M(\Omega \setminus i)$ in (2) is U_i-injective, where $M(\Omega \setminus i) = \bigoplus_{j \in \Omega \setminus i} U_j$. Let V be a submodule of U_i and g be a non-zero homomorphism of V to $M(\Omega \setminus i)$. Since $M(\Omega \setminus i)$ is M-nonsingular, $Ker(g)$ must be zero, that is, $V \simeq g(V)$, in particular $g(V)$ is a uniform submodule of $M(\Omega \setminus i)$. Since M is M-nonsingular, the closure of any uniform submodule H in M equals the closure of any non-zero cyclic submodule of H. Hence we may use Claim 1 to see that $g(V)$ is an essential submodule of a finitely generated direct summand W of $M(\Omega \setminus i)$. We have $M(\Omega \setminus i) = W \oplus M'$ for some submodule M' of $M(\Omega \setminus i)$. Hence $U_i \oplus W$ is a finitely generated direct summand of M. By assumption, $U_i \oplus W$ is quasi-continuous and so W is U_i-injective by [**11**, Corollary 2.14]. Since $g(V) \subseteq W$, it follows that g can be extended to a homomorphism from U_i to W. This implies the U_i-injectivity of $M(\Omega \setminus i)$.

Assume now that in (2) there is a U_j $(j \in \Omega)$ such that $Soc(U_j) \neq 0$. If U_j is simple then we are done. Assume that U_j is not simple. Let $S_j = End_R(U_j)$. If there is a non-zero element $f \in S_j$ such that $f(U_j)$ is small in U_j then $f(U_j)$ is contained in each maximal submodule of U_j, that is, $f(U_j) \subseteq J(U_j)$. Since $Soc(U_j)$ is contained in each non-zero submodule of U_j, we also have $Soc(U_j) \subseteq J(U_j)$. On the other hand, since U_j is a finitely generated quasi-projective WSI-module, $J(U_j) \subseteq Soc(U_j)$ by [**15**, Proposition 4]. Hence $J(U_j) = Soc(U_j)$. It follows that $f(U_j) = Soc(U_j)$. From this we must have $\mathrm{Ker}(f) \neq 0$, since U_j is not simple. This implies that $f(U_j)$ is M-singular, a contradiction. Thus for each $0 \neq f \in S_j$, $f(U_j)$ is not small in U_j. By [**19**, 22.2] we have $J(S_j) = 0$, and

$$S_j \simeq End_R(U_j/J(U_j)). \tag{3}$$

Furthermore, since each non-zero element of S_j must have zero kernel, S_j is a domain, in particular S_j has only one non-zero idempotent. Hence by (3) we see that $U_j/J(U_j)$ is indecomposable. On the other hand, any non-zero submodule of the M-singular module $U_j/J(U_j)$ contains a non-zero U_j-injective submodule. Thus $U_j/J(U_j)$ has to be simple. Since $J(U_j)$ $(= Soc(U_j))$ is a minimal submodule of U_j, U_j has composition length 2, proving the first statement in the second part of (b).

To prove the next assertion of (b) we write (2) in the form

$$M = \Big(\bigoplus_{\alpha\in\Omega_1} U_\alpha\Big) \oplus \Big(\bigoplus_{\beta\in\Omega_2} U_\beta\Big)$$

where any U_α and U_β are uniform (finitely generated) and $Soc(U_\alpha) \neq 0$, $Soc(U_\beta) = 0$ $(\alpha \in \Omega_1, \beta \in \Omega_2)$. Clearly, $\bigoplus_{\alpha\in\Omega_1} U_\alpha$ and $\bigoplus_{\beta\in\Omega_2} U_\beta$ are fully invariant submodules of M. Therefore to end this part we need only to show that each U_β is a fully invariant submodule of $U = \bigoplus_{\beta\in\Omega_2} U_\beta$. Since U is M-nonsingular, it is easy to see that for any $0 \neq f \in End_R(U)$, $f(U_\beta) \subseteq U_\beta$ or $f(U_\beta) \cap U_\beta = 0$. Now assume that $f(U_\beta) \cap U_\beta = 0$ for some $\beta \in \Omega_2$. Since $(f \mid U_\beta)$ is a monomorphism, $f(U_\beta) \simeq U_\beta$, and since M is M-nonsingular we may use Claim 1 (as explained above) to see that the uniform submodule $f(U_\beta)$ is an essential submodule of a finitely generated direct summand V of $\bigoplus_{\gamma\in\Omega_2\setminus\beta} U_\gamma$. Hence $U_\beta \oplus V$ is a direct summand of M, and so $U_\beta \oplus V$ is quasi-continuous. Therefore U_β is V-injective; consequently, U_β is $f(U_\beta)$-injective. It follows that U_β is quasi-injective.

On the other hand, since U_β is a finitely generated quasi-projective WSI-module with zero socle, each simple module E in $\sigma[U_\beta]$ is U_β-injective [**15**, Proposition 4]. Hence each such E is U_β -generated and so U_β is a generator of $\sigma[U_\beta]$ by [**19**, 18.5]. Therefore $\sigma[U_\beta]$ is Morita-equivalent to Mod-T where $T = End_R(U_\beta)$ (see [**19**, 46.2]).

Note that T is a domain. Since U_β is quasi-injective, T is a right self-injective domain. It follows that T is a division ring. Then by the above Morita-equivalence, U_β must be simple, a contradiction. Thus we only have $f(U_\beta) \subseteq U_\beta$, proving the fact that each U_β is a fully invariant submodule of M.

By a standard argument we see that $\bigoplus_{\alpha \in \Omega_1} U_\alpha$ is an SI-module and therefore the proof of (b) is complete.

(c) We assume now that each finitely generated direct summand of M is continuous. It follows that M has a decomposition of the form (2) such that each U_i is continuous. By (b), M is quasi-continuous. Hence by [**11**, Theorem 3.16], M is continuous.

To finish the first statement of (c) it is enough to show that $Soc(U_i) \neq 0$ for each $i \in \Omega$. Suppose there is a U_i $(i \in \Omega)$ with $Soc(U_i) = 0$, and let $S_i = End_R(U_i)$. By the argument in the proof of (b), $\sigma[U_i]$ is Morita - equivalent to Mod-S_i. On the other hand, since U_i is uniform, continuous and M-nonsingular, for each $0 \neq f \in S_i$, $\mathrm{Ker}(f) = 0$ and $f(U_i) = U_i$, that is, f is an isomorphism, proving that S_i is a division ring. By the previous Morita-equivalence, U_i must be a simple module, a contradiction to $Soc(U_i) = 0$. Thus each U_i $(i \in \Omega)$ has non-zero socle as desired. By (b), M is an SI-module.

Now assume that N is a finitely generated direct summand of M. By Claim 2, N has finite uniform dimension. Moreover, since N is then continuous and M-nonsingular, it is easy to see that for each $f \in End_R(N)$, $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are direct summands of N. Therefore $End_R(N)$ is a (von Neumann) regular ring. Since N has finite uniform dimension, $End_R(N)$ cannot have an infinite set of orthogonal idempotents. Thus $End_R(N)$ is a semisimple ring. ∎

The statement of Corollary 2 follows directly from (c) of Theorem 1.

Proof of Corollary 3: Let R be a right quasi-continuous right WSI-ring. By Theorem 1 we have a ring direct decomposition:

$$R = A \oplus B \tag{4}$$

where A is a direct sum of finitely many uniform right ideals with composition lengths at most 2 and B is a direct sum of finitely many fully invariant uniform right ideals B_i of R with zero socles. It follows that B is a direct sum of right Ore domains. Now express A in the form: $A = A_1 \oplus \ldots \oplus A_n$ where each A_i is uniform and of composition length $\leqslant 2$. Assume that, for example, A_1 is not simple and let S_1 be the minimal submodule of A_1. Since S_1 is projective (see [**15**, Corollary 5]) and cyclic, there is a minimal right ideal T of A with $T \simeq S_1$ and $A = T \oplus P$ for some right ideal P of A. Since A is right quasi-continuous, T is P-injective (see [**11**, Corollary 2.14]). Moreover, since $T \cap A_1 = 0$, A_1 is embedded in P, and so T is A_1-injective. But $T \simeq S_1$, and so S_1 is A_1-injective, hence S_1 is a direct summand of A_1, a contradiction. Thus A is a semisimple ring. $\square$

Concerning (a) of Theorem 1 we would like to ask the question:

(Q_1) Is a finitely generated CS, quasi-projective WSI-module a direct sum of uniform modules?

If the answer for (Q_1) is yes, it follows that a right CS right WSI-ring is the ring direct sum of a right Artinian ring and a semiprime right Goldie ring. Indeed, let R be a right CS right WSI-ring such that R_R has finite uniform dimension. Then by Corollary 4, R has a ring direct sum $R = A \oplus B$ where A is semiartinian and $Soc(B_B) = 0$. Since R_R has finite uniform dimension, $Soc(A_A)$ is finitely generated, and so by the argument for proving (b) of Theorem 1 we easily see that A is a direct sum of finitely many uniform right ideals of composition lengths at most 2; in particular A is right Artinian. Since B is also right WSI, B is right nonsingular (see [**15**]). From this and since B_B has finite uniform dimension, the maximal right quotient ring Q_{max} of B is semisimple by [**18**, Theorem XII.2.5]. Then by [**18**, Proposition XV.3.3] and since $J(B) \subseteq Soc(B_B) = 0$ (see [**15**]), Q_{max} is also the classical right quotient ring of B. Hence B is a semiprime right Goldie ring.

We should note also that if the answer of (Q_1) were yes, we would have the interesting consequence that any right CS, right semiartinian right V-ring is semisimple. From this surprising conclusion we have the feeling that the answer to (Q_1) might be no.

Furthermore, to our knowledge, it is unknown whether or not a ring as in Corollary 3 is right Noetherian. More generally we would like to ask the question:

(Q_2) Is a semiprime right Goldie right WSI-ring necessarily right Noetherian?

It is clear that any right WSI-ring with right Krull dimension is right Noetherian right SI. If we add to (a) of Theorem 1 the condition that $M/J(M)$ is an SI-module, then by the arguments presented above we obtain that M is SI and it is a direct sum of Noetherian uniform modules each of which is of composition length $\leqslant 2$ or of zero socle. For $M = R$ we get the fact that a right CS right WSI-ring is right Noetherian (and right SI) if and only if $R/J(R)$ is right SI.

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