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PAPER

Finitary monads on the category of posets

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Abstract

Finitary monads on Pos are characterized as precisely the free-algebra monads of varieties of algebras. These are classes of ordered algebras specified by inequations in context. Analogously, finitary enriched monads on Pos are characterized: here we work with varieties of coherent algebras which means that their operations are monotone.

Keywords: Posets; monad; algebraic theory

Dedicated to John Power on the occasion of his 60th birthday.

1. Introduction

Algebraic specifications of data types are often given in terms of operations and equations. The models of such equational specifications are (often many-sorted) finitary algebras satisfying those equations. The models of an equational specification form a variety of algebras over the category Set^S of S-sorted sets. Such varieties are well known to be equivalently described by finitary monads over Set^S, i.e. monads preserving filtered colimits: every variety $\mathcal{V}$ yields a free-algebra monad $\mathbb{T}_{\mathcal{V}}$ on Set^S which is finitary and whose Eilenberg–Moore category is isomorphic to $\mathcal{V}$. Conversely, every finitary monad $\mathbb{T}$ on Set^S defines a canonical S-sorted variety $\mathcal{V}$ whose free-algebra monad is isomorphic to $\mathbb{T}$.

There are cases in which algebraic specifications use operations and inequations; the corresponding models are then carried by partially ordered sets rather than sets without structure. In this article, we present an analogous characterization of finitary monads on the category Pos of partially ordered sets: we define varieties of ordered algebras which allow us to represent (a) all finitary monads on Pos and (b) all enriched finitary monads on Pos as the free-algebra monads of varieties. "Enriched" refers to Pos as a cartesian closed category: a monad is enriched if its underlying functor T is *locally monotone* ($f \leq g$ in $\mathsf{Pos}(A, B)$ implies $Tf \leq Tg$ in $\mathsf{Pos}(TA, TB)$). Case (b) works with algebras on posets whose operations are monotone (and as morphisms we take monotone homomorphisms), whereas Case (a) involves algebras on posets whose operations are

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not necessarily monotone (but whose morphisms are). To distinguish these cases, we shall call an algebra *coherent* if all of its operations are monotone.

A basic step, in which we follow the presentation of finitary monads on enriched categories due to Kelly and Power (1993), is to work with operation symbols whose arity is a finite poset rather than a natural number; we briefly recall the approach of *op. cit.* in Section 2. Just as natural numbers $n = \{0, 1, \ldots, n-1\}$ represent all finite sets up to isomorphism, we choose a representative set

$$\mathsf{Pos_f}$$

of finite posets up to isomorphism. Specializing the signatures of op. cit., we introduce the concept of a *discrete signature*. This is a set Σ of operation symbols equipped with an arity from $\mathsf{Pos_f}$. More precisely, Σ is a family of sets $(\Sigma_\Gamma)_{\Gamma \in \mathsf{Pos_f}}$. Thus, a Σ-algebra is a poset A together with an operation σ_A, for every $\sigma \in \Sigma_\Gamma$, which assigns to every monotone map $u\colon \Gamma \to A$ an element $\sigma_A(u)$ of A. For example, let $\mathbb{2}$ be the two-chain in $\mathsf{Pos_f}$. Then an operation symbol σ of arity $\mathbb{2}$ is interpreted in an algebra A as a partial function $\sigma_A\colon A \times A \to A$ whose domain of definition consists of all comparable pairs in A.

Given a signature Σ we form, for every finite poset Γ, the set $\mathscr{T}(\Gamma)$ of *terms in context* Γ. It is defined as usual in universal algebra, ignoring the order structure of Γ. Then, for every Σ-algebra A, whenever a monotone function $f\colon \Gamma \to A$ is given (i.e. whenever the variables of Γ are interpreted in A) we define an evaluation of terms in context Γ. This is a partial map $f^{\#}$ assigning a value to a term t provided that values of the subterms of t are defined and respect the order of Γ. This leads to the concept of *inequation in context* Γ: it is a pair (s, t) of terms in that context. An algebra A *satisfies* this inequation if for every monotone interpretation $f\colon \Gamma \to A$ we have that both $f^{\#}(t)$ and $f^{\#}(s)$ are defined and $f^{\#}(s) \le f^{\#}(t)$ holds in A. We use the following notation for inequations in context:

$$\Gamma \vdash s \le t.$$

By a *variety*, we understand a category $\mathcal{V}$ of Σ-algebras presented by a set $\mathcal{I}$ of inequations in context. Thus, the objects of $\mathcal{V}$ are all algebras satisfying each $\Gamma \vdash s \le t$ in $\mathcal{I}$, and morphisms are monotone homomorphisms. We prove that every variety $\mathcal{V}$ is strictly monadic over Pos, that is, for the monad $\mathbb{T}_\mathcal{V}$ of free $\mathcal{V}$-algebras, $\mathcal{V}$ is isomorphic to the category $\mathsf{Pos}^{\mathbb{T}_\mathcal{V}}$ of algebras for $\mathbb{T}_\mathcal{V}$. Moreover, $\mathbb{T}_\mathcal{V}$ is a finitary monad and, in case $\mathcal{V}$ consists of coherent algebras, $\mathbb{T}_\mathcal{V}$ is enriched.

Conversely, with every finitary monad $\mathbb{T}$ on Pos, we associate a canonical variety whose free-algebra monad is isomorphic to $\mathbb{T}$. This process from monads to varieties is inverse to the above assignment $\mathcal{V} \mapsto \mathbb{T}_\mathcal{V}$. Moreover, if $\mathbb{T}$ is enriched, the canonical variety consists of coherent algebras. This leads to a bijection between finitary enriched monads and varieties of coherent algebras up to isomorphism.

Is it really necessary to work with signatures of operations with partially ordered arities and inequations *in context*? There is a "natural" concept of a variety of ordered (coherent) algebras for classical signatures $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$. Here, terms are elements of free Σ-algebras on finite sets (of variables) and a variety is given by a set of inequations $s \le t$ between terms (with no context being used, which corresponds to using a discrete context). Such varieties were studied e.g. by Bloom (1976), Bloom and Wright (1983). Kurz and Velebil (2017) characterized these classical varieties as precisely the exact categories (in an enriched sense) with a 'suitable' generator. In a recent article, the first author, Dostál, and Velebil (2021) proved that for every such variety $\mathcal{V}$ the free-algebra monad $\mathbb{T}_\mathcal{V}$ is enriched and *strongly finitary* in the sense of Kelly and Lack (1993). This means that the functor $T_\mathcal{V}$ is the left Kan extension of its restriction along the full embedding $E\colon \mathsf{Pos_{fd}} \hookrightarrow \mathsf{Pos}$ of finite discrete posets:

$$T_\mathcal{V} = \mathsf{Lan}_E\,(T_\mathcal{V} \cdot E).$$

Conversely, every strongly finitary monad on Pos is isomorphic to the free-algebra monad of a variety in this classical sense. This answers our question above affirmatively: arities in $\mathsf{Pos_f}$ are necessary if *all* (possibly enriched) finitary monads are to be characterized via inequations.

For example, we have mentioned above a binary operation $\sigma(x, y)$ of arity $\mathbb{2}$. For the corresponding variety $\mathsf{Alg}\,\Sigma$ (with no specified inequations), the free-algebra monad is described in Example 4.3. This monad is not strongly finitary (Adámek et al., 2021, Ex. 3.17), thus no variety with a classical signature has this monad as the free-algebra monad.

Related work As we have already mentioned, the idea of using signatures with arities in $\mathsf{Pos_f}$ stems from work by Kelly and Power (1993) on the presentation of enriched monads by operations and equations. A signature in their sense is more general than what we use here: it is a family of *posets* $(\Sigma_\Gamma)_{\Gamma \in \mathsf{Pos_f}}$, and a Σ-algebra A is then a poset together with a monotone function from Σ_Γ to the poset of monotone functions from $\mathsf{Pos}(\Gamma, A)$ to A for every Γ in $\mathsf{Pos_f}$.

Whereas we deal with the monadic view on varieties of ordered algebras in this article, the view using algebraic theories has been investigated by Power with coauthors, e.g. Power (1999), Plotkin and Power (2001, 2002), Nishizawa and Power (2009); see Section 5. In particular, Nishizawa and Power (2009) work with enriched categories over a monoidal closed category $\mathscr{V}$ for which a $\mathscr{V}$-enriched base category $\mathscr{C}$ has been chosen.Then enriched algebraic $\mathscr{C}$-theories are shown to correspond to $\mathscr{V}$-enriched monads on $\mathscr{C}$. This is particularly relevant for this article: by choosing $\mathscr{V} = \mathsf{Set}$ and $\mathscr{C} = \mathsf{Pos}$ we treat non-enriched finitary monads on Pos, whereas the choice $\mathscr{V} = \mathscr{C} = \mathsf{Pos}$ covers the enriched case. An alternative proof of our main result has been presented by Rosický (2021) (after our paper was communicated to him).

Since the submission of this article, the results presented here have been generalized in at least two directions. First, Ford et al. (2021*a*) describe an extension of the notion of inequational theory for describing graded monads (with grades in the monoid $(\mathbb{N}, +, 0)$) on Pos, along with a sound and complete deduction system for graded inequational reasoning. Second, Ford et al. (2021*b*) establish a monad-theory correspondence between a notion of λ-ary relational algebraic theory and enriched λ-accessible monads given the choice of a locally λ-presentable category of relational structures specified by a set of infinitary Horn sentences; the results of this article are included there by instantiation. Furthermore, op. cit. describes a sound and complete sequent system for inequational reasoning, which yields an alternative description of the free-algebra monad of an inequational theory.

2. Equational Presentations of Monads

We now recall the approach to equational presentations of finitary monads introduced by Kelly and Power (1993); our aim here is to bring the rest of the article into this perspective. However, we note that the signatures used here are more general than those of the subsequent sections, and (unlike later) some enriched category theory is used. The reader can decide to skip this section without losing the connection.

For a locally finitely presentable category, $\mathscr{C}$ enriched over a symmetric monoidal closed category $\mathscr{V}$, Kelly and Power consider (enriched) monads on $\mathscr{C}$ that are finitary, i.e. the ordinary underlying endofunctors preserve filtered colimits. Below we specialize their approach to $\mathscr{C} = \mathsf{Pos}$ considered as an ordinary category ($\mathscr{V} = \mathsf{Set}$) or as a category enriched over itself ($\mathscr{V} = \mathsf{Pos}$) via its cartesian closed structure. In the first case, the hom-object $\mathsf{Pos}(A, B)$ is the *set* of all monotone functions from A to B; in the latter case, this is the *poset* of those functions, ordered pointwise. As in Section 1, a representative set $\mathsf{Pos_f}$ of finite posets (called *arities*) is chosen which is to be viewed as a full subcategory of Pos. We denote by

$$|\mathsf{Pos_f}|$$

the corresponding discrete category.

Definition 2.1. *A* signature *is a functor from* $|\mathsf{Pos_f}|$ *to* Pos. *In other words, a signature* Σ *is a family of posets* Σ_Γ *of* operation symbols of arity Γ *indexed by* $\Gamma \in \mathsf{Pos_f}$. *A morphism* $s\colon \Sigma \to \Sigma'$ *of signatures, being a natural transformation, is thus just a family of monotone maps* $s_\Gamma\colon \Sigma_\Gamma \to \Sigma'_\Gamma$ *indexed by arities.*

We denote by

$$\mathsf{Sig} = [|\mathsf{Pos_f}|, \mathsf{Pos}]$$

the category of signatures and their morphisms.

In the introduction, we have considered the special case of signatures where each poset Σ_Γ is discrete, i.e. we just have a *set* of operation symbols of arity Γ; for emphasis, we will call such signatures *discrete*. (N.B.: This terminology differs from the way the word *discrete* is used in the concept of *discrete Lawvere theory* (Power, 2005) where it refers to the arities Γ of operations rather than the objects Σ_Γ.)

Remark 2.2. Recall (Borceux, 1994, Def. 6.5.1) the concept of a *tensor* for objects $V \in \mathscr{V}$ and $C \in \mathscr{C}$: it is an object $V \otimes C$ of $\mathscr{C}$ together with an isomorphism

$$\mathscr{C}(V \otimes C, X) \cong \mathscr{V}(V, \mathscr{C}(C, X))$$

in $\mathscr{V}$ which is $\mathscr{V}$-natural in X; here $\mathscr{V}(-,-)$ denotes the internal hom-functor of $\mathscr{V}$.

In the case where $\mathscr{C} = \mathsf{Pos}$ and $\mathscr{V} = \mathsf{Set}$, the tensor is the copower

$$V \otimes C = \textstyle\coprod_V C,$$

and for $\mathscr{C} = \mathscr{V} = \mathsf{Pos}$, the tensor is just the product in Pos:

$$V \otimes C = V \times C.$$

Notation 2.3. (1) We denote by $\mathsf{Fin}(\mathsf{Pos})$ the enriched category of finitary enriched endofunctors on Pos. In the case where $\mathscr{V} = \mathsf{Set}$, these are all endofunctors preserving filtered colimits. For $\mathscr{V} = \mathsf{Pos}$, these are all locally monotone endofunctors preserving filtered colimits.

(2) The category of finitary enriched monads on Pos is denoted by $\mathsf{FinMnd}(\mathsf{Pos})$. We have a forgetful functor $U\colon \mathsf{FinMnd}(\mathsf{Pos}) \to \mathsf{Fin}(\mathsf{Pos})$.

By precomposing endofunctors with the non-full embedding $J\colon |\mathsf{Pos_f}| \to \mathsf{Pos}$, we obtain a forgetful functor from $\mathsf{Fin}(\mathsf{Pos})$ to Sig. It has a left adjoint assigning to every signature Σ the *polynomial functor* P_Σ given on objects by

$$P_\Sigma X = \textstyle\coprod_{\Gamma \in \mathsf{Pos_f}} \mathsf{Pos}(\Gamma, X) \otimes \Sigma_\Gamma, \tag{1}$$

and similarly on morphisms. As explained previously, the hom-object $\mathsf{Pos}(\Gamma, X)$ can have one of two meanings: for $\mathcal{V} = \mathsf{Set}$, it is regarded as a set and for $\mathcal{V} = \mathsf{Pos}$ as a poset. Henceforth, we will use that notation for hom-objects only in the latter case and write

$$\mathsf{Pos}_0(\Gamma, X)$$

for the set of monotone maps.

Observation 2.4. The usual category of algebras for the functor P_Σ, whose objects are posets A with a monotone map $\alpha\colon P_\Sigma A \to A$, has the following form for our two enrichments:

(1) Let $\mathscr{V} = \mathsf{Set}$. Then α as above is a monotone map

$$\big(\textstyle\coprod_{\Gamma \in \mathsf{Pos_f}} \coprod_{u \in \mathsf{Pos}_0(\Gamma, A)} \Sigma_\Gamma\big) \to A,$$

and as such has components assigning to every monotone function $u\colon \Gamma \to A$ (that is, a monotone interpretation of the variables in Γ) a monotone function $\Sigma_\Gamma \to A$. We denote this function by $\sigma \mapsto \sigma_A(u)$.

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In other words, the poset A is equipped with operations $\sigma_A\colon \mathsf{Pos}_0(\Gamma, A) \to A$ (which need not be monotone since $\mathsf{Pos}_0(\Gamma, A)$ is just a set) satisfying $\sigma_A(u) \le \tau_A(u)$ for all pairs $\sigma \le \tau$ in Σ_Γ and u in $\mathsf{Pos}_0(\Gamma, A)$. If Σ is discrete, this is precisely a Σ-algebra (see the Introduction).

(2) Now let $\mathscr{V} = \mathsf{Pos}$. Then $\alpha\colon P_\Sigma A \to A$ is a monotone map

$$\left(\coprod_{\Gamma\in\mathsf{Pos}_\mathsf{f}} \mathsf{Pos}(\Gamma, A) \times \Sigma_\Gamma\right) \to A,$$

and thus has as components monotone functions $(u, \sigma) \mapsto \sigma_A(u)$. That is, in addition to the condition that $\sigma_A(u) \le \tau_A(u)$ for all pairs $\sigma \le \tau$ in Σ_Γ and u in $\mathsf{Pos}(\Gamma, A)$ as above, we also see that each σ_A is monotone. Thus, if Σ is discrete, this is precisely a coherent algebra (again, see the Introduction).

Observe also that "homomorphism" has the usual meaning: a monotone function preserving the given operations. In fact, given algebras $\alpha\colon P_\Sigma A \to A$ and $\beta\colon P_\Sigma B \to B$ a homomorphism is a monotone function $h\colon A \to B$ such that $h \cdot \alpha = \beta \cdot P_\Sigma h$. This is equivalent to $h(\sigma_A(u)) = \sigma_B(h \cdot u)$ for all $u \in \mathsf{Pos}(\Gamma, A)$ and all $\sigma \in \Sigma_\Gamma$.

Remark 2.5. (1) As shown by Trnková et al. (1975) (see also Kelly 1980), every ordinary finitary endofunctor H on Pos generates a free monad whose underlying functor $\widehat{H}$ is a colimit of the ω-chain

$$\widehat{H} = \mathsf{colim}_{n<\omega} W_n$$

of functors, where

$$W_0 = \mathsf{Id} \qquad \text{and} \qquad W_{n+1} = HW_n + \mathsf{Id}.$$

Connecting morphisms are $w_0\colon \mathsf{Id} \to H + \mathsf{Id}$, the coproduct injection, and $w_{n+1} = Hw_n + \mathsf{Id}$. The colimit injections $c_n\colon W_nX \to \widehat{H}X$ in Pos have the property that if a parallel pair $u, v\colon \widehat{H}X \to A$ satisfies $c_n \cdot u \le c_n \cdot v$ for all $n < \omega$, then we have $u \le v$. It follows that $\widehat{H}$ is enriched if H is.

(2) The category of H-algebras is isomorphic to the Eilenberg-Moore category $\mathsf{Pos}^{\widehat{H}}$ (see Barr 1970).

(3) Lack (1999) shows that the composite functor

$$\mathsf{FinMnd}(\mathsf{Pos}) \xrightarrow{U} \mathsf{Fin}(\mathsf{Pos}) \xrightarrow{(-)\cdot J} \mathsf{Sig},$$

where $J\colon |\mathsf{Pos}_\mathsf{f}| \hookrightarrow \mathsf{Pos}$ is the canonical inclusion functor, is monadic; this means that the functor has a left adjoint and the Eilenberg-Moore category of the ensuing monad $\mathbb{M}$ on Sig is equivalent to $\mathsf{FinMnd}(\mathsf{Pos})$ via the comparison functor. The monad $\mathbb{M}$ assigns to every signature Σ the signature $\widehat{P_\Sigma} \cdot J\colon |\mathsf{Pos}_\mathsf{f}| \to \mathsf{Pos}$.

(4) It follows that every enriched finitary monad $\mathbb{T}$ on Pos can be regarded as an algebra for the monad $\mathbb{M}$. Therefore, $\mathbb{T}$ is a coequalizer in $\mathsf{FinMnd}(\mathsf{Pos})$ of a parallel pair of monad morphisms between free $\mathbb{M}$-algebras on signatures Δ, Σ:

$$\widehat{P_\Delta} \underset{r}{\overset{\ell}{\rightrightarrows}} \widehat{P_\Sigma} \xrightarrow{\;c\;} \mathbb{T}.$$

This is the equational presentation of $\mathbb{T}$ considered by Kelly and Lack (1993).

Example 2.6. (1) In the case where $\mathscr{V} = \mathsf{Set}$ and $\mathscr{C} = \mathsf{Pos}$, $\mathsf{FinMnd}(\mathsf{Pos})$ is the category of (non-enriched) finitary monads on Pos. Consider the above coequalizer in the special case that Δ consists of a single operation δ of arity Γ. That is, $\Delta_\Gamma = \{\delta\}$ and all $\Delta_{\bar\Gamma}$ for $\bar\Gamma \ne \Gamma$ are empty. By the Yoneda lemma, l and r simply choose two elements of $\widehat{P_\Sigma}\Gamma$, say t_ℓ and t_r. The above coequalizer means that $\mathbb{T}$ is presented by the signature Σ and the equation $t_\ell = t_r$.

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For Δ arbitrary, we do not get one equation, but a set of equations (one for every operation symbol in Δ) and $\mathbb{T}$ is presented by Σ and the corresponding set of equations, grouped by their respective arities.

(2) The case $\mathscr{V} = \mathscr{C} = \mathsf{Pos}$ yields as $\mathsf{FinMnd}(\mathsf{Pos})$ the category of enriched finitary monads on Pos. That is, the underlying endofunctor T is locally monotone.

Remark 2.7. The fact that every finitary (possibly enriched) monad on Pos has an *equational* presentation depends heavily on the fact that signatures are not restricted to be discrete. In contrast, we characterize finitary (possibly enriched) monads using discrete signatures and *in*equational presentations. While it is clear that the two specification formats are mutually convertible, inequational presentations seem natural for varieties of algebras on Pos.

Of course, it is possible to translate Σ-algebras for non-discrete signatures Σ as varieties of algebras for discrete ones (see Example 3.19(9)). Using the result of Kelly and Power, such a translation would lead to a correspondence between finitary monads and varieties. This article can be viewed as a detailed realization of this.

3. Varieties of Ordered Algebras

Recall that $\mathsf{Pos_f}$ is a fixed set of finite posets that represent all finite posets up to isomorphism. If $\Gamma \in \mathsf{Pos_f}$ has the underlying set $\{x_0, \ldots, x_{n-1}\}$, then we call the x_i the *variables* in Γ. Recall that all monotone functions from A to B form a set $\mathsf{Pos}_0(A, B)$ and a poset $\mathsf{Pos}(A, B)$ with the pointwise order.

Notation 3.1. The category Pos is cartesian closed, with hom-objects $\mathsf{Pos}(X, Y)$ given by all monotone functions $X \to Y$, ordered pointwise. That is, given monotone functions $f, g\colon X \to Y$, by $f \leq g$ we mean that $f(x) \leq g(x)$ for all $x \in X$.

We denote the underlying set of a poset X by $|X|$. We also often regard $|X|$ as the discrete poset on that set.

In the following, we will work with discrete signatures, which we already mentioned after Definition 2.1. Explicitly:

Definition 3.2. *A* discrete signature *is a set Σ of operation symbols each with a prescribed* arity. *That is, Σ is a family $(\Sigma_\Gamma)_{\Gamma \in \mathsf{Pos_f}}$ of sets Σ_Γ. A* Σ-algebra *is a poset A equipped with a function*

$$\sigma_A\colon \mathsf{Pos}_0(\Gamma, A) \to A.$$

for every $\sigma \in \Sigma_\Gamma$. That is, σ_A assigns to every monotone interpretation $f\colon \Gamma \to A$ of the variables in Γ an element $\sigma_A(f)$ of A. The algebra A is called coherent *if each σ_A is monotone, i.e. whenever $f \leq g$ in $\mathsf{Pos}(\Gamma, A)$, then $\sigma_A(f) \leq \sigma_A(g)$.*

Notation 3.3. We denote by $\mathsf{Alg}\,\Sigma$ the category of Σ-algebras. Its morphisms $A \to B$ are the *homomorphisms* in the expected sense; i.e. they are monotone functions $h\colon A \to B$ such that for every arity Γ and every operation symbol $\sigma \in \Sigma_\Gamma$, the square

$$\begin{array}{ccc} \mathsf{Pos}_0(\Gamma, A) & \xrightarrow{\sigma_A} & A \\ {\scriptstyle h\cdot(-)}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \mathsf{Pos}_0(\Gamma, B) & \xrightarrow{\sigma_B} & B \end{array}$$

commutes. Similarly, we have the category $\mathsf{Alg_c}\,\Sigma$ of all coherent Σ-algebras. For their homomorphisms we have the commutative squares

$$\begin{array}{ccc}
\mathsf{Pos}(\Gamma, A) & \xrightarrow{\sigma_A} & A \\
{\scriptstyle h\cdot(-)}\big\downarrow & & \big\downarrow{\scriptstyle h} \\
\mathsf{Pos}(\Gamma, B) & \xrightarrow{\sigma_B} & B
\end{array}$$

Example 3.4. Let Σ be the signature given by

$$\Sigma_{\mathbb{2}} = \{+\} \quad \text{and} \quad \Sigma_{\mathbb{1}} = \{@\},$$

where $\mathbb{2}$ is a 2-chain and $\mathbb{1}$ is a singleton. A Σ-algebra consists of a poset A with a (not necessarily monotone) unary operation $@_A$ and a partial binary operation $+_A$ whose domain of definition is formed by all comparable pairs. Moreover, A is coherent iff both $@_A$ and $+_A$ are monotone, the latter in the sense that $a + a' \leq b + b'$ whenever $a \leq a'$, $b \leq b'$, $a \leq b$, and $a' \leq b'$.

Similarly to the more general signatures discussed in Section 2, discrete signatures Σ can be represented as polynomial functors H_Σ (for Σ-algebras) and K_Σ (for coherent Σ-algebras), respectively, introduced next. These functors arise by specializing the corresponding instances of the polynomial functor P_Σ according to Observation 2.4 to discrete signatures.

Notation 3.5. The *polynomial* and *coherent polynomial* functors for a discrete signature Σ are the endofunctors $H_\Sigma\colon \mathsf{Pos} \to \mathsf{Pos}$ and $K_\Sigma\colon \mathsf{Pos} \to \mathsf{Pos}$ given by

$$H_\Sigma X = \coprod_{\Gamma \in \mathsf{Pos_f}} \Sigma_\Gamma \times \mathsf{Pos}_0(\Gamma, X) \qquad \text{and} \qquad K_\Sigma X = \coprod_{\Gamma \in \mathsf{Pos_f}} \Sigma_\Gamma \times \mathsf{Pos}(\Gamma, X),$$

respectively, where we regard the sets Σ_Γ and $\mathsf{Pos}_0(\Gamma, X)$ as discrete posets. Thus, the elements of both $H_\Sigma X$ and $K_\Sigma X$ are pairs (σ, f) where σ is an operation symbol of arity Γ and $f\colon \Gamma \to X$ is monotone. The action on monotone maps $h\colon X \to Y$ is then the same for both functors:

$$H_\Sigma h(\sigma, f) = (\sigma, h \cdot f) = K_\Sigma h(\sigma, f).$$

Remark 3.6. (1) Every Σ-algebra A induces an H_Σ-algebra $\alpha\colon H_\Sigma A \to A$ given by

$$\alpha(\sigma, f) = \sigma_A(f) \qquad \text{for } \sigma \in \Sigma_\Gamma \text{ and } f \in \mathsf{Pos}_0(\Gamma, X).$$

Conversely, every H_Σ-algebra $\alpha\colon H_\Sigma A \to A$ can be viewed as a Σ-algebra, putting $\sigma_A(f) = \alpha(\sigma, f)$. More conceptually, we have bijective correspondences between the following (families of) maps:

$$\begin{array}{ll}
\alpha\colon\ H_\Sigma A \to A & \\
\hline
\alpha_\Gamma\colon\ \Sigma_\Gamma \times \mathsf{Pos}_0(\Gamma, A) \to A & (\Gamma \in \mathsf{Pos_f}) \\
\hline
\sigma_A\colon\ \mathsf{Pos}_0(\Gamma, A) \to A & (\Gamma \in \mathsf{Pos_f}, \sigma \in \Sigma_\Gamma)
\end{array}$$

Thus, $\mathsf{Alg}\,\Sigma$ is isomorphic to the category $\mathsf{Alg}\,H_\Sigma$ of algebras for H_Σ whose morphisms from (A, α) to (B, β) are those monotone maps $h\colon A \to B$ for which the square below commutes:

$$\begin{array}{ccc}
H_\Sigma A & \xrightarrow{\alpha} & A \\
{\scriptstyle H_\Sigma h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\
H_\Sigma B & \xrightarrow{\beta} & B
\end{array}$$

Indeed, this is equivalent to h being a homomorphism of Σ-algebras. Shortly,

$$\mathsf{Alg}\,\Sigma \cong \mathsf{Alg}\,H_\Sigma.$$

Moreover, this isomorphism is *concrete*, i.e. it preserves the underlying posets (and monotone maps). That is, if $U\colon\ \mathsf{Alg}\,\Sigma \to \mathsf{Pos}$ and $\bar{U}\colon\ \mathsf{Alg}\,H_\Sigma \to \mathsf{Pos}$ denote the forgetful functors, the

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above isomorphism $I\colon \ \mathsf{Alg}\,\Sigma \to \mathsf{Alg}\,H_\Sigma$ makes the following triangle commutative:

$$\begin{array}{ccc} \mathsf{Alg}\,\Sigma & \xrightarrow{\ I\ } & \mathsf{Alg}\,H_\Sigma \\ & {\scriptstyle U}\searrow \quad \swarrow{\scriptstyle \bar{U}} & \\ & \mathsf{Pos} & \end{array}$$

(2) Similarly, every coherent Σ-algebra defines an algebra for K_Σ, and conversely. Indeed, giving an algebra structure $\alpha\colon K_\Sigma A \to A$ is the same as giving a Pos_f-indexed family of monotone maps

$$\alpha_\Gamma\colon \ \Sigma_\Gamma \times \mathsf{Pos}(\Gamma, A) \to A.$$

Equivalently, we have for every σ of arity Γ a monotone map $\sigma_A\colon \mathsf{Pos}(\Gamma, A) \to A$.

This leads to an isomorphism $I_c\colon \ \mathsf{Alg}_\mathsf{c}\,\Sigma \to \mathsf{Alg}\,K_\Sigma$, which is concrete:

$$\begin{array}{ccc} \mathsf{Alg}_\mathsf{c}\,\Sigma & \xrightarrow{\ I_c\ } & \mathsf{Alg}\,K_\Sigma \\ & {\scriptstyle U_c}\searrow \quad \swarrow{\scriptstyle \bar{U}_c} & \\ & \mathsf{Pos} & \end{array}$$

where U_c and $\bar{U}_c$ denote the forgetful functors, respectively.

Remark 3.7. Recall that an *embedding* in Pos is a map $m\colon A \to B$ such that for all $a, a' \in A$ we have $a \le a'$ iff $m(a) \le m(a')$. That is, embeddings are order-reflecting monotone functions. Given an ω-chain of embeddings in Pos, its colimit is simply their union (with inclusion maps as the colimit cocone).

Proposition 3.8. *Every poset X generates a free Σ-algebra $T_\Sigma X$. Its underlying poset is the union of the following ω-chain of embeddings in* Pos*:*

$$W_0 = X \xrightarrow{w_0} W_1 = H_\Sigma X + X \xrightarrow{w_1} W_2 = H_\Sigma W_1 + X \xrightarrow{w_3} \cdots \tag{2}$$

where w_0 is the right-hand coproduct injection $X \to H_\Sigma X + X$ and $w_{n+1} = Hw_n + \mathsf{id}_X\colon W_{n+1} = H_\Sigma W_n + X \to HW_{n+1} + X = W_{n+2}$ for every n. The universal map $\eta_X\colon X \to T_\Sigma X$ is the inclusion of W_0 into the union.

Proof. Observe first that the polynomial functor H_Σ can be rewritten, up to natural isomorphism, as

$$H_\Sigma X \cong \coprod_{\Gamma \in \mathsf{Pos}_\mathsf{f}} \coprod_{\Sigma_\Gamma} \mathsf{Pos}_0(\Gamma, X)$$

because every Σ_Γ is discrete. It follows that H_Σ is finitary, being a coproduct of functors $\mathsf{Pos}_0(\Gamma, -)$ (where each $\mathsf{Pos}_0(\Gamma, -)$ is finitary because Γ is finite). As shown by Adámek (1974), it follows that the free H_Σ-algebra over X is the colimit of the ω-chain (W_n) from (2) in Pos. The desired result thus follows from the concrete isomorphism $\mathsf{Alg}\,\Sigma \cong \mathsf{Alg}\,H_\Sigma$. □

A similar result can be proved for coherent Σ-algebras and the associated functor K_Σ, using the fact that like $\mathsf{Pos}_0(\Gamma, -)$, also the internal hom-functor $\mathsf{Pos}(\Gamma, -)$ is finitary:

Proposition 3.9. *Every poset X generates a free coherent Σ-algebra $T^\mathsf{c}_\Sigma X$. Its underlying poset is the union of the following ω-chain of embeddings in* Pos*:*

$$W_0 = X \xrightarrow{w_0} W_1 = K_\Sigma X + X \xrightarrow{w_1} W_2 = K_\Sigma W_1 + X \xrightarrow{w_3} \cdots$$

The universal morphism $\eta^\mathsf{c}_X\colon X \to T^\mathsf{c}_\Sigma X$ is the inclusion of W_0 into the union.

Definition 3.10. *For a finite poset Γ we define* terms in context Γ *as usual in universal algebra, ignoring the order structure of the context Γ; we write $\mathscr{T}(\Gamma)$ for the set of Σ-terms in variables*

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from $|\Gamma|$. Explicitly, the set $\mathscr{T}(\Gamma)$ of terms in context Γ is the least set containing $|\Gamma|$ such that given an operation σ with arity Δ and a function $f\colon |\Delta| \to \mathscr{T}(\Gamma)$, we obtain a term $\sigma(f) \in \mathscr{T}(\Gamma)$.

Convention 3.11. We denote by $u_\Gamma\colon \Gamma \to \mathscr{T}(\Gamma)$ the inclusion map. We will often silently assume that the elements of $|\Delta|$ are listed in some fixed sequence $x_1, \ldots, x_n$, and then write $\sigma(t_1, \ldots, t_n)$ in lieu of $\sigma(f)$ where $f(x_i) = t_i$ for $i = 1, \ldots, n$. In particular, in examples we will normally use arities Δ with $|\Delta| = \{1, \ldots, n\}$ for some n, and then assume the elements of Δ to be listed in the sequence $1, \ldots, n$. We will often abbreviate $(t_1, \ldots, t_n)$ as (t_i), in particular writing $\sigma(t_i)$ in lieu of $\sigma(t_1, \ldots, t_n)$. Every $\sigma \in \Sigma_\Gamma$ yields the term $\sigma(u_\Gamma) \in \mathscr{T}(\Gamma)$, which by abuse of notation we will occasionally write as just σ.

Example 3.12. Let Σ be a signature with a single operation symbol σ whose arity is a 2-chain. Then $\mathscr{T}(\Gamma)$ is the usual set of terms built from a binary operation σ and the variables from Γ, whereas $T_\Sigma\Gamma$ contains only those terms which either (a) are variables or (b) have the shape $\sigma(t, t)$ for a term t or (c) $\sigma(x, y)$ for variables $x \leq y$ in Γ. The order of $T_\Sigma\Gamma$ is such that the only comparable distinct terms are variables. On the other hand, $T^{\mathsf{c}}_\Sigma\Gamma$ not only has more comparable pairs of terms, but consequently also contains more terms. For instance, if $x \leq y$ in Γ, then $T^{\mathsf{c}}_\Sigma\Gamma$ contains the term $\sigma(\sigma(x, x), \sigma(x, y))$ (which is not present in $T_\Sigma\Gamma$).

Definition 3.13. *Let A be a Σ-algebra. Given a finite poset Γ and a monotone interpretation $f\colon \Gamma \to A$, the* evaluation *of terms in context Γ is the partial map*

$$f^\#\colon \mathscr{T}(\Gamma) \to |A|$$

defined recursively by

(1) *$f^\#(x) = f(x)$ for every $x \in |\Gamma|$, and*

(2) *$f^\#(\sigma(g))$ is defined for $\sigma \in \Sigma_\Delta$ and $g\colon |\Delta| \to \mathscr{T}(\Gamma)$ iff all $f^\#(g(i))$ are defined and $i \leq j$ in Δ implies $f^\#(g(i)) \leq f^\#(g(j))$ in A; then $f^\#(\sigma(g)) = \sigma_A(f^\# \cdot g)$.*

Example 3.14. (1) For the signature in Example 3.4, we have terms in $\mathscr{T}\{x, y\}$ (with $\{x, y\}$ ordered discretely) such as $@x$ and $y + @y$. Given a Σ-algebra A and an interpretation $f\colon \{x, y\} \to A$ we see that $@x$ is always interpreted as $f^\#(@x) = @_A(f(x))$, whereas $f^\#(y + @x)$ is defined if and only if $f(y) \leq @_A(f(x))$, and then $f^\#(y + @x) = f(y) +_A @_A(f(x))$.

(2) Every operation symbol $\sigma \in \Sigma_\Gamma$ considered as a term (see Convention 3.11) satisfies

$$f^\#(\sigma) = \sigma_A(f(x_i)).$$

Definition 3.15. *An* inequation in context Γ *is a pair (s, t) of terms in $\mathscr{T}(\Gamma)$, written in the form*

$$\Gamma \vdash s \leq t.$$

Furthermore, we denote by

$$\Gamma \vdash s = t$$

the conjunction of the inequations $\Gamma \vdash s \leq t$ and $\Gamma \vdash t \leq s$.

A Σ-algebra satisfies *$\Gamma \vdash s \leq t$ if for every monotone function $f\colon \Gamma \to A$, both $f^\#(s)$ and $f^\#(t)$ are defined and $f^\#(s) \leq f^\#(t)$.*

Example 3.16. For the signature of Example 3.4, consider the singleton context $\{x\}$ and the inequation

$$\{x\} \vdash x \leq @x. \tag{3}$$

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An algebra A satisfies this inequation iff $a \leq @_A(a)$ holds for every $a \in A$. In such algebras, the interpretation of the term $x + @x$ is defined everywhere.

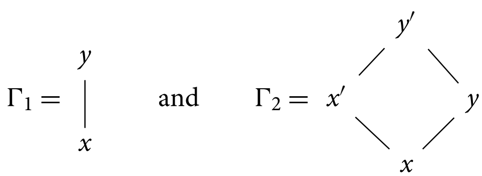
Example 3.17. For Σ as in Example 3.12, given a context Γ, the structure of $T^{\mathsf{c}}_{\Sigma}\Gamma$ is completely described as follows. The elements of $T^{\mathsf{c}}_{\Sigma}\Gamma$ are certain terms of *uniform depth*, where variables have uniform depth 0 and a term $\sigma(t, s)$ has uniform depth $n + 1$ if t and s have uniform depth n, and a term t has uniform depth if t has uniform depth n for some n. Given two such terms t, s in $T^{\mathsf{c}}_{\Sigma}\Gamma$, we have $t \leq s$ iff s arises from t by replacing any number of occurrences (maybe none) of x in t by y where $x \leq y$ in Γ (in particular, t and s have the same uniform depth). Finally, the terms of uniform depth actually contained in $T^{\mathsf{c}}_{\Sigma}\Gamma$ are determined by induction on the depth: Every term of uniform depth 0 is in $T^{\mathsf{c}}_{\Sigma}\Gamma$, and a term $\sigma(t, s)$ of uniform depth $n + 1$ is contained in $T^{\mathsf{c}}_{\Sigma}\Gamma$ iff $t, s \in T^{\mathsf{c}}_{\Sigma}\Gamma$ and $t \leq s$. This description is easily verified by noting that, on the one hand, for every t as per the above description, $f^{\#}(t)$ is defined for every monotone valuation f in a coherent Σ-algebra, and whenever $t \leq s$ according to the above description, then $f^{\#}(t) \leq f^{\#}(s)$; and that, on the other hand, the description actually yields a coherent Σ-algebra. We note in particular that $T_{\Sigma}\Gamma$ maps injectively into $T^{\mathsf{c}}_{\Sigma}\Gamma$.

Definition 3.18. *We denote by* $\mathsf{Alg}(\Sigma, \mathcal{I})$ *the full subcategory of* $\mathsf{Alg}\,\Sigma$ *that is specified by a set* $\mathcal{I}$ *of inequations in context. It consists of all* Σ*-algebras that satisfy all inequations in* $\mathcal{I}$*. A category of the form* $\mathsf{Alg}(\Sigma, \mathcal{I})$ *is called a* variety of Σ-algebras*. Analogously, a* variety of coherent Σ-algebras *is a full subcategory of* $\mathsf{Alg_c}\,\Sigma$ *specified by a set* $\mathcal{I}$ *of inequations in context, denoted by* $\mathsf{Alg_c}(\Sigma, \mathcal{I})$.

Example 3.19. We present some varieties of algebras.

(1) We have seen the variety $\mathcal{V}$ specified by (3) in Example 3.16.

(2) The subvariety of all coherent algebras in $\mathcal{V}$ as in the previous item can be specified as follows. Consider the contexts Γ_1 and Γ_2 given by



and the inequations

$$\Gamma_1 \vdash @x \leq @y \qquad \text{and} \qquad \Gamma_2 \vdash x + y \leq x' + y'. \tag{4}$$

It is clear that Σ-algebras satisfying (3) and (4) form precisely the full subcategory of $\mathcal{V}$ consisting of coherent algebras.

(3) In general, all coherent Σ-algebras form a variety of Σ-algebras. For every context Γ, form the context $\bar{\Gamma}$ with variables x and x' for every variable x of Γ, where the order is the least one such that the functions $e, e' \colon \Gamma \to \bar{\Gamma}$ given by $e(x) = x$ and $e'(x) = x'$ are embeddings satisfying $e \leq e'$. For every Γ and every $\sigma \in \Sigma_{\Gamma}$ consider the following inequation in context $\bar{\Gamma}$:

$$\bar{\Gamma} \vdash \sigma(e) \leq \sigma(e').$$

It is satisfied by precisely those Σ-algebras A for which σ_A is monotone.

(4) Ordered groups and ordered vector spaces are important examples of varieties that are not coherent. Recall that an ordered group is a group on a poset whose multiplication is monotone. But it is not required (and usually not true) that the operation of inverse elements be monotone. The situation is analogous for ordered vector spaces.

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(5) Recall that an *internal semilattice* in a category with finite products is an object A together with morphisms $+\colon A\times A\to A$ and $0\colon 1\to A$ such that

(a) 0 is a unit for $+$, i.e. the following triangles commute

$$\begin{array}{ccccc} A\cong 1\times A & \xrightarrow{0\times\mathsf{id}} & A\times A & \xleftarrow{\mathsf{id}\times 0} & A\times 1\cong A \\ & \searrow\!\!= & \big\downarrow{+} & =\!\!\swarrow & \\ & & A & & \end{array}$$

(b) $+$ is associative, commutative, and idempotent:

$$\begin{array}{ccc} A\times A\times A & \xrightarrow{+\times\mathsf{id}} & A\times A \\ {\scriptstyle \mathsf{id}\times +}\big\downarrow & & \big\downarrow{\scriptstyle +} \\ A\times A & \xrightarrow{\quad +\quad} & A \end{array} \qquad \begin{array}{ccc} A\times A & \xrightarrow{\mathsf{swap}} & A\times A \\ & {\scriptstyle +}\searrow & \big\downarrow{\scriptstyle +} \\ & & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\Delta} & A\times A \\ & = & \big\downarrow{\scriptstyle +} \\ & & A \end{array}$$

Here, $\mathsf{swap}=\langle \pi_r,\pi_\ell\rangle\colon A\times A\to A\times A$ is the canonical isomorphism commuting product components, and $\Delta=\langle\mathsf{id},\mathsf{id}\rangle\colon A\to A\times A$ is the diagonal.

Internal semilattices in Pos form a variety of coherent Σ-algebras. To see this, consider the signature Σ with $\Sigma_2=\{+\}$ and $\Sigma_\emptyset=\{0\}$, where 2 denotes the two-element discrete poset. The set $\mathcal{I}$ is formed by (in)equations specifying that $+$ is monotone, associative, commutative, and idempotent with unit 0. Note that this does *not* imply that $x+y$ is the join of x,y in X w.r.t. its given order (cf. Example 3.32).

(6) A related variety is that of classical join-semilattices (with 0). To specify those, we take the signature Σ from the previous item; but now we impose inequations in context specifying that 0 and $+$ are the least element and the join operation, respectively:

$$\{x\}\vdash 0\le x \qquad \{x,y\}\vdash x\le x+y \qquad \{x,y\}\vdash y\le x+y \qquad \{x\le z, y\le z\}\vdash x+y\le z.$$

It then follows that $+$ is monotone, associative, commutative, and idempotent, so these equations need not be included. Note that although all operations have discrete arities, the inequation stating that $x+y$ is below all upper bounds of $\{x,y\}$ needs a non-discrete context.

(7) *Bounded joins:* For a natural example of an operation with non-discrete arity, take the signature Σ consisting of a unary operation $\perp$ and an operation j (*bounded join*) of arity $\{0,1,2\}$ where $0\le 2$ and $1\le 2$ (but $0\not\le 1$). We then define a variety $\mathcal{V}$ by the following inequations in context

$$\begin{aligned} \{x,y\}&\vdash \perp(x)\le y \\ \{x\le z, y\le z\}&\vdash x\le j(x,y,z) \\ \{x\le z, y\le z\}&\vdash y\le j(x,y,z) \\ \{x\le z, y\le z, x\le w, y\le w\}&\vdash y\le j(x,y,z)\le w. \end{aligned}$$

That is, $j(x,y,z)$ is the join of elements x,y having a joint upper bound z. It follows that the value of $j(x,y,z)$, when defined, does not actually depend on z, which instead just serves as a witness for boundedness of $\{x,y\}$. The operation $\perp$ and its inequality specify that algebras are either empty or have a least element, i.e. the empty set has a join provided that it is bounded. Thus, $\mathcal{V}$ consists of the partial orders having all bounded finite joins, which we will refer to as *bounded-join semilattices*, and morphisms in $\mathcal{V}$ are monotone maps that preserve all existing finite joins.

(8) The theory of *subconvex algebras* (Pumplün and Röhrl, 1984, Definition 2.7) (or *positive convex modules* Pumplün 2003) has as operations $\sum_{i=1}^n p_i\cdot(-)$ (forming formal subconvex combinations) for all n-tuples of real numbers $p_i\ge 0$ such that $\sum p_i\le 1$, with discrete arity $\{1,\dots,n\}$.

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Its axioms are on the one hand all equations of the form

$$\sum_{k=1}^{n} \delta_{ik} \cdot x_k = x_i,$$

where δ_{ik} is the Kronecker symbol (i.e. $\delta_{ik} = 1$ if $i = k$, and 0 otherwise), and on the other hand all equations of the form

$$\sum_{i=1}^{n} p_i \cdot \sum_{k=1}^{m} q_{ik} \cdot x_k = \sum_{k=1}^{m} \left(\sum_{i=1}^{n} p_i q_{ik} \right) \cdot x_k.$$

The theory of *ordered subconvex algebras* additionally has inequational axioms

$$\sum_{i=1}^{n} p_i \cdot x_i \leq \sum_{i=1}^{n} q_i \cdot x_i$$

for coefficients p_i, q_i satisfying $p_i \leq q_i$ for all $i = 1, \ldots n$. This is an example of a theory where inequations are naturally presented in the format of Kelly and Power (1993), i.e. the inequations are effectively among operation symbols only.

(9) Let a collection of posets Σ_Γ ($\Gamma \in \mathsf{Pos_f}$), i.e. a signature in the sense of Kelly and Power (1993) (cf. Section 2), be given. We obtain the corresponding discrete signature $\Sigma^d = (|\Sigma_\Gamma|)_{\Gamma \in \mathsf{Pos_f}}$ by disregarding the order of Σ_Γ. Now consider the set $\mathcal{I}$ consisting of all inequations in context of the form

$$\Gamma \vdash \sigma(x_i) \leq \tau(x_i)$$

where $|\Gamma| = \{x_1, \ldots, x_n\}$ and $\sigma \leq \tau$ in Σ_Γ. Then the variety $\mathsf{Alg}\,(\Sigma, \mathcal{I})$ is precisely the category of algebras for the non-discrete signature Σ (see Definition 2.1).

Remark 3.20. We will now discuss limits and directed colimits in $\mathsf{Alg}\,\Sigma$.

(1) It is easy to see that for every endofunctor H on Pos the category $\mathsf{Alg}\,H$ of algebras for H is complete. Indeed, the forgetful functor $V\colon \mathsf{Alg}\,H \to \mathsf{Pos}$ creates limits. This means that for every diagram $D\colon \mathscr{D} \to \mathsf{Alg}\,H$ with VD having a limit cone $(\ell_d\colon L \to VDd)_{d \in \mathrm{obj}(\mathscr{D})}$, there exists a unique algebra structure $\alpha\colon HL \to L$ making each ℓ_d a homomorphism in $\mathsf{Alg}\,H$. Moreover, the cone (ℓ_d) is a limit of D.

(2) Analogously, it is easy to see that for every finitary endofunctor H of Pos the category $\mathsf{Alg}\,H$ has filtered colimits created by V.

(3) We conclude from $\mathsf{Alg}\,\Sigma \cong \mathsf{Alg}\,H_\Sigma$ that limits and filtered colimits of Σ-algebras exist and are created by the forgetful functor into Pos; similarly for $\mathsf{Alg_c}\,\Sigma$.

(4) Moreover, we note that $\mathsf{Alg}\,H_\Sigma$ is a locally finitely presentable category; this was shown by Bird (1984, Prop. 2.14), see also the remark given by the first author and Rosický (1994, 2.78).

Lemma 3.21. *Let $h\colon A \to B$ be a homomorphism of Σ-algebras, and let $f\colon \Gamma \to A$ be a monotone interpretation. Then for every term $t \in \mathscr{T}(\Gamma)$ we have that*

(1) *if $f^\#(t)$ is defined, then $(h \cdot f)^\#(t)$ is also defined, and $(h \cdot f)^\#(t) = h(f^\#(t))$.*

(2) *if $h(f^\#(t))$ is defined and h is an embedding, then $f^\#(t)$ is defined, too.*

Proof. (1) We proceed by induction on the structure of t. If t is a variable, then the claim is immediate from the definition of $(-)^\#$. For the inductive step, let $t \in \mathscr{T}(\Gamma)$ be a term of the form $t = \sigma(t_1, \ldots, t_n)$ such that $f^\#(t)$ defined, where $\sigma \in \Sigma_\Delta$ and Δ has cardinality n. Then, by definition of $(-)^\#$, it follows that $f^\#(t_i)$ is defined for all $i = 1, \ldots, n$ and $f^\#(t_i) \leq f^\#(t_j)$ for all $i \leq j$ in

Δ (i.e. the map $i \mapsto f^{\#}(t_i)$ is monotone). Combining this with our assumption that $h\colon A \to B$ is a homomorphism, we obtain that

$$h \cdot f^{\#}(\sigma(t_1, \ldots, t_n)) = \sigma_B(h \cdot f^{\#}(t_1), \ldots, h \cdot f^{\#}(t_n)).$$

Moreover, since $f^{\#}(t_i)$ is defined for all $i = 1, \ldots, n$, the inductive hypothesis implies that $h \cdot f^{\#}(t_i) = (h \cdot f)^{\#}(t_i)$ for all $i \leq n$, hence also

$$(h \cdot f)^{\#}(t_i) = h \cdot f^{\#}(t_i) \leq h \cdot f^{\#}(t_j) = (h \cdot f)^{\#}(t_j)$$

for all $i \leq j$ in Δ. Thus $\sigma_B((h \cdot f)^{\#}(t_1), \ldots, (h \cdot f)^{\#}(t_n))$ is defined and equal to $h \cdot f^{\#}(\sigma(t_1, \ldots, t_n))$, as desired.

(2) Suppose now that h is an embedding. We use a similar inductive proof. In the inductive step suppose that $(h \cdot f)^{\#}(t)$ is defined. Then by the definition of $(-)^{\#}$, it follows that $(h \cdot f)^{\#}(t_i)$ is defined for all $i = 1, \ldots, n$ and $(h \cdot f)^{\#}(t_i) \leq (h \cdot f)^{\#}(t_j)$ holds for all $i \leq j$ in Δ. By induction we know that all $f^{\#}(t_i)$ are defined and by item (1) that

$$h \cdot f^{\#}(t_i) = (h \cdot f)^{\#}(t_i) \leq (h \cdot f)^{\#}(t_j) = h \cdot f^{\#}(t_i)$$

holds for all $i \leq j$ in Δ. Since h is an embedding, we therefore obtain $f^{\#}(t_i) \leq f^{\#}(t_j)$ for all $i \leq j$ in Δ, whence $f^{\#}(t)$ defined. □

Proposition 3.22. *Every variety is closed under filtered colimits in* $\mathsf{Alg}\,\Sigma$.

In other words, the full embedding $E\colon \mathcal{V} \hookrightarrow \mathsf{Alg}\,\Sigma$ creates filtered colimits.

Proof. Let $\mathcal{V}$ be a variety of Σ-algebras. Let $D\colon \mathscr{D} \to \mathsf{Alg}\,\Sigma$ be a filtered diagram having colimit $c_d\colon Dd \to A$ ($d \in \mathsf{obj}\,\mathscr{D}$). It suffices to show that every inequation in context $\Gamma \vdash s \leq t$ satisfied by every algebra Dd is also satisfied by A. Let $f\colon \Gamma \to A$ be a monotone interpretation. Since Γ is finite, f factorizes, for some $d \in \mathsf{obj}\,\mathscr{D}$, through c_d via a monotone map $\bar{f}\colon \Gamma \to Dd$: in symbols, $c_d \cdot \bar{f} = f$. Since Dd satisfies the given inequation in context, we know that $\bar{f}^{\#}(s)$ and $\bar{f}^{\#}(t)$ are defined and that $\bar{f}^{\#}(s) \leq \bar{f}^{\#}(t)$ in Dd. By Lemma 3.21 we conclude that

$$f^{\#}(s) = (c_d \cdot \bar{f})^{\#}(s) = c_d \cdot \bar{f}^{\#}(s) \qquad \text{and} \qquad f^{\#}(t) = (c_d \cdot \bar{f})^{\#}(t) = c_d \cdot \bar{f}^{\#}(t)$$

are defined. Using the monotonicity of c_d, we obtain

$$f^{\#}(s) = c_d \cdot \bar{f}^{\#}(s) \leq c_d \cdot \bar{f}^{\#}(t) = f^{\#}(t)$$

as desired. □

Corollary 3.23. *The forgetful functor of a variety into* Pos *creates filtered colimits.*

Indeed, the forgetful functor of a variety $\mathcal{V}$ is a composite of the inclusion $\mathcal{V} \hookrightarrow \mathsf{Alg}\,\Sigma$ and the forgetful functor of $\mathsf{Alg}\,\Sigma$, both of which create filtered colimits.

Proposition 3.24. *Every variety of Σ-algebras is a reflective subcategory of* $\mathsf{Alg}\,\Sigma$ *closed under subalgebras.*

Proof. We are going to prove below that every variety $\mathcal{V} = \mathsf{Alg}\,(\Sigma, \mathcal{I})$ is closed in $\mathsf{Alg}\,\Sigma$ under products and subalgebras, whence it is closed under all limits. We also know from Proposition 3.22 that $\mathcal{V}$ is closed under filtered colimits in $\mathsf{Alg}\,\Sigma$. Being a full subcategory of the locally finitely presentable category $\mathsf{Alg}\,\Sigma$ (Remark 3.20(4)), $\mathcal{V}$ is reflective by the reflection theorem for locally presentable categories (Adámek and Rosický, 1994, Cor. 2.48).

(1) $\mathsf{Alg}\,(\Sigma, \mathcal{I})$ is closed under products in $\mathsf{Alg}\,\Sigma$. Indeed, given $A = \prod_{i \in I} A_i$ with projections $\pi_i\colon A \to A_i$ and a monotone interpretation $f\colon \Gamma \to A$, we prove for every term $s \in \mathscr{T}(\Gamma)$ that $f^{\#}(s)$ is defined if and only if so is $(\pi_i \cdot f)^{\#}(s)$ for all $i \in I$. This is done by structural induction: for $s \in |\Gamma|$ there is nothing to prove. Suppose that $s = \sigma(t_j)$ for some $\sigma \in \Sigma_\Delta$ and $t_j \in \mathscr{T}(\Gamma)$, $j \in \Delta$.

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Then $f^{\#}(s)$ is defined iff $j \leq k$ in Δ implies $f^{\#}(t_j) \leq f^{\#}(t_k)$ in A. Equivalently, $j \leq k$ in Δ implies $\pi_i \cdot f^{\#}(t_j) \leq \pi_i \cdot f^{\#}(t_k)$ in A_i for all $i \in I$ because the π_i are monotone and jointly order-reflecting, i.e. for every $x, y \in A$ we have $x \leq y$ iff $\pi_i(x) \leq \pi_i(y)$ for all $i \in I$. By Lemma 3.21, we have, again equivalently, that $(\pi_i \cdot f)^{\#}(t_j) \leq (\pi_i \cdot f)^{\#}(t_k)$ since every π_i is a homomorphism.

We now prove that A satisfies every inequation $\Gamma \vdash s \leq t$ in $\mathcal{I}$, as claimed. Let $f\colon \Gamma \to A$ be a monotone interpretation. We have that $(\pi_i \cdot f^{\#})(s)$ and $(\pi_i \cdot f^{\#})(t)$ are defined and $\pi_i \cdot f^{\#}(s) \leq \pi_i \cdot f^{\#}(t)$ for all $i \in I$, using Lemma 3.21 and since all A_i satisfy the given inequation in context. Using again that the π_i are jointly order-reflecting, we obtain $f^{\#}(s) \leq f^{\#}(t)$, as required.

(2) $\mathsf{Alg}\,(\Sigma, \mathcal{I})$ is closed under subalgebras in $\mathsf{Alg}\,\Sigma$. Indeed, let $m\colon B \hookrightarrow A$ be a Σ-homomorphism carried by an embedding. For every inequation $\Gamma \vdash s \leq t$ in $\mathcal{I}$, we prove that B satisfies it. For a monotone interpretation $f\colon \Gamma \to B$, we see that $(m \cdot f)^{\#}(s)$ and $(m \cdot f)^{\#}(t)$ are defined and $(m \cdot f)^{\#}(s) \leq (m \cdot f)^{\#}(t)$ since A satisfies the given inequation in context. By Lemma 3.21, we obtain that $f^{\#}(s)$ and $f^{\#}(t)$ are defined and

$$m \cdot f^{\#}(s) = (m \cdot f)^{\#}(s) \leq (m \cdot f)^{\#}(t) = m \cdot f^{\#}(s).$$

Since m is an embedding, it follows that $f^{\#}(s) \leq f^{\#}(t)$. $\square$

Corollary 3.25. *The category* $\mathsf{Alg}_{\mathsf{c}}\,\Sigma$ *of all coherent* Σ*-algebras is a reflective subcategory of* $\mathsf{Alg}\,\Sigma$*.*

Indeed, this follows using Example 3.19(3).

Example 3.26. Unlike in classical general algebra a variety need not be regular-epireflective in $\mathsf{Alg}\,\Sigma$. To see this recall from Example 3.12 the signature Σ with a binary operation symbol σ whose arity is a 2-chain. Consider $\mathsf{Alg}_{\mathsf{c}}\,\Sigma$ as a variety of Σ-algebras (see Example 3.19(3)). Then the reflection of the free Σ-algebra $T_\Sigma \Gamma$ in $\mathcal{V}$ is its embedding in the free coherent Σ-algebra $T^{\mathsf{c}}_\Sigma \Gamma$ (see Example 3.17), which is not a regular epimorphism being a monomorphism but not an isomorphism, as explained in Example 3.12.

Remark 3.27. (1) A concrete category over Pos is a category $\mathcal{V}$ together with a faithful functor $U_{\mathcal{V}}\colon \mathcal{V} \to \mathsf{Pos}$. We say that $\mathcal{V}$ is *concretely isomorphic* to a concrete category $U_{\mathcal{W}}\colon \mathcal{W} \to \mathsf{Pos}$ if there is an isomorphism $I\colon \mathcal{V} \to \mathcal{W}$ such that $U_{\mathcal{V}} = W_{\mathcal{W}} \cdot I$ (cf. Remark 3.6(1)).

(2) In the proof of Theorem 3.28, we will apply Beck's Monadicity Theorem (MacLane, 1998, Thm. VI.7.1). This makes use of the notion of a *split coequalizer*: a morphism $c\colon B \to C$ is a split coequalizer of a parallel pair $f, g\colon A \rightrightarrows B$ if there are morphisms s and t with types vizualized as

$$A \underset{\underset{t}{\longleftarrow}}{\overset{f}{\underset{g}{\rightrightarrows}}} B \underset{\underset{s}{\longleftarrow}}{\overset{c}{\longrightarrow}} C \qquad \text{such that} \qquad \begin{array}{ll} c \cdot f = c \cdot g, & c \cdot s = \mathsf{id}_C, \\ f \cdot t = \mathsf{id}_B, & g \cdot t = s \cdot c. \end{array} \tag{5}$$

Note that this implies that c is an absolute coequalizer of f and g (i.e. a coequalizer that is preserved by every functor).

Beck's Monadicity Theorem states that for a right adjoint functor $U\colon \mathscr{C} \to \mathsf{Pos}$ with induced monad $\mathbb{T}$, the category $\mathscr{C}$ is concretely isomorphic to $\mathsf{Pos}^{\mathbb{T}}$ (i.e. U is *strictly monadic*; note that MacLane simply calls this monadic) if and only if U creates coequalizers of U*-split pairs*; these are parallel pairs $f, g\colon A \to B$ in $\mathscr{C}$ such that the pair Uf, Ug has a split coequalizer c in Pos. In more detail, there exists a unique morphism $c'\colon B \to C$ in $\mathscr{C}$ such that $Uc' = c$ and, moreover, c' is a coequalizer of the pair f, g in $\mathscr{C}$.

Theorem 3.28. *For every variety, the forgetful functor to* Pos *is strictly monadic.*

Proof. Given a variety $\mathcal{V}$ of Σ-algebras we prove that the forgetful functor $U\colon \mathcal{V} \to \mathsf{Pos}$ is strictly monadic (cf. Remark 3.27(2)).

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(1) The functor U is a right adjoint because it is the composite of the embedding $E\colon \mathcal{V} \to \mathsf{Alg}\,\Sigma$ and the forgetful functor $V\colon\ \mathsf{Alg}\,\Sigma \to \mathsf{Pos}$: the functor E is a right adjoint by Proposition 3.24 and V is one by Proposition 3.8.

(2) Let $f, g\colon A \to B$ be a U-split pair of homomorphisms in $\mathcal{V}$. For every $\sigma \in \Sigma_\Gamma$, there exists a unique operation $\sigma_C\colon \mathsf{Pos}_0(\Gamma, C) \to C$ making c a homomorphism:

$$\begin{array}{ccc} \mathsf{Pos}_0(\Gamma,B) & \xrightarrow{\ \sigma_B\ } & B \\ {\scriptstyle c\cdot(-)}\big\downarrow & & \big\downarrow{\scriptstyle c} \\ \mathsf{Pos}_0(\Gamma,C) & \xrightarrow{\ \sigma_C\ } & C \end{array}$$

Indeed, let us define σ_C by

$$\sigma_C(h) = c \cdot \sigma_B(i \cdot h) \qquad \text{for all } h\colon \Gamma \to C.$$

Then c is a homomorphism since $\sigma_C(c \cdot k) = c \cdot \sigma_B(k)$ for every $k\colon \Gamma \to B$:

$$\begin{aligned} c \cdot \sigma_B(k) &= c \cdot \sigma_B(f \cdot j \cdot k) && \text{since } f \cdot j = \mathsf{id} \\ &= c \cdot f \cdot \sigma_A(j \cdot k) && f \text{ a homomorphism} \\ &= c \cdot g \cdot \sigma_A(j \cdot k) && \text{since } c \cdot f = c \cdot g \\ &= c \cdot \sigma_B(g \cdot j \cdot k) && g \text{ a homomorphism} \\ &= c \cdot \sigma_B(i \cdot c \cdot k) && \text{since } g \cdot j = i \cdot c \\ &= \sigma_C(c \cdot k). \end{aligned}$$

Conversely, if C has an algebra structure making c a homomorphism, then the above formula holds since $c \cdot i = \mathsf{id}$:

$$\sigma_C(h) = \sigma_C(c \cdot i \cdot h) = c \cdot \sigma_B(i \cdot h).$$

Furthermore, C lies in $\mathcal{V}$. To verify this, we prove that whenever an inequation $\Gamma \vdash s \le t$ is satisfied by B, then the same holds for the algebra C. Given a monotone interpretation $h\colon \Gamma \to C$ such that $h^\#(s)$ and $h^\#(t)$ are defined, we prove $h^\#(s) \le h^\#(t)$.

For the monotone interpretation $i \cdot h\colon \Gamma \to B$ we have that $(i \cdot h)^\#(s)$ and $(i \cdot h)^\#(t)$ are defined and that $(i \cdot h)^\#(s) \le (i \cdot h)^\#(t)$ since B lies in $\mathcal{V}$. Since c is a homomorphism, we conclude using Lemma 3.21 and that $c \cdot i = \mathsf{id}_C$ that

$$h^\#(s) = (c \cdot i \cdot h)^\#(s) = c \cdot (i \cdot h)^\#(s)$$

is defined and similarly for $h^\#(t)$. Then we have

$$h^\#(s) = c \cdot (i \cdot h)^\#(s) \le c \cdot (i \cdot h)^\#(t) = h^\#(t)$$

since c is monotone, as desired.

Finally, we prove that c is a coequalizer of f and g in $\mathcal{V}$. We already know that c is a coequalizer in Pos. Given a homomorphism $d\colon B \to D$ such that $d \cdot f = d \cdot g$ we therefore obtain a unique monotone map $d'\colon C \to D$ such that $d' \cdot c = d$. It remains to prove that d' is a homomorphism. Given $\sigma \in \Sigma_\Gamma$ we consider the following diagram:

$$\begin{array}{ccccc} & \mathsf{Pos}_0(\Gamma,B) & \xrightarrow{\ \sigma_B\ } & B & \\ & {\scriptstyle c\cdot(-)}\big\downarrow & & \big\downarrow{\scriptstyle c} & \\ {\scriptstyle d\cdot(-)} & \mathsf{Pos}_0(\Gamma,C) & \xrightarrow{\ \sigma_C\ } & C & {\scriptstyle d} \\ & {\scriptstyle d'\cdot(-)}\big\downarrow & & \big\downarrow{\scriptstyle d'} & \\ & \mathsf{Pos}_0(\Gamma,D) & \xrightarrow{\ \sigma_D\ } & D & \end{array}$$

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The left-hand and right-hand parts clearly commute, and the upper square and outside do since c and d are homomorphisms. Thus, the desired lower square commutes when precomposed by $c \cdot (-)$. This is an epimorphism since it is a coequalizer, being the image of the absolute coequalizer c under the hom-functor $\mathsf{Pos}_0(\Gamma, -)$. Hence, the desired lower square commutes. □

Definition 3.29. *Given a variety $\mathcal{V}$, the left adjoint of $U\colon \mathcal{V} \to \mathsf{Pos}$ assigns to every poset X the free algebra of $\mathcal{V}$ on X. The ensuing monad is called the* free-algebra monad *of the variety and is denoted by $\mathbb{T}_{\mathcal{V}}$.*

Remark 3.30. The monad $\mathbb{T}_{\mathcal{V}}$ is finitary, which means that its underlying endofunctor preserves filtered colimits. Indeed, the underlying endofunctor is UF, where $F\colon \mathsf{Pos} \to \mathcal{V}$ is the free-algebra functor. Since F is left adjoint, it preserves (filtered) colimits, and U is finitary by Corollary 3.23.

Corollary 3.31. *Every variety $\mathcal{V}$ is concretely isomorphic to the Eilenberg-Moore category $\mathsf{Pos}^{\mathbb{T}_{\mathcal{V}}}$.*

Example 3.32. (1) Recall the variety of internal semilattices considered in Example 3.19(5). It is well known (and easy to show) that the free internal semilattice on a poset X is formed by the poset $C_\omega X$ of its finitely generated convex subsets. Here, a subset $S \subseteq X$ is *convex* if $x, y \in S$ implies that every z such that $x \leq z \leq y$ lies in S, too, and *finitely generated* means that S is the convex hull of a finite subset of X. The order on $C_\omega X$ is the Egli-Milner order, which means that for $S, T \in C_\omega X$ we have

$$S \leq T \quad \text{iff} \quad \forall s \in S.\, \exists t \in B.\, s \leq t \wedge \forall t \in T.\, \exists s \in S.\, s \leq t.$$

The constant 0 is the empty set, and the operation $+$ is the join w.r.t. inclusion, explicity, $S + T$ is the convex hull of $S \cup T$ for all $S, T \in C_\omega X$. One readily shows that $+$ is monotone w.r.t. the Egli-Milner order and that $C_\omega X$ with the universal monotone map $x \mapsto \{x\}$ is a free internal semilattice on X. Thus we see that C_ω is a monad on Pos and Pos^{C_ω} is (isomorphic to) the category of internal semilattices in Pos.

(2) Denote by D_ω the monad of free join-semilattices. It assigns to every poset X the set of all finitely generated, downwards closed subsets of X ordered by inclusion. Here a downwards closed subset $S \subseteq X$ is *finitely generated* if there are $x_1, \ldots, x_n \in S$, $n \in \mathbb{N}$, such that $S = \bigcup_{i=1}^n x_i{\downarrow}$. The category Pos^{D_ω} is equivalent to that of join-semilattices, see Example 3.19(6).

(3) Similarly, the monad D_ω^b generated by the variety of bounded-join semilattices (Example 3.19(7)) assigns to a poset X the set of finitely generated downwards closed *bounded* subsets of X, ordered by inclusion.

(4) The *subdistribution monad* $\mathcal{S}$ on Pos assigns to each poset X the set of finitely supported subdistributions on X, i.e. finitely supported $[0, 1]$-valued measures; these may be represented as maps $\mu\colon X \to [0, 1]$ such that $\{x \in X \mid \mu(x) > 0\}$ is finite and $\sum_{x \in X} \mu(x) \leq 1$. The ordering on $\mathcal{S}X$ is given by $\mu \leq \nu$ iff $\mu(x) \leq \nu(x)$ for all $x \in X$. This monad is generated by the variety of ordered subconvex algebras as described in Example 3.19(8). A variant of this claim with complete partial orders instead of Pos as the base category has been proved by Jones and Plotkin (1989); a direct proof for Pos is given by Ford et al. (2021*a*).

Corollary 3.33. *The forgetful functors $U\colon \mathsf{Alg}\,\Sigma \to \mathsf{Pos}$, $U_c\colon \mathsf{Alg_c}\,\Sigma \to \mathsf{Pos}$ are strictly monadic.*

Note that the corresponding monads are the free-(coherent-)Σ-algebra monads given by $T_\Sigma X$ and $T_\Sigma^{\mathsf{c}} X$, respectively (cf. Propositions 3.8 and 3.9).

4. Finitary Monads

Let $\mathbb{T}$ be a finitary monad on Pos. We present a variety $\mathcal{V}_{\mathbb{T}}$ such that the mapping $\mathbb{T} \mapsto \mathcal{V}_{\mathbb{T}}$ is inverse to the assignment $\mathcal{V} \to \mathbb{T}_{\mathcal{V}}$ of a variety to its free-algebra monad (up to isomorphism). Moreover, we prove that there is a completely analogous bijection between enriched finitary monads and varieties of coherent algebras.

Remark 4.1. Recall, e.g. from Moggi (1991), that monads can, equivalently, be presented by Kleisli triples; this notion goes back to Manes (1976, Exercise 12), who called it *algebraic theory in extension form*.

(1) A *Kleisli triple* on Pos consists of (a) a self map $X \mapsto TX$ on the class of all posets, (b) an assignment of a monotone map $\eta_X : X \to TX$ to every poset, and (c) an assignment of a monotone map $f^* : TX \to TY$ to every monotone map $f : X \to TY$, which satisfies

$$\eta_X^* = \mathsf{id}_{X^*} \tag{6}$$

$$f^* \cdot \eta_X = f \tag{7}$$

$$g^* \cdot f^* = (g^* \cdot f)^* \tag{8}$$

for all posets X and all monotone maps $f : X \to TY$ and $g : Y \to TZ$.

(2) A morphism into another Kleisli triple $(T', \eta', (-)^+)$ is a family $\varphi_X : TX \to T'X$ of monotone maps such that the diagrams below commute for all posets X and all monotone functions $f : X \to TY$:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\eta_X} & & \searrow^{\eta'_X} & \\ TX & & \xrightarrow{\varphi_X} & & T'X \end{array} \qquad \begin{array}{ccc} TX & \xrightarrow{\varphi_X} & T'X \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (\varphi_Y \cdot f)^+} \\ TY & \xrightarrow{\varphi_Y} & T'Y \end{array}$$

(3) Every monad $\mathbb{T}$ defines a Kleisli triple $(T, \eta, (-)^*)$ by

$$f^* = TX \xrightarrow{Tf} TTY \xrightarrow{\mu_Y} TY.$$

Every monad morphism $\varphi : \mathbb{T} \to \mathbb{T}'$ defines a morphism $\varphi_X : TX \to T'X$ of Kleisli triples. The resulting functor from the category of monads to the category of Kleisli triples is an equivalence functor.

Definition 4.2. *Let $\mathbb{T}$ be a finitary monad on* Pos. *The* variety $\mathcal{V}_{\mathbb{T}}$ associated to $\mathbb{T}$ *on* Pos *has the signature*

$$\Sigma_\Gamma = |T\Gamma| \qquad \textit{for every } \Gamma \in \mathsf{Pos_f}.$$

That is, operations of arity Γ are elements of the poset $T\Gamma$. For each $\Gamma \in \mathsf{Pos_f}$, we impose inequations of the following types:

(1) *$\Gamma \vdash \sigma \le \tau$ for all $\sigma \le \tau$ in $T\Gamma$ (with operations used as terms as per Convention 3.11);*

(2) *$\Gamma \vdash k^*(\sigma) = \sigma(k)$ for all $\Delta \in \mathsf{Pos_f}$, monotone maps $k : \Delta \to T\Gamma$ and $\sigma \in T\Delta$;*

(3) *$\Gamma \vdash \eta_\Gamma(x) = x$ for all $x \in \Gamma$ (again with the operation $\eta_\Gamma(x) \in T\Gamma$ used as a term).*

Example 4.3. For every poset X, the poset TX carries the following structure of an algebra of $\mathcal{V}_{\mathbb{T}}$. Given $\sigma \in T\Gamma$, we define the operations $\sigma_{TX} : \mathsf{Pos}_0(\Gamma, TX) \to TX$ by

$$\sigma_{TX}(f) = f^*(\sigma) \qquad \text{for } f : \Gamma \to TX.$$

It then follows that the evaluation map $f^\# : \mathscr{T}(\Gamma) \to |TX|$ coincides with f^* on operation symbols (used as terms as per Convention 3.11):

$$f^\#(\sigma) = f^*(\sigma) \tag{9}$$

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for all $\sigma \in T\Gamma$. Indeed, for $|\Gamma| = \{x_1, \ldots, x_n\}$ we have

$$\begin{aligned} f^\#(\sigma) &= f^\#(\sigma(x_1, \ldots, x_n)) && \text{Conv. 3.11} \\ &= \sigma_{TX}(f^\#(x_1), \ldots, f^\#(x_n)) && \text{def. of } f^\# \\ &= \sigma_{TX}(f(x_1), \ldots, f(x_n)) && \text{def. of } f^\# \\ &= \sigma_{TX}(f) \\ &= f^*(\sigma) && \text{def. of } \sigma_{TX}. \end{aligned}$$

We now verify that the Σ-algebra TX lies in $\mathcal{V}_{\mathbb{T}}$. It satisfies the inequations of type (1) because f^* is monotone: given $\sigma \le \tau$ in $T\Gamma$, we have $f^\#(\sigma) = f^*(\sigma) \le f^*(\tau) = f^\#(\tau)$. Further, it satisfies the inequations of type (2) since for every monotone map $k\colon \Delta \to T\Gamma$ we know that $f^\#(k^*(\sigma))$ is defined by Example 3.14(2), and we have

$$\begin{aligned} f^\#(k^*(\sigma)) &= f^* \cdot k^*(\sigma) && \text{by (9)} \\ &= (f^* \cdot k)^*(\sigma) && \text{by (8)} \\ &= \sigma_{TX}(f^* \cdot k) && \text{def. of } \sigma_{TX} \\ &= \sigma_{TX}(f^\# \cdot k) && \text{by (9)} \\ &= f^\#(\sigma(k)) && \text{def. of } f^\#. \end{aligned}$$

Finally, we verify that TX satisfies the inequations of type (3). Indeed, given a monotone interpretation $k\colon \Gamma \to TX$, we know that $k^\#(\eta_\Gamma(x))$ and $k^\#(x)$ are defined, and the desired equality $k^\#(\eta_\Gamma(x)) = k^\#(x)$ follows immediately from Equation (9) using that $k^* \cdot \eta_\Gamma = k$ (see Remark 4.1(1)). We conclude that TX lies in $\mathcal{V}_{\mathbb{T}}$, as claimed.

Theorem 4.4. *Every finitary monad $\mathbb{T}$ on* Pos *is the free-algebra monad of its associated variety $\mathcal{V}_{\mathbb{T}}$.*

Proof. (1) We first prove that the algebra TX of Example 4.3 is a free algebra of $\mathcal{V}_{\mathbb{T}}$ w.r.t. the monad unit $\eta_X\colon X \to TX$.

(1a) First, suppose that $X = \Gamma$ is an object of Pos_f. Given an algebra A of $\mathcal{V}_{\mathbb{T}}$ and a monotone map $f\colon \Gamma \to A$, we are to prove that there exists a unique homomorphism $\bar{f}\colon T\Gamma \to A$ such that $f = \bar{f} \cdot \eta$.

Indeed, given $\sigma \in T\Gamma$, define $\bar{f}$ by

$$\bar{f}(\sigma) = \sigma_A(f).$$

Then $\bar{f} \cdot \eta_\Gamma = f$ since for every $x \in \Gamma$, we have

$$\begin{aligned} \bar{f} \cdot \eta_\Gamma(x) &= \bar{f}(\eta_\Gamma(x)) \\ &= \eta_\Gamma(x)_A(f) && \text{def. of } \bar{f} \\ &= \eta_\Gamma(x)_A(f^\# \cdot u_\Gamma) && \text{def. of } f^\# \\ &= f^\#(\eta_\Gamma(x)(u_\Gamma)) && \text{def. of } f^\# \\ &= f^\#(x) && A \text{ satisfies } \Gamma \vdash \eta_\Gamma(x) = x \\ &= f(x) && \text{def. of } f^\#. \end{aligned}$$

Moreover, $\bar{f}$ is a monotone function: if $\sigma \le \tau$ in $T\Gamma$, then use the fact that A satisfies the inequation $\Gamma \vdash \sigma \le \tau$ to obtain

$$\sigma_A(f) = f^\#(\sigma) \le f^\#(\tau) = \tau_A(f).$$

We now verify that $\bar{f}$ is a homomorphism: given $\tau \in \Sigma_\Delta$, we will prove that the following square commutes:

$$\begin{array}{ccc} \mathsf{Pos}_0(\Delta, T\Gamma) & \xrightarrow{\tau_{T\Gamma}} & T\Gamma \\ {\scriptstyle \bar{f}\cdot(-)}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ \mathsf{Pos}_0(\Delta, A) & \xrightarrow[\tau_A]{} & A \end{array}$$

Indeed, for every monotone map $k\colon \Delta \to T\Gamma$ we have that $f^\#$ is defined in $k^*(\tau)$ by Example 3.14(2), and we therefore obtain:

$$\begin{aligned} \bar{f}(\tau_{T\Gamma}(k)) &= \bar{f}(k^*(\tau)) && \text{def. of } \tau_{T\Gamma} \\ &= (k^*(\tau))_A(f) && \text{def. of } \bar{f} \\ &= f^\#(k^*(\tau)) && \text{Def. 3.13} \\ &= f^\#(\tau(\hat{k})) && A \text{ satisfies } \Gamma \vdash k^*(\tau) = \tau(\hat{k}) \\ &= \tau_A(f^\#(k)) && \text{def. of } f^\# \\ &= \tau_A(\bar{f}\cdot k). \end{aligned}$$

For the last step, we use again the definition of $f^\#$ to obtain that for every $x \in |\Delta|$ the operation symbol $\sigma = k(x)$, considered as the term $\sigma(y_1, \ldots, y_k)$ where $|\Gamma| = \{y_1, \ldots, y_k\}$ (Convention 3.11), satisfies

$$\begin{aligned} f^\#(\sigma(y_1, \ldots, y_k)) &= \sigma_A(f^\#(y_1), \ldots, f^\#(y_k)) \\ &= \sigma_A(f(y_1), \ldots, f(y_k)) \\ &= \sigma_A(f) = \bar{f}(\sigma). \end{aligned}$$

Since $\sigma = k(x)$, this gives the desired $\bar{f}\cdot k$ when we let x range over Δ.

As for uniqueness, suppose that $\bar{f}\colon T\Gamma \to A$ is a homomorphism such that $f = \bar{f}\cdot\eta_\Gamma$. The above square commutes for $\Delta = \Gamma$ which applied to $\eta_\Gamma \in \mathsf{Pos}(\Gamma, T\Gamma)$ yields for every $\sigma \in |T\Gamma|$:

$$\begin{aligned} \bar{f}(\sigma) &= \bar{f}(\eta_\Gamma^*(\sigma)) && \text{by (6)} \\ &= \bar{f}(\eta_\Gamma^\#(\sigma)) && \text{by (9)} \\ &= \bar{f}(\sigma_{T\Gamma}(\eta_\Gamma)) && \text{def. of } \eta_\Gamma^\# \\ &= \sigma_A(\bar{f}\cdot\eta_\Gamma) && \bar{f} \text{ a homomorphism} \\ &= \sigma_A(f) && \text{since } \bar{f}\cdot\eta_\Gamma = f, \end{aligned}$$

as required.

(1b) Now, let X be an arbitrary poset. Express it as a filtered colimit $X = \mathrm{colim}_{i\in I}\,\Gamma_i$ of objects from $\mathsf{Pos_f}$. The free algebra on X is then a filtered colimit of the corresponding diagram of the Σ-algebras $T\Gamma_i$ ($i \in I$). Indeed, that $TX = \mathrm{colim}\, T\Gamma_i$ in Pos follows from T preserving filtered colimits. That this colimit lifts to $\mathcal{V}$ follows from the forgetful functor of $\mathcal{V}$ creating filtered colimits (Corollary 3.23).

(2) To conclude the proof, we apply Remark 4.1. Our given monad and the monad $\mathbb{T}_\mathcal{V}$ of the associated variety share the same object assignment $X \mapsto TX = T_\mathcal{V}X$ for an arbitrary poset X, and the same universal map η_X, as shown in part (1). It remains to prove that for every morphism $f\colon X \to TY$ in Pos the homomorphism $h^* = \mu_Y \cdot Th$ extending h in $\mathsf{Pos}^\mathbb{T}$ is a Σ-homomorphism $h^*\colon TX \to TY$ of the corresponding Σ-algebras of Example 4.3. Then $\mathbb{T}$ and $\mathbb{T}_\mathcal{V}$ also share the operator $h \mapsto h^*$. Thus given $\sigma \in \Sigma_\Gamma$ we are to prove that the following square commutes:

$$\begin{array}{ccc} \mathsf{Pos}_0(\Gamma, TX) & \xrightarrow{\sigma_{TX}} & TX \\ {\scriptstyle h^*\cdot(-)}\downarrow & & \downarrow{\scriptstyle h^*} \\ \mathsf{Pos}_0(\Gamma, TY) & \xrightarrow[\sigma_{TY}]{} & TY \end{array}$$

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Indeed, given $f\colon \Gamma \to TX$, we have

$$\begin{aligned} h^* \cdot \sigma_{TX}(f) &= h^* \cdot f^*(\sigma) && \text{def. of } \sigma_A \\ &= (h^* \cdot f)^*(\sigma) && \text{by (8)} \\ &= \sigma_{TY}(h^* \cdot f) && \text{def. of } \sigma_{TY} \end{aligned}$$

This completes the proof. □

In the following corollary, we consider varieties independently of their presentation. In other words, concretely isomorphic varieties (Remark 3.27(1)) are identified. For example, join semilattices form the same variety as meet semilattices or as commutative idempotent monoids.

Corollary 4.5. *Finitary monads on* Pos *correspond bijectively, up to monad isomorphism, to finitary varieties of ordered algebras.*

Indeed, the assignment of the associated variety $\mathcal{V}_{\mathbb{T}}$ to every finitary monad $\mathbb{T}$ is essentially inverse to the asignment of the free-algebra monad $\mathbb{T}_{\mathcal{V}}$ to every variety $\mathcal{V}$. To see this, recall that every variety $\mathcal{V}$ is concretely isomorphic to the category $\mathsf{Pos}^{\mathbb{T}_{\mathcal{V}}}$ (Corollary 3.31). Conversely, every finitary monad $\mathbb{T}$ is isomorphic to $\mathbb{T}_{\mathcal{V}}$ for the associated variety (Theorem 4.4).

Proposition 4.6. *If* $\mathbb{T}$ *is an enriched finitary monad on* Pos, *then the algebras of its associated variety* $\mathcal{V}_{\mathbb{T}}$ *are coherent. Conversely, for every variety* $\mathcal{V}$ *of coherent algebras, the free-algebra monad* $\mathbb{T}_{\mathcal{V}}$ *is enriched.*

Proof. For the first claim, let $\mathbb{T}$ be enriched. Then the Σ-algebra TX of Example 4.3 is coherent: Given an operation symbol $\sigma \in \Sigma_\Gamma$ and monotone interpretations $f \le g$ in $\mathsf{Pos}(\Gamma, TX)$, we have $Tf \le Tg$, and hence $f^* = \mu_{TX} \cdot Tf \le \mu_{TX} \cdot Tg = g^*$ because $\mathbb{T}$ is enriched. Therefore, $f^*(\sigma) \le g^*(\sigma)$. That is,

$$\sigma_{TX}(f) \le \sigma_{TX}(g).$$

For every algebra A of the variety $\mathcal{V}_{\mathbb{T}}$ we have the unique Σ-homomorphism $k\colon TA \to A$ such that $k \cdot \eta_A = \mathsf{id}_A$ (since TA is a free Σ-algebra in $\mathcal{V}_{\mathbb{T}}$; see item (1) in the proof of Theorem 4.4). The coherence of TA implies the coherence of A: given $f_1 \le f_2$ in $\mathsf{Pos}(\Gamma, A)$, we verify $\sigma_A(f_1) \le \sigma_A(f_2)$ by applying the commutative square below to $\eta_A \cdot f_i$:

$$\begin{array}{ccc} \mathsf{Pos}(\Gamma, TA) & \xrightarrow{\sigma_{TA}} & TA \\ {\scriptstyle k\cdot(-)}\big\downarrow & & \big\downarrow{\scriptstyle k} \\ \mathsf{Pos}(\Gamma, A) & \xrightarrow[\sigma_A]{} & A \end{array}$$

We obtain $\sigma_A(f_i) = \sigma_A(k \cdot \eta_A \cdot f_i) = k \cdot \sigma_{TA}(\eta_A \cdot f_i)$; by monotonicity of composition in Pos and of σ_{TA} as established above, this implies $\sigma_A(f_1) \le \sigma_A(f_2)$ as desired.

Conversely, let $\mathcal{V}$ be a variety of coherent Σ-algebras. Given $f_1 \le f_2$ in $\mathsf{Pos}(X, Y)$, we prove that the free-algebra monad $\mathbb{T}_{\mathcal{V}}$ fulfils $T_{\mathcal{V}} f_1 \le T_{\mathcal{V}} f_2$. Let $e\colon E \hookrightarrow T_{\mathcal{V}}X$ be the subposet of all elements $t \in |T_{\mathcal{V}}X|$ such that $T_{\mathcal{V}} f_1(t) \le T_{\mathcal{V}} f_2(t)$. Since for $x \in X$ we know that $f_1(x) \le f_2(x)$, the poset E contains all elements $\eta_X(x)$. Moreover, E is closed under the operations of $T_{\mathcal{V}}X$: Suppose that $\sigma \in \Sigma_\Gamma$ and that $h\colon \Gamma \to T_{\mathcal{V}}X$ is a monotone map such that $h[\Gamma] \subseteq E$; we have to show that $\sigma_{T_{\mathcal{V}}X}(h) \in E$. Applying the commutative square

$$\begin{array}{ccc} \mathsf{Pos}(\Gamma, T_{\mathcal{V}}X) & \xrightarrow{\sigma_{T_{\mathcal{V}}X}} & T_{\mathcal{V}}X \\ {\scriptstyle T_{\mathcal{V}}f_i\cdot(-)}\big\downarrow & & \big\downarrow{\scriptstyle T_{\mathcal{V}}f_i} \\ \mathsf{Pos}(\Gamma, T_{\mathcal{V}}Y) & \xrightarrow[\sigma_{T_{\mathcal{V}}Y}]{} & T_{\mathcal{V}}Y \end{array}$$

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to h, we obtain

$$\begin{aligned} T_{\mathcal{V}}f_1(\sigma_{T_{\mathcal{V}}X}(h)) &= \sigma_{T_{\mathcal{V}}Y}(T_{\mathcal{V}}f_1 \cdot h) \\ &\le \sigma_{T_{\mathcal{V}}Y}(T_{\mathcal{V}}f_2 \cdot h) \\ &= T_{\mathcal{V}}f_2(\sigma_{T_{\mathcal{V}}X}(h)) \end{aligned}$$

using in the inequality that $\sigma_{T_{\mathcal{V}}Y}$ is monotone and, by assumption, $T_{\mathcal{V}}f_1(h) \le T_{\mathcal{V}}f_2(h)$; that is, $\sigma_{T_{\mathcal{V}}X}(h) \in E$, as desired.

We thus see that E is a Σ-subalgebra of $T_{\mathcal{V}}X$. Since $T_{\mathcal{V}}X$ is the free algebra of $\mathcal{V}$ w.r.t. η_X and the subalgebra E contains $\eta_X[X]$, it follows that $E = T_{\mathcal{V}}X$. This proves that $Tf_1 \le Tf_2$, as desired. □

Corollary 4.7. *Enriched finitary monads on* Pos *correspond bijectively, up to monad isomorphism, to finitary varieties of coherent ordered algebras.*

5. Enriched Lawvere Theories

Power (1999) proves that enriched finitary monads on Pos bijectively correspond to Lawvere Pos-theories. This is another way of proving Corollary 4.7. However, we believe that a precise verification of all details would not be simpler than our proof. Here we indicate this alternative proof.

Dual to Remark 2.2, *cotensors* $P \pitchfork X$ in a Pos-enriched category $\mathscr{T}$ are characterized by an enriched natural isomorphism $\mathscr{T}(-, P \pitchfork X) \cong \mathsf{Pos}(P, \mathscr{T}(-, X))$. If we restrict ourselves to finite posets P we speak about *finite cotensors*.

Definition 5.1 (Power 1999). *A* Lawvere Pos-theory *is a small enriched category* $\mathscr{T}$ *with finite cotensors together with an enriched identity-on-objects functor* $\iota\colon \mathsf{Pos}_{\mathsf{f}}^{\mathsf{op}} \to \mathscr{T}$ *which preserves finite cotensors.*

Example 5.2. Let $\mathcal{V}$ be a variety, and denote by $\mathbb{T}_{\mathcal{V}}$ its free-algebra monad on Pos. The following theory $\mathscr{T}_{\mathcal{V}}$ is the restriction of the Kleisli category of $\mathbb{T}_{\mathcal{V}}$ to $\mathsf{Pos}_{\mathsf{f}}$: objects are all arities, and morphisms from Γ to Γ' form the poset $\mathsf{Pos}(\Gamma', T_{\mathcal{V}}\Gamma)$. A composite of $f\colon \Gamma' \to \mathbb{T}_{\mathcal{V}}\Gamma$ and $g\colon \Gamma'' \to T_{\mathcal{V}}\Gamma'$ is $f^* \cdot g\colon \Gamma'' \to \mathbb{T}_{\mathcal{V}}\Gamma$ where $(-)^*$ is the Kleisli extension (see Remark 4.1(3)).

Theorem 5.3 (Power, 1999, Thm. 4.3). *There is a bijective correspondence between enriched finitary monads on* Pos *and Lawvere* Pos*-theories.*

Example 5.4. By inspecting Power's proof, we see that for the theory $\mathscr{T}_{\mathcal{V}}$ of Example 5.2, the corresponding monad is precisely the free-algebra monad $\mathbb{T}_{\mathcal{V}}$.

Remark 5.5. With every Lawvere Pos-theory $\mathscr{T}$, Power associates the category $\mathsf{Mod}\ \mathscr{T}$ of *models*, which are enriched functors $\bar{A}\colon \mathscr{T} \to \mathsf{Pos}$ preserving finite cotensors. Morphisms are all enriched natural transformations between models.

In Example 5.2, every algebra A of $\mathcal{V}$ yields a model $\bar{A}$ of $\mathscr{T}_{\mathcal{V}}$ by putting $\bar{A}(\Gamma) = \mathcal{V}(\mathbb{T}_{\mathcal{V}}\Gamma, A)$ and for $f\colon \Gamma' \to T_{\mathcal{V}}\Gamma$ we have

$$\bar{A}(f) = f^* \cdot (-)\colon \mathcal{V}(T_{\mathcal{V}}\Gamma, A) \to \mathcal{V}(\mathbb{T}_{\mathcal{V}}\Gamma', A).$$

The proof of Theorem 5.3 implies that these are, up to isomorphism, all models of $\mathscr{T}_{\mathcal{V}}$ and this yields an equivalence between $\mathcal{V}$ and $\mathsf{Mod}\ \mathscr{T}_{\mathcal{V}}$.

Thus, Corollary 4.7 can be proved by verifying that every Lawvere Pos-theory $\mathscr{T}$ is naturally isomorphic to $\mathscr{T}_{\mathcal{V}}$ for a variety of algebras, and the passage from $\mathbb{T}$ to $\mathcal{V}$ is inverse to the passage $\mathcal{V} \mapsto \mathscr{T}_{\mathcal{V}}$ of Example 5.4.

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In addition, Nishizawa and Power (2009) generalize the concept of Lawvere theory to a setting in which one may obtain an alternative proof of the non-coherent case (Corollary 4.5); we briefly indicate how. Again we believe that that proof would not be simpler than ours. The setting of op. cit. includes a symmetric monoidal closed category $\mathcal{V}$ that is locally finitely presentable in the enriched sense and a locally finitely presentable $\mathcal{V}$-category $\mathscr{A}$. For our purposes, $\mathcal{V} = \mathsf{Set}$ and $\mathscr{A} = \mathsf{Pos}$.

Definition 5.6 (Nishizawa and Power, 2009, Def. 2.1)**.** *A* Lawvere Pos-theory *for* $\mathcal{V} = \mathsf{Set}$ *is a small ordinary category* $\mathscr{T}$ *together with an ordinary identity-on-objects functor* $\iota\colon \mathsf{Pos}_{\mathsf{f}}^{\mathsf{op}} \to \mathscr{T}$ *preserving finite limits.*

Example 5.7. Every variety of (not necessarily coherent) algebras yields a theory $\mathscr{T}$ analogous to Example 5.2: the hom-set $\mathscr{T}(\Gamma, \Gamma')$ is $\mathsf{Pos}_0(\Gamma', \mathbb{T}_{\mathcal{V}}\Gamma)$.

Remark 5.8. Here, a model of a theory $\mathscr{T}$ is an ordinary functor $A\colon \mathscr{T} \to \mathsf{Set}$ such that $A \cdot \iota\colon \mathsf{Pos}_{\mathsf{f}}^{\mathsf{op}} \to \mathsf{Set}$ is naturally isomorphic to $\mathsf{Pos}(-, X)/\mathsf{Pos}_{\mathsf{f}}^{\mathsf{op}}$ for some poset X. The category $\mathsf{Mod}\,\mathscr{T}$ of models has ordinary natural transformations as morphisms.

Theorem 5.9 (Nishizawa and Power, 2009, Cor. 5.2)**.** *There is a bijective correspondence between ordinary finitary monads on* Pos *and Lawvere* Pos*-theories in the sense of Definition 5.6.*

6. Conclusion and Future Work

Classical varieties of algebras are well known to correspond to finitary monads on Set. We have investigated the analogous situation for the category of posets. It turns out that there are two reasonable variants: one considers either all (ordinary) finitary monads, or just the enriched ones, whose underlying endofunctor is locally monotone. (An orthogonal restriction, not considered here, is to require the monad to be strongly finitary, which corresponds to requiring the arities of operations to be discrete, see Adámek et al. 2021.) We have defined the concept of a variety of ordered algebras using signatures where arities of operation symbols are finite posets. We have proved that these varieties bijectively correspond to

(1) all finitary monads on Pos, provided that algebras are not required to have monotone operations,

(2) all enriched finitary monads on Pos for varieties of coherent algbras, i.e. those with monotone operations.

In both cases, “term” has the usual meaning in universal algebra, and varieties are classes presented by inequations in context.

Although we have concentrated entirely on posets, many features of our article can clearly be generalized to enriched locally λ-presentable categories and the question of a semantic presentation of (ordinary or enriched) λ-accessible monads. For example, what type of varieties corresponds to countably accessible monads on the category of metric spaces with distances at most one (and nonexpanding maps)? Such varieties will be related to Mardare et al.’s (2016) quantitative varieties (also called c-varieties by Mardare et al. 2017; Milius and Urbat 2019), probably extended by allowing non-discrete arities of operation symbols.

Rosický (2021) suggests another possibility of presenting finitary monads on Pos: by applying the functorial semantics by Linton (1969) to functors into Pos and taking the appropriate finitary variation in the case where those functors are finitary. We intend to pursue this idea in future work.

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