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POWER-ASSOCIATIVE ALGEBRAS IN WHICH EVERY SUBALGEBRA IS A LEFT IDEAL

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1. **Introduction.** By an L-algebra we mean a power-associative nonassociative algebra (not necessarily finite-dimensional) over a field F in which every subalgebra generated by a single element is a left ideal. An H-algebra is a power-associative algebra in which every subalgebra is an ideal. The H-algebras were characterized by D. L. Outcalt in [2]. Let S_α be the semigroup with cardinality α such that if $x, y \in S_\alpha$ then $xy=y$. Consider the algebra over a field F with basis S_α. Such an algebra is an L-algebra that is not an H-algebra unless S_α contains only one element. In this paper we will prove that an algebra A over a field F with char. $\neq 2$ is an L-algebra if and only if it is either an H-algebra or has a basis S_α where α is the dimension of A. Also, we will show that an algebra A has basis S_α for $\alpha>1$ if and only if A is the vector space sum $\{e\}+B$ where $e^2=e\neq 0$ and B is a zero algebra such that $be=eb-b=0$ for b in B.

2. **Preliminaries.** It is convenient to denote the algebra generated by x as $\{x\}$. If every $\{x\}$ is an ideal then for x in B a subalgebra of A and y in A, we have xy, yx in $\{x\}\subseteq B$. Hence, A is an H-algebra and we have proved

LEMMA 2.1. *If A is a power-associative algebra then A is an H-algebra if and only if every subalgebra generated by a single element is an ideal.*

Some of our results can be derived in a more general setting than that of L-algebras. Thus, we define a T-algebra as a power-associative algebra in which every subalgebra generated by a single element is either a right or a left ideal.

LEMMA 2.2. *If A is a T-algebra with identity element* 1 *then $A=\{1\}$.*

Proof. For y in A, we have $y=y1=1y$ so y is in $\{1\}$.

LEMMA 2.3. *If A is a T-algebra then $\{a\}$ is finite-dimensional.*

Proof. Suppose $a, a^2, \ldots, a^n$ are linearly independent for any n. Now $a^3=a^2a=aa^2$ is in $\{a^2\}$. But then a^3 is a linear combination of a finite number of elements of the form a^{2m}.

LEMMA 2.4. *If a is a nilpotent element in a T-algebra then $a^3=0$.*

Proof. Suppose $a^n=0$, $a^{n-1}\neq 0$ with $n\geq 4$. Let $m=n/2$ if n is even and $m=(n+1)/2$

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if n is odd. Then $m+1 \le n-1$ and $2m \ge n$. Hence, $(a^m)^2=0$ so $\{a^m\}$ is one-dimensional. Now $a^{m+1}=aa^m=a^m a$ is in $\{a^m\}$ so $a^{m+1}=\alpha a^m$ with $\alpha \ne 0$, α in F (the base field). Thus, $a^{m+t}=\alpha^t a^m$ and $0=a^{2m}=\alpha^m a^m$ a contradiction.

If A is power-associative with char. $\ne 2$, and if e is an idempotent in A, then $A=A_e(1)+A_e(\frac{1}{2})+A_e(0)$ where $A_e(\lambda)=\{x: xe+ex=2\lambda x\}$ (see [1]). Also, from [1] we have for x in $A_e(\lambda)$, $\lambda \ne \frac{1}{2}$ then $xe=ex=\lambda x$.

Define $x\cdot y=(xy+yx)/2$, $(x, y)=xy-yx$ and $(x, y, z)=(xy)z-x(yz)$. From [1], we know that:

$$A_e(1)A_e(0) = A_e(0)A_e(1) = 0.$$
$$A_e(\lambda)\cdot A_e(\lambda) \subseteq A_e(\lambda), \quad \lambda \ne \tfrac{1}{2}.$$
$$A_e(\tfrac{1}{2})\cdot A_e(\tfrac{1}{2}) \subseteq A_e(1)+A_e(0).$$
$$A_e(\lambda)\cdot A_e(\tfrac{1}{2}) \subseteq A_e(\tfrac{1}{2})+A_e(1-\lambda), \quad \lambda \ne \tfrac{1}{2}.$$

In any ring,

(1) $$(xy, z)+(yz, x)+(zx, y) = (x, y, z)+(y, z, x)+(z, x, y).$$

Furthermore, if char. $\ne 2$ in a power-associative ring then

(2) $$(x, x, y)+(x, y, x)+(y, x, x) = 0.$$

Consequently,

(3) $$(xy, x)+(yx, x)+(x^2, y) = 0.$$

If $\{x\}$ is a left ideal, we then have $\{yx\}$ is in $\{x\}$ so

(4) $$(xy, x)+(x^2, y) = 0.$$

We shall now establish the following result.

THEOREM 2.1. *If A is a non-nil L-algebra over a field F of* char. $\ne 2$ *then either $A=\{e\}\oplus B$ for e an idempotent and B a nil L-algebra or A has a basis which under multiplication forms a semigroup S_α.*

Proof. Suppose A is non-nil and let a be not nilpotent. Then $\{a\}$ is finite-dimensional so there is an idempotent e in $\{a\}$. Now,

$$A = A_e(1)+A_e(\tfrac{1}{2})+A_e(0).$$

Also, for x in $A_e(1)$, $xe=ex=x$ so x is in $\{e\}$. Therefore $A_e(1)=\{e\}$.

We will now prove that $A_e(0)$ is a nil L-algebra. Since $A_e(0)\cdot A_e(0) \subseteq A_e(0)$ then $\{x\} \subseteq A_e(0)$ for any x in $A_e(0)$. Hence, y in $A_e(0)$ implies yx is in $\{x\} \subseteq A_e(0)$ so $A_e(0)$ is a subalgebra. It is clearly an L-algebra. If x is not nilpotent then there is an idempotent f in $\{x\} \subseteq A_e(0)$. Hence $f^2=f$, $ef=fe=0$. Obviously $g=e+f$ is an idempotent so $\{e+f\}$ is one-dimensional. But, $e(e+f)=e$, $f=f(e+f)$ are both in $\{e+f\}$. This contradiction establishes the fact that there can be no idempotent in $A_e(0)$ so $A_e(0)$ is nil.

Now, let x be in $A_e(\frac{1}{2})$. We have $xe=\alpha e$ for xe is in $\{e\}$. Hence $ex=xe+ex-xe=x-\alpha e$ is in $\{x\}$. From this $0=(ex, x)=(x-\alpha e, x)=-\alpha(e, x)$ so either $\alpha=0$ or

$ex=xe=(\frac{1}{2})x$ which is impossible since $xe=\alpha e$. We have shown that $ex=x$ and $xe=0$. Now x^2 is in $A_e(\frac{1}{2})\cdot A_e(\frac{1}{2})\subseteq A_e(1)+A_e(0)$ so $x^2=\beta e+z$ with z in $A_e(0)$. Since $A_e(0)$ is nil, $z^3=0$ and $(x^2)^2=(\beta e+z)^2=\beta^2 e+z^2$, $(x^2)^3=\beta^3 e$. But A is power-associative so $\beta^3 x=(x^2)^3 x=x(x^2)^3=0$ and $\beta=0$. Therefore $x^2=z$ and x is nilpotent. But this implies $x^3=0$. If $x^2\neq 0$ then $(e+x+x^2)^2=e+x+x^2$ for x^2 in $A_e(0)$. Also, e, x, x^2 are linearly independent. Now $\{e+x+x^2\}$ is one-dimensional but $e+x=e(e+x+x^2)$ and $x^2=x(e+x+x^2)$ are in $\{e+x+x^2\}$. Hence, $x^2=0$. If y is in $A_e(0)$ then xy is in $\{y\}$ in $A_e(0)$ and $yx=\alpha x$ is in $A_e(\frac{1}{2})$. But $xy+yx=2x\cdot y$ is in $A_e(\frac{1}{2})+A_e(1)$ so $xy=0$.

If $y^2=0$ then (4) implies $\alpha(x, y)=(yx, y)=0$. Thus, $yx=0$. Now, $(x+y)^2=0$ so $\{x+y\}$ is one-dimensional. Since $x=e(x+y)$ and $x+y$ are in $\{x+y\}$ we conclude that either $x=0$ or $y=0$.

If $y^2\neq 0$ then $y^3=0$. Now, interchanging x and y in (3) gives

$$(yx, y)+(y^2, x) = 0$$

so $(y^2, x)=\alpha^2 x$. But, $xy^2=0$ so $y^2x=\alpha^2 x$. Now, letting $z=y^2$ we have $z^2=0$ so we have shown $zx=xz=0$. Hence, $\alpha^2=0$ and $yx=xy=0$. Now, $(x+y)^3=y^2(x+y)=0$ so $\{x+y\}$ has dimension two. However, $x=e(x+y)$, $x+y$, $y^2=y(x+y)$ are in $\{x+y\}$. This contradiction shows that $x=0$ or $y=0$.

We conclude that $A_e(\frac{1}{2})\neq 0$ implies $A_e(0)=0$. If $A_e(\frac{1}{2})=0$ then either $A=\{e\}$ which has basis S_α for $\alpha=1$ or $A=\{e\}\oplus A_e(0)$ where $A_e(0)$ is a nil L-algebra.

If $A_e(\frac{1}{2})\neq 0$, let $\{x_\beta\}$ be a basis for $A_e(\frac{1}{2})$. Clearly, e, $\{y_\beta\}$ form a basis for A where $y_\beta=e+x_\beta$. Now $y_\beta y_\gamma=(e+x_\beta)(e+x_\gamma)=e+x_\gamma+x_\beta x_\gamma$. We have $x_\beta x_\gamma=ax_\gamma$ and $x_\gamma x_\beta=bx_\beta$ with $x_\beta x_\gamma+x_\gamma x_\beta$ in $A_e(1)+A_e(0)$. Hence $a=b=0$ and $y_\beta y_\gamma=y_\gamma$. Also, $y_\beta e=e$, $ey_\beta=y_\beta$ so e, $\{y_\beta\}$ forms a semigroup S_α under multiplication. The proof of the theorem is now complete.

3. **Nil L-algebras.** Throughout this section, we will assume that A is a nil algebra over a field F of char. $\neq 2$.

LEMMA 3.1. *If* $x^2=0$ *then* $xA=Ax=0$.

Proof. We will first prove that $xy=yx=0$ when $y^2=0$. Indeed, (4) implies $(xy, x)=0=(yx, y)$. If $xy=ky$ then $k=0$ or $xy=yx$. Also, $yx=mx$ implies $m=0$ or $xy=yx$. Now, if $xy=yx$ then $mx=ky$. Hence, in any case $xy=yx=0$.

Now, let $y^2\neq 0$, $y^3=0$. From above, $xy^2=y^2x=0$ so (4) implies $(yx, y)=0=(xy, x)$. Let $yx=kx$ and $xy=my+ny^2$. Hence, $(xy, x)=0$ implies $mkx=(xy)x=x(xy)=x(my+ny^2)=mxy=m^2y+mny^2$. If $m\neq 0$, we have $kx=my+ny^2$ so $0=kx^2=x(my+ny^2)=m^2y+mny^2$. Since y, y^2 are linearly independent this is impossible so $m=0$. Also, $0=(yx, y)=k(x, y)=kny^2-k^2x$. Hence, $0=y(kny^2-k^2x)=-k^3x$. Therefore, $k=0$ and $yx=0$. Consider $\{x-ny\}$. We have $(x-ny)^2=x^2-nxy-nyx+n^2y^2=0$. Therefore $\{x-ny\}$ is one-dimensional. If $n\neq 0$ then $y(x-ny)=-ny^2$ so $y^2=\alpha(x-ny)$ since $y^2\neq 0$. We then have $0=y^3=\alpha(yx-ny^2)=-\alpha ny^2$. This is impossible so $n=0$ and $xy=yx=0$.

LEMMA 3.2. *If* $x^2 \neq 0 \neq y^2$ *then for* $\alpha \neq 0$, α, β, γ *in* F *we have* $x^2 = \alpha y^2$, $xy = \beta y^2$ *and* $yx = \gamma x^2$.

Proof. From Lemma 3.1, $x^2y = yx^2 = y^2x = xy^2 = 0$. Now, (4) implies $(xy, x) = 0 = (yx, y)$. Write $xy = cy + dy^2$ and $yx = mx + nx^2$. Now $0 = (yx, y) = m(x, y) + n(x^2, y) = m(x, y)$ and $0 = (xy, x) = c(y, x) + d(y^2, x) = c(y, x)$. Hence, either $xy = yx$ or $m = c = 0$. If $xy = yx$ then $xy = (\frac{1}{2})[(x+y)^2 - x^2 - y^2]$ so $(xy)^2 = 0$. If $c \neq 0$ then $x = (xy - dy^2)/c$ so $x^2 = 0$ which contradicts our assumption that $x^2 \neq 0$. Therefore $c = 0$. Similarly $m = 0$. We have $xy = dy^2$ and $yx = nx^2$ as desired.

If $(x+y)^2 = 0$ then $x^2(1+n) + y^2(1+d) = (x+y)^2 = 0$ so $x^2 = \alpha y^2$ with $\alpha \neq 0$ unless $n = -1$, $d = -1$. In this case, $dn = 1$.

If $(x+y)^2 \neq 0$ then, since $(x+y)^3 = 0$, $\{x+y\}$ is two-dimensional. Now, $x^2 + dy^2 = x(x+y)$ and $y^2 + nx^2 = y(x+y)$ are in $\{x+y\}$ so there exist r, s, and t not all zero with $r(x^2 + dy^2) + s(y^2 + nx^2) + t(x+y) = 0$. If $t \neq 0$ then $(x+y)^2 = 0$. Hence, $t = 0$. If x^2 and y^2 are linearly dependent, we are done; so assume that x^2 and y^2 are linearly independent. Then $r + sn = dr + s = 0$ and $r = -sn = -drn$. If $r = 0$ then $s = 0$. Hence $r \neq 0$ and $dn = 1$ in this case as well.

Now, $(x - dy)^2 = x^2 - dxy - dyx + d^2y^2 = 0$ so $\{x - dy\}$ is one-dimensional. Therefore $x(x - dy) = x^2 - d^2y^2 = a(x - dy)$ for a in F. Hence $0 = x(x^2 - d^2y^2) = a(x^2 - d^2y^2)$. If $a = 0$, we have $x^2 = d^2y^2$ with $d \neq 0$. If $a \neq 0$ then $x^2 = d^2y^2$ with $d \neq 0$ and the proof of the lemma is complete.

THEOREM 3.1. *If A is a nil algebra over a field of* char. $\neq 2$ *then A is an L-algebra if and only if A is an H-algebra.*

Proof. Clearly, if A is an H-algebra then A is an L-algebra. Now let A be a nil L-algebra. If $x^2 = 0$ then $\{x\}$ is an ideal by Lemma 3.1. If $x^2 \neq 0$ then $x^3 = 0$ and $yx = \gamma x^2$. Also, $xy = \beta y^2 = (\beta/\alpha)x^2$ and $\{x\}$ is an ideal. Hence, every subalgebra of the form $\{x\}$ is an ideal and we are done by Lemma 2.1.

4. Proof of the main theorem.

THEOREM 4.1. *If A is an algebra over a field F of* char. $\neq 2$ *then A is an L-algebra if and only if A is an H-algebra or has a basis* S_α *where* α *is the dimension of A.*

Proof. Let A be an L-algebra. If A is nil then A is an H-algebra by Theorem 3.1. If A is non-nil, then by Theorem 2.1 either $A = \{e\} \oplus B$ with B a nil L-algebra or A has a basis S_α. We claim that $\{e\} \oplus B$ is an H-algebra. If $x \in B$ then $\{x\}$ is an ideal in B. Since $ey = ye = 0$ for y in B then $\{x\}$ is an ideal in A. Now, $\{e\}$ is an ideal in A. Finally, let $x = \alpha e + y$ with y in B and α in F, $\alpha \neq 0$. Now, $x^2 = \alpha^2 e + y^2$ and $x^3 = \alpha^3 e$. If $y^2 = 0$, $\{x\}$ is spanned by e and y while if $y^2 \neq 0$ then $\{x\}$ is spanned by e, y and y^2. Now $zx = \alpha ze + zy$ is in $\{e\} + \{y\} = \{x\}$ and $xz = \alpha ez + yz$ is in $\{e\} + \{y\} = \{x\}$ and A is an H-algebra.

Conversely, an H-algebra is an L-algebra. Suppose A is an algebra with basis S_α. If x and y are in A then x and y are linear combinations of a finite number of

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elements of S_α. Call this set $\{z_i\}_{i=1}^n$. Hence,

$$x = \sum_{i=1}^n \alpha_i z_i$$

$$y = \sum_{i=1}^n \beta_i z_i.$$

Now,

$$\begin{aligned} yx &= \left(\sum_{i=1}^n \beta_i z_i\right)\left(\sum_{i=1}^n \alpha_i z_i\right) \\ &= \sum_{j=1}^n \sum_{i=1}^n \alpha_i \beta_j z_j z_i \\ &= \sum_{j=1}^n \sum_{i=1}^n \alpha_i \beta_j z_i \\ &= \left(\sum_{j=1}^n \beta_j\right)\left(\sum_{i=1}^n \alpha_i z_i\right) \\ &= \left(\sum_{j=1}^n \beta_j\right) x. \end{aligned}$$

Hence, A is an L-algebra. Now, if $\alpha=1$ then A is also an H-algebra. Suppose $\alpha>1$. Then $z_1 z_2 = z_2$ which is not in $\{z_1\}$. We also have proved

THEOREM 4.2. *If an algebra A has a basis S_α then A is an H-algebra if and only if* $\alpha=1$.

Finally, we prove

THEOREM 4.3. *An algebra A over a field F has basis S_α with $\alpha>1$ if and only if A is a vector space sum $\{e\}+B$ where $e^2=e\neq 0$ and B is a zero algebra such that $be=eb-b=0$ for b in B.*

Proof. Let e be a fixed element in S_α and let $\{x_\beta\}_{\beta\in C}$ be the complement of e in S_α. Define $y_\beta=x_\beta-e$ for β in C. Now let B be the algebra over F with basis $\{y_\beta\}_{\beta\in C}$. We have $y_\beta y_\gamma=(x_\beta-e)(x_\gamma-e)=0$ so B is a zero algebra. Also $ey_\beta=ex_\beta-ee=y_\beta$ and $y_\beta e=x_\beta e-e=0$. Conversely, if $A=\{e\}+B$ where B is a zero algebra and $be=eb-b=0$, let $\{y_\beta\}_{\beta\in C}$ be a basis of B. Then e, $\{x_\beta\}_{\beta \text{ in } C}$ is a basis for A where $x_\beta=e+y_\beta$ and this set is a semigroup of the form S_α.

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