The Dirichlet boundary value problem for two non-overlapping spheres

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A new method is found for solving the general Dirichlet problem
for two non-overlapping spheres of different radius. The
expression for the external potential involves hypergeometric
functions and is obtained from an infinite set of linear
equations. In essence the method makes use of the fact that

$$(I-A)^{-1} = I + A + A^2 + A^3 + \ldots ,$$

where A belongs to a certain class of infinite matrices and I
is the unit matrix.

1. Introduction

The electrostatic potential exterior to two conducting spheres has
been obtained by Kottler [1] and Mitra [2]. These and other authors have
made use of Kelvin's method of images which, together with an ingenious set
of transformations, leads to the final result. More recently Shail [3]
solved the Dirichlet problem for two spheres. He reduced the problem to an
integral equation which, after using the transformations proposed by
Kottler [1] and Mitra [2], is found to have a simple iterative solution.
Now Ross [4], who was concerned with the potential near two dielectric
spheres, suggested that a more direct approach could be successful. He
expanded the potentials as series of spherical harmonics and showed that
the coefficients of these series satisfy an infinite set of linear
equations. These can, in principle, be solved by a matrix inversion. In
this paper it is found that the general Dirichlet problem for two

non-overlapping spheres leads to an infinite set of linear equations which can be solved *explicitly* by a matrix inversion. The final expression for the potential is found to be consistent with that given by Shail [3] but the present series is more rapidly convergent.

2. Expansions for the potential

Consider two spheres S_1 and S_2 which do not overlap, are of radius a_1 and a_2 and at separation $R = O_1O_2$, where O_1 and O_2 are the centres of the spheres. [Here the suffix 1 and 2 is used to refer to the first and second sphere respectively.] Let P be a typical point in the vicinity such that $O_iP = r_i$ for $i = 1$ or 2 and let θ_1 and θ_2 be the angles PO_1O_2 and PO_2O_1 . These coordinates are such that $r_1 = r_2$ and $\theta_1 = \theta_2$ for any point P on the median plane. If we suppose that the electrostatic potential Φ vanishes at infinity and that there are no charges in the region exterior to the two spheres then we may apply Green's theorem to find the potential at the typical point P outside the spheres. Thus

$$\Phi(P) = \frac{1}{4\pi} \int \left\{ \Phi(Q_i) \frac{\partial}{\partial n} \left(\frac{1}{\rho_i}\right) - \frac{1}{\rho_i} \frac{\partial \Phi(Q_i)}{\partial n} \right\} dS_i \ ,$$

where the integration is over $S_1 + S_2$, Q_i is on the boundary of S_i and has spherical polar coordinates $\left(a_i,\ \theta_i^*,\ \phi_i^*\right)$, $\rho_i = PQ_i$, $\Phi(Q_i)$ is the potential on the sphere S_i and $\partial/\partial n$ denotes differentiation along the inward normal n . An expression for $\Phi(P)$ can be found by integrating the above equation with respect to ϕ_i^* , remembering that the potential is independent of the azimuthal angle ϕ and using the identities

$$\rho_i^2 = a_i^2 + r_i^2 - 2a_i r_i \cos\gamma_i$$

with

$$\cos\gamma_i = \cos\theta_i \cos\theta_i^* + \sin\theta_i \sin\theta_i^* \cos\left(\phi_i - \phi_i^*\right)$$

together with the addition formula.

$$P_n(\cos\gamma) = P_n(\cos\theta)P_n(\cos\theta^*) + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\ P_n^m(\cos\theta)P_n^m(\cos\theta^*)\cos m(\phi-\phi^*)\ .$$

Here $P_n^m(x)$ is the associated Legendre function as defined in Hobson [5].
In this way we obtain

$$(1)\quad \Phi(P) = \sum_{n=0}^{\infty}\left[A_n^{(1)}(a_1/r_1)^{n+1}P_n(\cos\theta_1) + A_n^{(2)}(a_2/r_2)^{n+1}P_n(\cos\theta_2)\right]$$

where

$$A_n^{(i)} = \tfrac{1}{2}\int_0^{\pi} X_n(Q_i)P_n(\cos\theta_i^*)\sin\theta_i^*d\theta_i^*$$

and

$$X_n(Q_i) = a_i\partial\Phi(Q_i)/\partial n - n\Phi(Q_i)\ .$$

By Schwarz's inequality it follows that

$$\left(A_n^{(i)}\right)^2 \le \tfrac{1}{4}\int_0^{\pi} X_n^2(Q_i)\sin\theta_i^*d\theta_i^*\ .\ \int_0^{\pi} P_n^2(\cos\theta_i^*)\sin\theta_i^*d\theta_i^*$$

$$= [1/(4n+2)]\int_0^{\pi} X_n^2(Q_i)\sin\theta_i^*d\theta_i^*\ .$$

Since the potential and its gradient normal to the boundary of S_i are
integrable we obtain the result $\left|n^{-\frac{1}{2}}A_n^{(i)}\right|$ is bounded for all n. Thus
the two series in (1) representing $\Phi(P)$ are absolutely and uniformly
convergent in the region exterior to the two spheres, except perhaps at
the points where $\theta_i = 0$ or π.

3.   The boundary value problem

We shall consider the Dirichlet boundary value problem where the
potential outside the two spheres satisfies the conditions:

$$(2a)\qquad \Phi = B_iP_{k_i}(\cos\theta_i)\quad \text{on}\quad r_i = a_i\quad \text{for}\quad i = 1, 2\ ,$$

and

(2b) $\Phi \to 0$ as r_1 and $r_2 \to \infty$.

In order to apply these conditions in equation (1) we shall need the shift
formula of zonal harmonics, namely

$$\frac{P_n(\cos\theta_2)}{r_2^{n+1}} = \frac{1}{R^{n+1}} \sum_{m=0}^{\infty} \binom{m+n}{n} \left(\frac{r_1}{R}\right)^m P_m(\cos\theta_1) ,$$

which is valid for $|r_1| < R$, and a similar expression with the suffices
1 and 2 interchanged. These equations are given in Hobson [5] although
he defined the angles θ_i in a slightly different way.

From the boundary conditions (2a) we get, after substitution of the
above result into equation (1), rearranging the absolutely convergent
series and making use of the orthogonality of the Legendre polynomials

$$(3) \qquad\qquad B_1 \delta_{m,k_1} = A_m^{(1)} + t_2 \sum_{s=0}^{\infty} p_{m,s} A_s^{(2)}$$

and

$$(4) \qquad B_2 \delta_{m,k_2} = A_m^{(2)} + t_1 \sum_{s=0}^{\infty} p_{s,m} A_s^{(1)} \quad \text{for} \quad m = 0, 1, 2, \ldots ,$$

where

$$(5) \qquad p_{m,n} = \binom{m+n}{n} t_1^m t_2^n , \quad t_i = a_i/R \quad \text{and} \quad 0 \le t_1 + t_2 < 1 .$$

Here $\delta_{i,j}$ is the Kronecker delta function. Elimination of $A_m^{(1)}$ from
equations (3) and (4) leads to

$$(6) \qquad B_2 \delta_{n,k_2} - B_1 t_1 p_{k_1,n} = \sum_{s=0}^{\infty} \left\{ \delta_{n,s} - q_{n,s}^{(2)} \right\} A_s^{(2)}$$

where

$$(7) \qquad q_{n,s}^{(2)} = t_1 t_2 \sum_{m=0}^{\infty} p_{m,n} p_{m,s} .$$

Similarly, on elimination of $A_m^{(2)}$ we obtain

(8) $$B_1 \delta_{n,k_1} - B_2 t_2 p_{n,k_2} = \sum_{s=0}^{\infty} \left\{ \delta_{n,s} - q_{n,s}^{(1)} \right\} A_s^{(1)}$$

where

$$q_{n,s}^{(1)} = t_1 t_2 \sum_{m=0}^{\infty} p_{n,m} p_{s,m} .$$

4.   Inverse of the infinite matrices

Now it is a simple matter to verify that the sums of the rows and columns of the matrices $q_{n,s}^{(i)}$ are each of modulus less than 1 . Hence, from Theorem 2.4,I of Cooke [6], it follows that $\delta_{n,s} - q_{n,s}^{(i)}$ has the unique and two-sided reciprocal

$$I + \sum_{m=1}^{\infty} Q_i^m , \quad Q_i = \left[q_{n,s}^{(i)} \right] , \quad I \text{ the unit matrix .}$$

The next step is to find a sufficiently simple expression for the various powers of the matrices Q_i . We begin with three lemmas.

LEMMA 1. *Let* $\{U_n\}$ *with* $n = 1, 2, 3, \ldots$ *be a set of functions of* (t_1, t_2) *which satisfy the difference equations*

(10) $$U_{2n+1} = U_{2n} - t_1^2 U_{2n-1}$$

and

(11) $$U_{2n} = U_{2n-1} - t_2^2 U_{2n-2}$$

where t_1 *and* t_2 *are positive constants such that* $0 \leq t_1 + t_2 < 1$ *and the initial conditions are*

$$U_1 = 1 , \quad U_2 = 1 - t_2^2 .$$

Then the following deductions can be made:

(a) *if* $t_1 = t_2 = 0$ *then* $U_n = 1$ *for all* n ,

(b) *if* t_1 *and* t_2 *are not zero then*

(12) $$U_{2n-1} = (t_1 t_2)^{n-1} \sinh n\mu / \sinh \mu$$

and

(13) $U_{2n} = t_1^n t_2^{n-1}\{t_1\sinh n\mu + t_2\sinh(n+1)\mu\}/\sinh\mu$

with μ *positive and given by*

(14) $\cosh\mu = (1-t_1^2-t_2^2)/(2t_1t_2)$.

   (c) $U_{2n+1} = (1-t_2^2)U_{2n-1} - t_1^2\Big\{U_{2n-1}^2 - t^{2n-2}t_2^{2n}\Big\}/U_{2n}$,

   (d) $0 \le t^2 U_{2n-1} < (1-t_2^2)U_{2n}$,

where the equality is only possible when $t_1 = 0$.

   The proof of *(b)* is based on the general theory of linear difference equations as discussed in the book by Durell and Robson [7] and the conclusion in *(c)* may be obtained by a direct substitution of *(b)*. The inequalities in *(d)* follow from the fact that

$0 \le t_2^2 U_{2n}/U_{2n+1} = t_2\{t_1\sinh n\mu + t_2\sinh(n+1)\mu\}/\sinh(n+1)\mu < t_2(t_1+t_2) < 1$.

Combining this with equation (10) gives the required result.

   LEMMA 2.

$$\sum_{k=0}^{\infty} \binom{i+k}{k} x^k \, {}_2F_1(-k, -j; 1; z) = \frac{(1-x+xz)^j}{(1-x)^{i+j+1}} \, {}_2F_1\left(-i, -j; 1; \frac{xz}{1-x+xz}\right) ,$$

where ${}_2F_1(a, b; c; z)$ *is the hypergeometric function,* i *and* j *are positive integers and* $|x|$ *and* $|z|$ *are each less than* 1 . *This identity can be proved by using the power series expansion*

$$ {}_2F_1(-k, -j; 1; z) = \sum_{n=0}^{\min(k, j)} \frac{(-k)_n(-j)_n z^n}{(-1)_n n!} ,$$

given in Bateman [8], *and changing the order of the summation on the left-hand side of the identity.*

   LEMMA 3.

$$\sum_{k=0}^{\infty} x^k \, {}_2F_1(-i, -k; 1; A) \, {}_2F_1(-k, -j; 1; B)$$

$$= (1-x+Ax)^i(1-x+Bx)^j \, {}_2F_1(-i, -j; 1; S)/(1-x)^{i+j+1}$$

where

$$S = xAB/[(1-x+Ax)(1-x+Bx)] ,$$

i and j are positive integers and $|x|$, $|A|$ *and* $|B|$ *are each less than* 1 .

This identity can be proved by using the expansion

$$_2F_1(-k, -j; 1; B) = (1-B)^{j+k+1} \sum_{m=0}^{\infty} \binom{k+m}{m}\binom{j+m}{m} B^m ,$$

changing the order of the summation and applying Lemma 2 appropriately.

We shall now prove, by induction, that the matrix of coefficients of the n-th power of $q_{n,s}^{(1)}$ is given by

$$(15) \qquad t_1^{m+s}(t_1t_2)^n U_{2n-1}^{m+s} \, _2F_1\left(-m, -s; 1; t_1^{2n-2}t_2^{2n}/U_{2n-1}^2\right)/U_{2n}^{m+s+1}$$

for $n = 1, 2, 3, \ldots$.

This result is trivial for $n = 1$ since

$$q_{m,s}^{(1)} = t_1^{m+s}(t_1t_2) \sum_{n=0}^{\infty} \binom{m+n}{n}\binom{n+s}{s} t_2^{2n}$$

$$= t_1^{m+s} (t_1t_2) \, _2F_1(-m, -s; 1; t_2^2)/(1-t_2^2)^{m+s+1} .$$

This follows from a transformation formula for the hypergeometric function given in Bateman [8].

Let us consider the matrix product

$$\sum_{k=0}^{\infty} \left\{q_{m,k}^{(1)}\right\}^n q_{k,s}^{(1)} = \frac{t_1^{m+s}(t_1t_2)^{n+1} U_{2n-1}^m}{U_{2n}^{m+1} U_2^{s+1}} \sum_{k=0}^{\infty} \left\{\frac{t_1 U_{2n-1}}{U_{2n} U_2}\right\}^k$$

$$_2F_1\left(-m, -k; t_1^{2n-2}t_2^{2n}/U_{2n-1}^2\right) \, _2F_1(-k, -s; 1; t_2^2);$$

then from Lemmas 1 and 3 it follows, after a considerable amount of algebra, that

$$\sum_{k=0}^{\infty} \left\{q_{m,k}^{(1)}\right\}^n q_{k,s}^{(1)} = \left\{q_{m,s}^{(1)}\right\}^{n+1} .$$

This completes the proof.

Equation (15) may now be used to find the inverse of the matrix $\delta_{n,s} - q_{n,s}^{(1)}$ which, when applied to equation (8), determines the set of constants $A_s^{(1)}$. In particular, if we make use of the symbol

(16) $$U_m^* = U_m(t_2, t_1) \quad \text{when} \quad U_m = U_m(t_1, t_2)$$

then

(17)
$$
\begin{cases}
A_s^{(1)} = B_1 \delta_{s,k_1} - B_2 \binom{s+k_2}{s} t_1^s t_2^{1+k_2} + B_1 t^{s+k_1} \sum_{n=1}^{\infty} (t_1 t_2)^n U_{2n-1}^{s+k_1} \\
\quad {}_2F_1\left(-s, -k_1; 1; t_1^{2n-2} t_2^{2n}/U_{2n-1}^2\right)/U_{2n}^{1+s+k_1} - B_2 t_1^s t_2^{1+k_2} \\
\quad \sum_{n=1}^{\infty} (t_1 t_2)^n U_{2n}^{*s} U_{2n}^{k_2} \; {}_2F_1\left(-s, -k_2; 1; t_1^{2n} t_2^{2n}/U_{2n} U_{2n}^*\right)/U_{2n+1}^{1+s+k_2} \; .
\end{cases}
$$

The formula for $A_s^{(2)}$ can be deduced from the above by interchanging the suffices 1 and 2 . The equations for $A_s^{(i)}$ can be simplified by making use of the transformations

(18) $$\sinh\lambda_1 = t_1\sinh\mu \quad \text{and} \quad \sinh\lambda_2 = t_2\sinh\mu .$$

If we combine these with equation (14) then we find that

$$\cosh\lambda_1 = t_2 + t_1\cosh\mu , \quad \cosh\lambda_2 = t_1 + t_2\cosh\mu \quad \text{and} \quad \mu = \lambda_1 + \lambda_2 ,$$

and on substitution of these identities into (13) we obtain

$$U_{2n} = t_1^n t_2^{n-1}\sinh(n\mu+\lambda_2)/\sinh\mu .$$

Finally equation (17) becomes

$$A_s^{(1)} = \sinh\lambda_2\left[B_1 \sum_{n=1}^{\infty} \frac{\sinh^{s+k_1}n\mu}{\sinh^{1+s+k_1}(n\mu+\lambda_2)} \; {}_2F_1\left(-s, -k_1; 1; \frac{\sinh^2\lambda_2}{\sinh^2 n\mu}\right)\right.$$

$$\left. - B_2 \sum_{n=1}^{\infty} \frac{\sinh^s(n\mu+\lambda_1)\sinh^{k_2}(n\mu+\lambda_2)}{\sinh^{1+s+k_2}(n+1)\mu} {}_2F_1\left(-s, -k_2; 1; \frac{\sinh\lambda_1\sinh\lambda_2}{\sinh(n\mu+\lambda_1)\sinh(n\mu+\lambda_2)}\right)\right] .$$

It is a simple matter to verify that this result is consistent with the paper of Mitra [2] who treated the case of two conducting spheres whence $k_1 = k_2 = 0$ and the hypergeometric function disappears

altogether. This work is also in agreement with the result obtained by Shail [3] who did not express the coefficients $A_s^{(i)}$ in terms of hypergeometric functions and did not make use of their properties in order to improve the convergence rate.

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