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COPRODUCTS OF ALGEBRAS AND DERIVATIONS ON CATEGORIES

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Coproducts and tensor products of algebras for essentially algebraic theories are exhibited as Kan extensions and relationships with derivations on monoidal closed categories are described.

1. Coproducts and tensor products

We shall assume some familiarity with the notation, terminology and results of G. M. Kelly [3] (except that we dualise the notion of a theory). Let $V = (V_o, I, \otimes, [-,-], \ldots)$ be a (cocomplete) locally finitely presentable closed category and let T be an essentially algebraic theory over V. Let A denote the category $\mathrm{Lex}(T^{op}, V)$ of T-algebras.

PROPOSITION 1. *If* A *and* B *are* T*-algebras then*

$$A + B \cong \int^{mn} Am \otimes Bn \otimes T(-, m+n)$$

Proof. Both A and B are filtered colimits $\mathrm{colim}\, T(-, m_\phi)$ and $\mathrm{colim}\, T(-, n_\psi)$ of representables, respectively. Thus $A + B$ in $[T^{op}, V]$ actually lies in A. Moreover

$$A(\int^{mn} Am \otimes Bn \otimes T(-, m+n), C)$$

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$$\cong \int_{mn} [\operatorname{colim} T(m, m_\phi) \otimes \operatorname{colim} T(n, n_\psi), A(T(-, m+n), C)]$$

$$\cong \lim_\phi \lim_\psi C(m_\phi + n_\psi) \quad \text{by the Yoneda lemma,}$$

$$\cong \lim_\phi \lim_\psi (Cm_\phi \times Cn_\psi) \quad \text{since } C \text{ is an algebra,}$$

$$\cong (\lim_\phi Cm_\phi) \times (\lim_\psi Cn_\psi) \quad \text{since the limits are connected,}$$

$$\cong A(A, C) \times A(B, C) \quad \text{by the Yoneda lemma.}$$

This isomorphism is natural in $C \in A$ so the result follows from the Yoneda lemma. □

Let us call T a *commutative* theory if it has a symmetric monoidal structure with tensor product $\otimes$ such that $- \otimes t : T \to T$ preserves finite colimits for all $t \in T$.

PROPOSITION 2. *If T is a commutative theory then A is closed under tensor product and exponentiation in $[T^{op}, V]$.*

Proof. The tensor product on $[T^{op}, V]$ is given by:

$$A \otimes B = \int^{mn} Am \otimes Bn \otimes T(-, m \otimes n).$$

Since $A \otimes B$ is then a filtered colimit of representable functors by the Yoneda lemma, it follows that $A \otimes B$ is left exact (hence is an algebra). Also A is closed under exponentiation in $[T^{op}, V]$ since $A(t \otimes -)$ lies in A for all $A \in A$ and $t \in T$. □

One basic example of this situation is V itself which can be represented at $\text{Lex}(V_f^{op}, Set)$ where V_f denotes the finitely presented objects in V. Conversely, it can be seen that if T is any commutative essentially algebraic theory (over V) then $A = \text{Lex}(T^{op}, V)$ is an l.f.p. closed category. This follows from the fact that $[M, V]$ (as constructed in [1]) is an l.f.p. closed category for any small symmetric monoidal category M (over V), together with the observation that if $T : W \to V$

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is the right adjoint in a symmetric monoidal closed adjunction with V an l.f.p. closed category, W cocomplete, and T conservative and filtered-colimit preserving then W is again an l.f.p. closed category.

The analogies of all these results hold with $|\mathbb{N}|$ replaced by any infinite regular cardinal.

2. Examples of derivations

F. W. Lawvere called a V-endofunctor D, on a symmetric monoidal closed category V with finite colimits, a *derivation* if the canonical morphism

$$DX \otimes Y + X \otimes DY \to D(X \otimes Y)$$

is an isomorphism and $DX + DY \cong D(X + Y)$. He gives, as an example, the derivation arising in A. Joyal [2]; such derivations are of interest in the study of formal series.

We have the following additional examples:

EXAMPLE 1. Let C be a small category with universal disjoint finite coproducts and let $C \in C$ be such that

$$C(C,A + B) \cong C(C,A) + C(C,B)$$

for all $A,B \in C$. Let G denote the category (groupoid) whose objects are those of C and whose morphisms are the isomorphisms in C. Then the internal-hom functor $[G(-,C),-] : [G^{op},V] \to [G^{op},V]$ is a derivation, where $[G^{op},V]$ has the convolution structure with respect to the trace of finite coproducts on G (see B. J. Day [1]). There are also various "internal" instances of this example.

EXAMPLE 2. Given V we can consider the formal power series category $V^{\mathbb{N}}$ with derivation D given by $(DX)_n = (n + 1)X_{n+1}$. There is also a "many-variable" form of this example.

EXAMPLE 3. Given a derivation D on V, we can form the *dual* derivation $\hat{D}$ on $\mathrm{Lex}(V^{op},SET)$ whose values are given by left Kan extension:

$$\hat{D}(A) = \int^{X} A(X) \times V(-,DX)$$

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It is easily verified, using Propositions 1 and 2 and the Yoneda lemma, that this is a derivation.

References

[1] Brian Day, "On closed categories of functors", *Reports of the Midwest Category Seminar IV*, 1-38 (Lecture Notes in Mathematics 137. Springer-Verlag, Berlin, Heidelberg, New York, 1970).

[2] André Joyal, "Une théorie combinatoire des séries formelles", *Advances in Mathematics* 42 (1981), 1-82.

[3] G. M. Kelly, "Structures defined by finite limits in the enriched context I", *Cahiers Topologie Géom. Différentielle* 23 (1982), 3-42.

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